Bull. Aust. Math. Soc. **93** (2016), 447–453
doi:10.1017/S0004972715001252

ON THE PROBABILITY OF GENERATING NILPOTENT SUBGROUPS IN A FINITE GROUP

S. M. JAFARIAN AMIRI✉, H. MADADI and H. ROSTAMI

(Received 13 July 2015; accepted 28 July 2015; first published online 20 November 2015)

Abstract

Let G be a finite group. We denote by $\nu(G)$ the probability that two randomly chosen elements of G generate a nilpotent subgroup and by $\mathrm{Nil}_G(x)$ the set of elements $y \in G$ such that $\langle x, y\rangle$ is a nilpotent subgroup. A group G is called an $\mathcal{N}$-group if $\mathrm{Nil}_G(x)$ is a subgroup of G for all $x \in G$. We prove that if G is an $\mathcal{N}$-group with $\nu(G) > \frac{1}{12}$, then G is soluble. Also, we classify semisimple $\mathcal{N}$-groups with $\nu(G) = \frac{1}{12}$.

2010 *Mathematics subject classification*: primary 20D60; secondary 20P05.

Keywords and phrases: nilpotent group, probability, $\mathcal{N}$-group.

1. Introduction

Throughout this paper G will be a finite group. Probability theory has made significant contributions to the study of finite groups. An early example concerns the commutativity degree of a finite group. The commutativity degree of G, denoted by $\mathrm{cp}(G)$, is the probability that two randomly chosen elements of G commute: that is

$$\mathrm{cp}(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

This concept was introduced in 1968 by Erdős and Turàn [4]. They showed that $\mathrm{cp}(G) = k(G)/|G|$, where $k(G)$ is the number of conjugacy classes of G. A few years later, Gustafson [9] showed that $\mathrm{cp}(G) \le \frac{5}{8}$ for any nonabelian finite group G and that equality holds when $|G : Z(G)| = 4$, where $Z(G)$ is the centre of G.

In 1992, Dubose-Schmidt *et al.* [3] took the idea in another direction. For every positive integer i, they defined $\nu_i(G)$ as the probability that two randomly chosen elements of a group G generate a subgroup of nilpotency class i. Fulman *et al.* [5] introduced $\nu_0(G)$ as the proportion of ordered pairs of G that generate a nonnilpotent subgroup. Here, we denote by $\nu(G)$ the probability that two randomly chosen elements

© 2015 Australian Mathematical Publishing Association Inc. 0004-9727/2015 $16.00

https://doi.org/10.1017/S0004972715001252 Published online by Cambridge University Press

of G generate a nilpotent subgroup. Clearly, $\nu(G) = 1 - \nu_0(G) = \sum_{i=1}^{\infty} \nu_i(G)$ and $\mathrm{cp}(G) \le \nu(G)$.

If H is an arbitrary subgroup of the group G, then $\mathrm{cp}(H) \ge \mathrm{cp}(G)$ and this is a powerful tool to estimate $\mathrm{cp}(G)$. But a similar result does not hold for $\nu(G)$ (see Remark 2.5). So investigating the structure of groups G by $\nu(G)$ is much harder than it is with $\mathrm{cp}(G)$. In [8] Guralnick and Wilson proved that if G is a nonnilpotent group, then $\nu(G) \le \frac{1}{2}$.

It is easy to see that $\mathrm{cp}(A_5) = \nu(A_5) = \frac{1}{12}$, where A_5 is the alternating group of degree five. J. D. Dixon observed that $\mathrm{cp}(G) \le \frac{1}{12}$ for any finite nonabelian simple group G. This was submitted by Dixon as a problem in the *Canadian Mathematical Bulletin* **13** (1970), with his own solution appearing in 1973. Guralnick and Robinson [7] extended this result to nonsoluble groups and determined precisely for which nonsoluble groups the equality happens. So it is natural to pose the following question.

Question 1.1. If G is a finite group with $\nu(G) > \frac{1}{12}$, then is G soluble?

We conjecture that the answer is, in general, positive. In this paper we will answer the question for certain groups. Let G be a finite group and $x \in G$. We denote by $\mathrm{Nil}_G(x)$ the subset of all elements y of G such that the subgroup $\langle x, y\rangle$ is nilpotent. We notice that $\mathrm{Nil}_G(x)$ is not necessarily a subgroup of G (see the second paragraph of the proof of Theorem 1.3 in Section 2). It is easy to see that $\nu(G) = \sum_{x\in G} |\mathrm{Nil}_G(x)|/|G|^2$. A group G is called an $\mathcal{N}$-group if $\mathrm{Nil}_G(x)$ is a subgroup of G for every $x \in G$. In the last section of the paper we give some examples of $\mathcal{N}$-groups. In Section 2 we give a positive answer to Question 1.1, when G is an $\mathcal{N}$-group.

Theorem 1.2. *Let G be an $\mathcal{N}$-group. If $\nu(G) > \frac{1}{12}$, then G is soluble.*

We also determine the semisimple $\mathcal{N}$-groups with $\nu(G) = \frac{1}{12}$.

Theorem 1.3. *Let G be a semisimple $\mathcal{N}$-group with $\nu(G) = \frac{1}{12}$. Then $G \cong A_5$.*

2. Proofs of the main results

To prove our main results, we need the following lemmas.

Lemma 2.1 [5, Proof of Lemma 6]. *Let G be a group and $N \lhd G$. Then $\nu(G) \le \nu(G/N)$.*

Lemma 2.2 [5, Corollary 3]. *Let G be a group. Then $\nu(G) = \nu(G/Z^*(G))$, where $Z^*(G)$ is the hypercentre of G.*

Lemma 2.3. *Let G and H be finite groups. Then $\nu(G \times H) = \nu(G)\nu(H)$.*

Proof. The proof is straightforward. □

Proposition 2.4. *Let G be a finite $\mathcal{N}$-group and H be a subgroup of G. Then H is an $\mathcal{N}$-group and $\nu(G) \le \nu(H)$.*

PROOF. The first assertion is clear. If $g_1, g_2 \in \mathrm{Nil}_G(x)$ for some $x \in G$ such that $g_1 \mathrm{Nil}_H(x) \neq g_2 \mathrm{Nil}_H(x)$, then $g_1 g_2^{-1} \notin \mathrm{Nil}_H(x) = \mathrm{Nil}_G(x) \cap H$ and so $g_1 H \neq g_2 H$. We conclude that $|\mathrm{Nil}_G(x) : \mathrm{Nil}_H(x)| \leq |G : H|$. Therefore $|\mathrm{Nil}_G(x)| \leq |G : H|\,|\mathrm{Nil}_H(x)|$ and so $\sum_{x\in G} |\mathrm{Nil}_G(x)| \leq |G : H| \sum_{x\in G} |\mathrm{Nil}_H(x)|$. Also

$$\sum_{x\in G} |\mathrm{Nil}_H(x)| = |\{(x, y) \in G : \langle x, y\rangle \text{ is nilpotent and } x \in H \text{ or } y \in H\}| = \sum_{y\in H} |\mathrm{Nil}_G(y)|.$$

We conclude that $\sum_{x\in G} |\mathrm{Nil}_G(x)| \leq |G : H|^2 \sum_{x\in H} |\mathrm{Nil}_H(x)|$ from which it follows that $\nu(G) \leq \nu(H)$. This completes the proof. □

REMARK 2.5. The second assertion of Proposition 2.4 is not true in general. For example, it can be checked by GAP [6] that $\nu(\mathrm{SmallGroup}(96, 229)) = \frac{7}{24}$, but this group has a normal subgroup H of order 48 with $\nu(H) = \frac{1}{6}$.

In the three following lemmas, we compute $\nu(G)$ for some simple groups.

LEMMA 2.6. *Let G be the Suzuki group* $\mathrm{Sz}(q)$ *where* $q = 2^{2n+1}$ *for some* $n \geq 1$. *Then* $\nu(G) = (2q + 1)/q^2(q^2 + 1)(q - 1)$.

PROOF. It is well known that $\mathrm{Sz}(q)$ has a partition $\Psi = \{A^x, B^x, C^x, F^x \mid x \in G\}$, where A, B, C are cyclic groups of orders, say, $a = q - 1, b = q - 2r + 1$ and $c = q + 2r + 1$ respectively, and F is a Sylow 2-subgroup G of order q^2. Also, if $T \in \Psi$, then $C_G(y) \leq T$ for each $y \in T$ (see [10, Satz 3.10, Satz 3.11 and pages 192–193]). If there are $1 \neq t_1 \in T_1$, $1 \neq t_2 \in T_2$ and $T_1 \neq T_2 \in \Psi$ such that the subgroup $\langle t_1, t_2\rangle$ is nilpotent, then $Z(\langle t_1, t_2\rangle) \subseteq C_G(t_1) \cap C_G(t_2) = 1$, which is a contradiction. Therefore, for any two non-identity elements x, y, we see that $\langle x, y\rangle$ is a nilpotent subgroup of G if and only if there is a $T \in \Psi$ such that $x, y \in T$. Since the number of conjugates of A, B, C and F in G are respectively

$$m := \frac{q^2(q^2 + 1)}{2}, \quad l := \frac{q^2(q^2 + 1)(q - 1)}{4(q - 2^{n+1} + 1)}, \quad k := \frac{q^2(q^2 + 1)(q - 1)}{4(q + 2^{n+1} + 1)}, \quad t := q^2 + 1,$$

we find

$$\nu(G) = \frac{ma(a - 1) + lb(b - 1) + kc(c - 1) + tq^2(q^2 - 1) + |G|}{q^4(q^2 + 1)^2(q - 1)^2},$$

which gives the result. □

For any prime power q, we denote the general linear group, the projective general linear group, the special linear group and the projective special linear group of degree two over the finite field of size q by $\mathrm{GL}(2, q), \mathrm{PGL}(2, q), \mathrm{SL}(2, q)$ and $\mathrm{PSL}(2, q)$, respectively.

LEMMA 2.7. *Let* $G = \mathrm{GL}(2, q), \mathrm{PGL}(2, q), \mathrm{SL}(2, q)$, *or* $\mathrm{PSL}(2, q)$, *where* $q = 2^m$ *and* $m > 1$. *Then* $\nu(G) = 1/q(q - 1)$.

https://doi.org/10.1017/S0004972715001252 Published online by Cambridge University Press

Proof. By Lemma 2.2 and since $\mathrm{PGL}(2, 2^m)$ is isomorphic to $\mathrm{PSL}(2, 2^m)$, it is enough to prove the result for the group $G := \mathrm{PSL}(2, 2^m)$. It is well known that $\mathrm{PSL}(2, 2^m)$ has a partition $\Pi = \{P^x, D^x, I^x \mid x \in G\}$, where P is an elementary abelian Sylow 2-subgroup of order q, and D and I are cyclic subgroups of G of orders $q-1$ and $q+1$ respectively (see [10, pages 191–193]). If a is a nontrivial element of G, then it is easy to see that

$$\mathrm{Nil}_G(a) = \begin{cases} P^x & \text{if } a \in P^x, \\ D^x & \text{if } a \in D^x, \\ I^x & \text{if } a \in I^x. \end{cases}$$

Therefore $\langle x, y\rangle$ is a nilpotent subgroup of G if and only if there is $X \in \Pi$ such that $x, y \in X$. Since the number of conjugates of P, D and I in G are $a := q+1$, $b := \frac{1}{2}q(q+1)$ and $c := \frac{1}{2}q(q-1)$ respectively,

$$\nu(\mathrm{PSL}(2, q)) = \frac{aq(q-1) + b(q-1)(q-2) + cq(q-1) + q(q-1)(q+1)}{q^2(q-1)^2(q+1)^2} = \frac{1}{q(q-1)}.$$

This completes the proof. □

Lemma 2.8. *If $q > 5$ is odd and $16 \nmid q^2 - 1$, then*

$$\nu(\mathrm{SL}(2, q)) = \nu(\mathrm{PSL}(2, q)) = \frac{q+5}{q(q-1)(q+1)}.$$

Proof. Since $\nu(G) = \nu(G/Z^*(G))$, by Lemma 2.2, it is sufficient to compute $\nu(G)$ for $G = \mathrm{PSL}(2, q))$. Assume $q \equiv 1 \pmod 4$. It is well known that $\Lambda = \{A^x, B^x, C^x \mid x \in G\}$ is a partition for G, where C is elementary abelian of order q with $\gamma := q+1$ conjugates in G, A is a cyclic subgroup of order $\frac{1}{2}(q-1)$ with $\alpha := q(q+1)(q-1)/2(q-1)$ conjugates in G, and B is a cyclic subgroup of order $\frac{1}{2}(q+1)$ for which the number of conjugates in G is $\beta := q(q+1)(q-1)/2(q+1)$. Now we claim that $\mathrm{Nil}_G(x) = C_G(x)$ for every $x \in G$.

Suppose, for a contradiction, that there is an element $y \in \mathrm{Nil}_G(x)\backslash C_G(x)$ for some $x \in G$. Since the subgroup $\langle x, y\rangle$ is nilpotent, we see that $\langle x, y\rangle \le C_G(a)$ for some nonidentity element $a \in Z(\langle x, y\rangle)$. If $a^2 \neq 1$, then $C_G(a) \in \Lambda$ and so $y \in C_G(x)$, which is a contradiction. Now assume that $a^2 = 1$. Then $C_G(a)$ is isomorphic to the dihedral group of order $q-1$ and so $\langle x, y\rangle$ is abelian or a 2-subgroup of G. On the other hand, since $16 \nmid q^2 - 1$, every Sylow subgroup of G is abelian (by [10, Satz 8.10]). Therefore $\langle x, y\rangle$ is abelian, which is a contradiction. Consequently, $\mathrm{Nil}_G(x) = C_G(x)$ and it is sufficient to compute the number of centralisers of G. Now, for every $1 \neq a \in G$,

$$C_G(a) = \begin{cases} N_G(\langle a\rangle) & \text{if } a^2 = 1 \text{ and } a \in A^x \text{ for some } x \in G, \\ A^x & \text{if } a^2 \neq 1 \text{ and } a \in A^x \text{ for some } x \in G, \\ B^x & \text{if } a \in B^x \text{ for some } x \in G, \\ C^x & \text{if } a \in C^x \text{ for some } x \in G. \end{cases}$$

So

$$\nu(G) = \frac{2\alpha|A| + \alpha\frac{1}{2}|A|(\frac{1}{2}|A| - 2) + \beta\frac{1}{2}|B|(\frac{1}{2}|B| - 1) + \gamma|C|(|C| - 1) + |G|}{|G|^2}$$

and this is equal to $(q+5)/q(q-1)(q+1)$, as desired.

https://doi.org/10.1017/S0004972715001252 Published online by Cambridge University Press

Now let $q \equiv 3 \pmod 4$. Then $\mathrm{Nil}_G(x) = C_G(x)$ for all $x \in G$. If $1 \neq a \in G$, then

$$C_G(a) = \begin{cases} N_G(\langle a\rangle) & \text{if } a^2 = 1 \text{ and } a \in B^x \text{ for some } x \in G,\\ B^x & \text{if } a^2 \neq 1 \text{ and } a \in B^x \text{ for some } x \in G,\\ A^x & \text{if } a \in A^x \text{ for some } x \in G,\\ C^x & \text{if } a \in C^x \text{ for some } x \in G \end{cases}$$

and the desired result follows by a similar argument. □

Now we are ready to prove our main results.

Proof of Theorem 1.2. Let G be a nonsoluble counterexample of minimal order. We can assume that G is a minimal nonsoluble group by Proposition 2.4. By [12], G contains a normal soluble subgroup N such that G/N is one of the following groups:

$\mathrm{PSL}(3, 3)$;
$\mathrm{PSL}(2, 2^m)$;
$\mathrm{PSL}(2, 3^m)$ or $\mathrm{Sz}(2^m)$ where m is an odd prime; or
$\mathrm{PSL}(2, p)$ where $3 < p$ is prime.

First, let $G/N \cong \mathrm{PSL}(3, 3)$. By using GAP [6] we have that $\nu(\mathrm{PSL}(3, 3)) = \frac{6631}{31539456}$ and since $\nu(G) \leq \nu(G/N)$, by Lemma 2.1, we reach a contradiction. Now assume G/N is isomorphic to one of the remaining groups. This is again impossible, by Lemmas 2.6–2.8, 2.1 and [1, Lemma 3.10]. Therefore G is soluble.

Proof of Theorem 1.3. We claim that A_5 is the only simple $\mathcal{N}$-group with $\nu(G) = \frac{1}{12}$. To prove this, suppose, on the contrary, that there exists a nonabelian finite simple $\mathcal{N}$-group of the least possible order such that $\nu(G) = \frac{1}{12}$ and G is not isomorphic to A_5. Then every proper nonabelian simple section of G is isomorphic to A_5. It follows from [2, Proposition 4] that G is isomorphic to one of the following groups:

$\mathrm{PSL}(2, 2^m)$ where $m = 4$ or a prime;
$\mathrm{PSL}(2, 3^p), \mathrm{PSL}(2, 5^p), \mathrm{PSL}(2, 7^p)$ where p is a prime;
$\mathrm{PSL}(2, p)$ with $p \geq 7$;
$\mathrm{PSL}(3, p)$ with $p = 3, 5$, or 7;
$\mathrm{PSU}(3, m)$ with $m = 3, 4$, or 7; or
$\mathrm{Sz}(2^m)$ where m is an odd prime.

By Lemmas 2.6–2.8 and [1, Lemma 3.10], we reach a contradiction for each of the projective special linear groups of degree two and the Suzuki groups. For each of the remaining groups, we have a contradiction by Proposition 2.4, after checking by GAP [6] that each has a subgroup isomorphic to S_4 (the symmetric group of degree four), which is not an $\mathcal{N}$-group (since $\mathrm{Nil}_{S_4}((12)(34))$ is not a subgroup of S_4). So the claim is proved.

Now let G be semisimple and $\nu(G) = \frac{1}{12}$. If R is the centreless CR-Radical of G, then $R \cong R_1 \times \cdots \times R_k$, where R_i is simple for each i. It follows from Lemma 2.3 and Proposition 2.4 that $R \cong A_5$, since $\nu(A_5) = \frac{1}{12}$. On the other hand, we know that G is embedded in $\mathrm{Aut}(R)$. Since $\mathrm{Aut}(A_5) \cong S_5$, we have either $G \cong A_5$ or S_5. But S_5 is not an $\mathcal{N}$-group, by Proposition 2.4. This completes the proof.

https://doi.org/10.1017/S0004972715001252 Published online by Cambridge University Press

3. Examples of $\mathcal{N}$-groups

It is clear that every nilpotent group is an $\mathcal{N}$-group. Now we present some examples of $\mathcal{N}$-groups which are not nilpotent. First, we determine Frobenius $\mathcal{N}$-groups.

Proposition 3.1. *Let G be a Frobenius group with the Frobenius complement H. Then G is an $\mathcal{N}$-group if and only if H is an $\mathcal{N}$-group.*

Proof. If G is an $\mathcal{N}$-group, then so is H by Proposition 2.4. Conversely, assume that H is an $\mathcal{N}$-group. It is well known that G has a partition as $\Pi = \{K, x^{-1}Hx \mid x \in K\}$. We claim that if $1 \neq g \in G$, then either $\mathrm{Nil}_G(g) \subseteq K$ or $\mathrm{Nil}_G(g) \subseteq w^{-1}Hw$ for some $w \in K$.

Suppose that $1 \neq y \in \mathrm{Nil}_G(g)$. Then there is a nonidentity element t in the centre of $\langle g, y\rangle$ by definition of $\mathrm{Nil}_G(g)$. Now we consider two cases.

Case 1. Suppose that $g \in K$. If $y \in x^{-1}Hx$ for some $x \in K$, then $t \in C_G(g) \cap C_G(y) \subseteq K \cap x^{-1}Hx$, which is a contradiction. It follows that $y \in K$ and so $\mathrm{Nil}_G(g) = K$, since K is nilpotent by [11, Theorem 10.5.6].

Case 2. Suppose that $g \in w^{-1}Hw$ for some $w \in K$. If $y \in K$ or $x^{-1}Hx$ for some $x \in K\backslash\{w\}$, then $t \in w^{-1}Hw \cap K$ or $w^{-1}Hw \cap x^{-1}Hx$, which is impossible. Therefore $\mathrm{Nil}_G(g) \subseteq w^{-1}Hw$. This completes the proof of the claim.

Since K is nilpotent and $w^{-1}Hw$ is an $\mathcal{N}$-group by hypothesis for every $w \in K$, the result follows. □

The next corollary, which follows directly from the above proposition and the next two propositions, gives some further examples of N-groups.

Corollary 3.2. *Let G be a nonnilpotent group such that every proper subgroup of G is nilpotent. Then G is an $\mathcal{N}$-group.*

Proposition 3.3. *Let G_1 and G_2 be finite groups. Then $G_1 \times G_2$ is an $\mathcal{N}$-group if and only if G_1 and G_2 are $\mathcal{N}$-groups.*

Proof. Put $G = G_1 \times G_2$. If G is an $\mathcal{N}$-group, then both G_1 and G_2 are $\mathcal{N}$-groups by Proposition 2.4. Conversely, assume that G_1 and G_2 are $\mathcal{N}$-groups. Suppose $\langle x, y\rangle$ and $\langle x, z\rangle$ are nilpotent subgroup of G, where $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$ with $x_1, y_1, z_1 \in G_1$ and $x_2, y_2, z_2 \in G_2$. Then we show that $\langle x, yz\rangle$ is nilpotent. Since $\langle x, y\rangle$ and $\langle x, z\rangle$ are nilpotent, so are $\langle (x_1, 1), (y_1, 1)\rangle$ and $\langle (x_1, 1), (z_1, 1)\rangle$. Next, $\langle (x_1, 1), (y_1z_1, 1)\rangle$ is nilpotent because $\mathrm{Nil}_{G_1}(x_1)$ is a subgroup of G_1. In a similar way, $\langle (1, x_2), (1, y_2z_2)\rangle$ is nilpotent. Hence $\langle (x_1, 1), (y_1z_1, 1)\rangle \times \langle (1, x_2), (1, y_2z_2)\rangle$ is nilpotent, which shows that $\langle x, yz\rangle$ is nilpotent and completes the proof. □

Proposition 3.4. *Let $G = PGL(2, q)$. Then G is an $\mathcal{N}$-group if and only if $q > 3$ is prime and $8 \nmid (q \pm 3)$.*

Proof. Suppose that $8 \nmid (q \pm 3)$. Then all nilpotent subgroups of G are abelian. Now we show that $\mathrm{Nil}_G(x) = C_G(x)$ for all $x \in G$. If $C_G(a) \neq \mathrm{Nil}_G(a)$ for some $a \in G$, then there

https://doi.org/10.1017/S0004972715001252 Published online by Cambridge University Press

is an element $b \in \mathrm{Nil}_G(a) \backslash C_G(a)$. It follows that $\langle a, b \rangle$ is nilpotent and so $b \in C_G(a)$, which is a contradiction. Hence G is an $\mathcal{N}$-group.

Conversely if $q \equiv \pm 3 \pmod 8$, then G has some subgroups isomorphic to S_4, which is not an $\mathcal{N}$-group. Therefore we have the result by Proposition 2.4. □

Acknowledgement

The authors are indebted to the referee for his/her careful reading and valuable comments.

References

[1] A. Abdollahi and M. Zarrin, 'Non-nilpotent graph of a group', *Comm. Algebra* **38**(12) (2010), 4390–4403.
[2] R. D. Blyth and D. J. S. Robinson, 'Semisimple groups with the rewritting property Q_5', *Comm. Algebra* **23**(6) (1995), 2171–2180.
[3] R. Dubose-Schmidt, M. D. Galloy and D. L. Wilson, Counting nilpotent pairs in finite groups: some conjectures, Rose-Hulman Technical Report MS TR 92-05. (1992).
[4] P. Erdős and P. Turàn, 'On some problems of statistical group-theory IV', *Acta Math. Acad. Sci. Hungar.* **19** (1968), 413–435.
[5] J. E. Fulman, M. D. Galloy, G. J. Sherman and J. M. Vanderkam, 'Counting nilpotent pairs in finite groups', *Ars Combin.* **54** (2000), 161–178.
[6] The GAP Group, *GAP-Groups, Algorithms and Programming*, version 4.4.10, 2007, (http://www.gap-system.org).
[7] R. M. Guralnick and G. R. Robinson, 'On the commuting probability in finite groups', *J. Algebra* **300**(2) (2006), 509–528.
[8] R. M. Guralnick and J. S. Wilson, 'The probability of generating a finite soluble group', *Proc. Lond. Math. Soc.* (3) **81**(2) (2000), 405–427.
[9] W. H. Gustafson, 'What is the probability that two group elements commute?', *Amer. Math. Monthly* **80**(9) (1973), 1031–1034.
[10] B. Huppert, *Endlich Gruppen III* (Springer, Berlin, 1982).
[11] D. J. S. Robinson, *A Course in the Theory of Groups* (Springer, New York, 1996).
[12] J. G. Thompson, 'Nonsolvable finite groups all of whose local subgroups are solvable', *Bull. Amer. Math. Soc.* **74**(3) (1968), 383–437.

S. M. JAFARIAN AMIRI, Department of Mathematics,
Faculty of Sciences, University of Zanjan,
PO Box 45371-38791, Zanjan, Iran
e-mail: sm_jafarian@znu.ac.ir

H. MADADI, Department of Mathematics,
Faculty of Sciences, University of Zanjan,
PO Box 45371-38791, Zanjan, Iran
e-mail: halime_madadi@yahoo.com

H. ROSTAMI, Department of Mathematics,
Faculty of Sciences, University of Zanjan,
PO Box 45371-38791, Zanjan, Iran
e-mail: h.rostami5991@gmail.com

https://doi.org/10.1017/S0004972715001252 Published online by Cambridge University Press