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ON 2-ADJACENCY RELATION OF TWO-BRIDGE KNOTS AND LINKS

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Abstract

We give a necessary condition for a two-bridge knot or link $S(p, q)$ to be 2-*adjacent* to another two-bridge knot or link $S(r, s)$. In particular, we show that if the trivial knot or link is 2-adjacent to $S(p, q)$, then $S(p, q)$ is trivial, that if $S(p, q)$ is 2-adjacent to its mirror image, then $S(p, q)$ is amphicheiral, and that for a prime integer p, if $S(p, q)$ is 2-adjacent to $S(r, s)$, then $S(p, q) = S(r, s)$ or $S(r, s) = S(1, 0)$.

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1. Introduction

Let K and K' be knots in S^3. We say that K is n-*adjacent* to K' for some $n \in \mathbf{N}$ if K admits a diagram containing n (generalized) crossings such that changing any nonempty subset of them yields a diagram of K'. We write $K \xrightarrow{n} K'$. By definition, $K \xrightarrow{n} K'$ implies that $K \xrightarrow{n'} K'$ for all $0 < n' \le n$. We remark that if $K \xrightarrow{n} K'$, all of the finite type invariants of K and K' of orders at most $n - 1$ agree. For example, both the trefoil knot and the figure-eight knot are 2-adjacent to the trivial knot O (Figure 1). In fact, if K is a two-bridge knot or genus-one knot, $K \xrightarrow{2} O$ implies that K is a two-bridge knot of the form $S(p, q)$ and $p/q = [2q_1, 2q_2]$ in Conway's notation (see [9, Theorem 1.1; 4, Theorem 5.1]). In [5], Kalfagianni and Lin proved that if $K \xrightarrow{n} K'$ and $g(K) > g(K')$, then $n \le 6g(K) - 3$, where $g(\cdot)$ is the Seifert genus. Note that in the case where K' is trivial, this theorem was first obtained by Howards and Luecke in [2]. On the other hand, in [3] Kalfagianni recently proved that if K' is a fibred knot, $K \xrightarrow{2} K'$ implies $g(K) > g(K')$ or $K = K'$. It follows

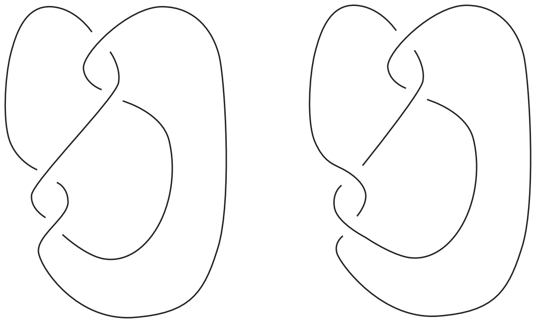

FIGURE 1. Two-bridge knots 2-adjacent to the trivial knot.

that the trivial knot is 2-adjacent to neither the trefoil knot nor the figure-eight knot [3, Remark 7.5]. Thus, n-adjacency is not an equivalence relation on the set of knots.

We can naturally extend the definition of n-adjacency to a link L in S^3 with more than one component (see [10, 11, 12]). In [12], Tsutsumi proved that if L is 2-adjacent to a trivial link, then L is a *boundary link*. The author does not know whether the converse is true. That is, it is a problem whether there is a nonboundary link L such that a trivial link is 2-adjacent to L.

In this paper, we give a necessary condition for a two-bridge knot or link $S(p, q)$ to be 2-adjacent to another two-bridge knot or link $S(r, s)$ (Theorem 1). In particular, we show that if the trivial knot or link is 2-adjacent to $S(p, q)$, then $S(p, q)$ is trivial, that if $S(p, q)$ is 2-adjacent to its mirror image, then $S(p, q)$ is amphicheiral, and that for a prime integer p, if $S(p, q)$ is 2-adjacent to $S(r, s)$, then $S(p, q) = S(r, s)$ or $S(r, s) = S(1, 0)$. To prove our results, we use double-cover and Dehn surgery techniques as discussed in [7, 8, 9, 11]. Therefore the main ingredients of the proof are the Montesinos trick [6] and the cyclic surgery theorem of Culler *et al.* [1].

2. The statement of result

Let L be a link in S^3. If the number of components of L is one (that is, L is a knot), we may denote L by K. A *generalized crossing* of order $t \in \mathbf{Z}$ on a diagram of L is a set C of $|t|$ twist crossings on two strings that inherit opposite orientations from any orientation of L. If L' is obtained from L by changing all of the crossings in C simultaneously, we say that L' is obtained from L by a *generalized crossing change* of order t (see Figure 2). Note that if $|t| = 1$, L and L' differ by an ordinary crossing change while if $t = 0$, then $L = L'$. Throughout this paper, we assume that $t \neq 0$.

Let $S(p, q)$ be a two-bridge knot or link whose two-fold branched cover is the lens space $L(p, q)$, where p, q are relatively prime integers. Then $S(p, q)$ is a two-component link for an even p and a knot for an odd p. In particular $S(0, 1)$ is the two-component trivial link and $S(1, 0)$ is the trivial knot.

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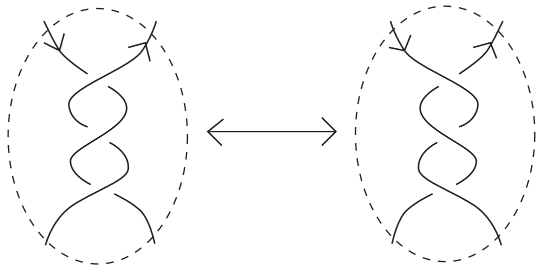

FIGURE 2. A generalized crossing change of order −3.

Recall that L is n-adjacent to L' (denoted by $L \xrightarrow{n} L'$), if L admits a diagram containing n generalized crossings such that changing any nonempty subset of them yields a diagram of L'.

THEOREM 1. *Suppose that $S(p, q)$ is 2-adjacent to $S(r, s)$. Then:*
(i) *p is factorized by r; and*
(ii) *if $p = r$, $S(p, q) = S(r, s)$.*

COROLLARY 2. *We have the following.*
(i) *Suppose that the trivial knot is 2-adjacent to $S(p, q)$. Then $S(p, q)$ is the trivial knot $S(1, 0)$.*
(ii) *Suppose that the two-component trivial link is 2-adjacent to $S(p, q)$. Then $S(p, q)$ is the trivial link $S(0, 1)$.*
(iii) *Suppose that $S(p, q)$ is 2-adjacent to $S(p, -q)$. Then $S(p, q) = S(p, -q)$, that is, $S(p, q)$ is amphicheiral.*
(iv) *For a prime integer p, suppose that $S(p, q)$ is 2-adjacent to $S(r, s)$. Then $S(p, q) = S(r, s)$ or $S(r, s) = S(1, 0)$.*

3. Proof of Theorem 1

Let $N(K)$ be a regular neighbourhood of a knot K in a closed orientable 3-manifold M, with μ a meridian for $N(K)$ and let $E(K) = M - \text{int}N(K)$ be the exterior of K in M. Now let $K(\gamma)$ denote the manifold obtained by attaching a solid torus V to $E(K)$ so that a curve of slope γ on $\partial E(K)$ bounds a disc in V, where γ indicates the isotopy class of an essential simple closed curve on the 2-torus. We say that $K(\gamma)$ is the result of γ-Dehn surgery on K. Dehn surgery on a 2-component link $K \cup K'$ is also defined in the same way and denoted by $K(\gamma) \cup K'(\gamma')$ for slopes γ and γ' of K and K', respectively. For two slopes γ and δ in $\partial E(K)$, let $\Delta(\gamma, \delta)$ be their minimal geometric intersection number.

First of all, the Montesinos trick in [6] connects a generalized crossing change with Dehn surgery.

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LEMMA 3 [6]. *Let L and L' be links in S^3 and let M_L and $M_{L'}$ be the two-fold branched covering spaces of S^3 along L and L', respectively. Suppose that the result of a generalized crossing change of order t on L is L'. Then $M_{L'}$ is obtained by γ-Dehn surgery on some knot K in M_L, where $\Delta(\gamma, \mu) = 2|t|$. Moreover, $N(K)$ is obtained by the lift of a crossing ball as illustrated in Figure 2 by the dotted circle.*

A *torus knot* in a lens space is a knot isotopic to a Heegaard torus of the lens space. For $i = 1, 2$, let V_i be a solid torus standardly embedded in S^3 and let μ_i and λ_i be a meridian and a longitude of V_i, respectively. Let h be an orientation-reversing homeomorphism from ∂V_1 to ∂V_2 such that $h(\mu_1) = s\mu_2 + r\lambda_2$. Then the space $V_1 \bigcup_h V_2$ obtained from V_1 and V_2 by identifying their boundaries by h is the lens space $L(r, s)$. Let $C_{m,n}$ be a (m, n)-curve on ∂V_1, that is, a simple closed curve which is isotopic to $m\mu_1 + n\lambda_1$. Let a, b be integers such that $rb - sa = 1$. Then we may assume $h(\lambda_1) = b\mu_2 + a\lambda_2$ and $C_{m,n}$ is equal to $(sm + bn)\mu_2 + (rm + an)\lambda_2$ on $\partial V_2 = \partial V_1$. Note that if $n = 0$ or $rm + an = 0$, then $C_{m,n}$ is the trivial knot in $L(r, s)$. Then it is not hard to see that every torus knot in $L(r, s)$ is isotopic to some $C_{m,n}$. We may push $C_{m,n}$ into $\mathrm{int} V_1$. Then, for a slope γ on $\partial N(C_{m,n})$, using the usual preferred meridian-longitude coordinates of $\partial N(C_{m,n})$ in $V_1 \subset S^3$, we identify γ with $c/d \in \mathbf{Q} \cup \{\infty\}$ where c and d are relatively prime.

We need the following calculations from [7] and [8].

THEOREM 4 [7]. *The space $C_{m,n}(c/d)$ is homeomorphic to a lens space if and only if there is a pair of coprime integers m', n' such that:*

(i) *$C_{m,n}$ is isotopic to $C_{m',n'}$ in $L(r, s)$;*

(ii) *$c = dm'n' \pm 1$; and*

(iii) *$C_{m',n'}(c/d)$ is orientation-preserving homeomorphic to $L(dan'^2 + r(dm'n' \pm 1), dbn'^2 + s(dm'n' \pm 1))$, where a, b are as above for $C_{m',n'}$.*

COROLLARY 5 [8, Proof of Theorem 2.2]. *Let K_T be a torus knot in $L(r, s)$. Suppose that $K_T(\gamma)$ is orientation-preserving homeomorphic to $L(r, s)$ for some slope γ with $\Delta(\gamma, \mu) \geq 2$. Then K_T is the trivial knot, that is, K_T bounds a disc in $L(r, s)$.*

PROOF OF COROLLARY 5. Put $K_T = C_{m,n}$ and $\gamma = c/d$ with $|d| \geq 2$. By Theorem 4 and the classification of lens spaces, we conclude that for coprime m', n', $r = dan'^2 + r(dm'n' \pm 1)$ and either s is congruent to $dbn'^2 + s(dm'n' \pm 1)$ modulo r, or $s(dbn'^2 + s(dm'n' \pm 1))$ is congruent to 1 modulo r. Then an elementary congruence argument shows that $n' = 0$ or $rm' + an' = 0$. Therefore, $C_{m',n'}$ and, hence, $C_{m,n}$ is trivial.

PROPOSITION 6. *Suppose that $S(p, q)$ is 2-adjacent to $S(r, s)$ by two generalized crossing changes of orders t_1 and t_2. Then there is a two-component link $K_1 \cup K_2$ in $L(r, s)$ such that $K_1(\gamma_1)$ and $K_2(\gamma_2)$ are orientation-preserving homeomorphic to $L(r, s)$ and $K_1(\gamma_1) \cup K_2(\gamma_2)$ is orientation-preserving homeomorphic to $L(p, q)$, where each slope γ_i satisfies $\Delta(\gamma_i, \mu) = 2|t_i|$. Moreover, each K_i is the trivial knot in the original $L(r, s)$ and a torus knot in $K_j(\gamma_j) = L(r, s)$ $(i \neq j)$.*

Proof of Proposition 6. By definition, there are two generalized crossings of orders t_1, t_2 in a diagram of $S(p, q)$ such that both each generalized crossing change and simultaneous generalized crossing changes yield $S(r, s)$. The two-fold cover of $S(r, s)$ in S^3 is $L(r, s)$. Therefore, by repeated use of Lemma 3, there exists a 2-component link $K_1 \cup K_2$ in $L(r, s)$ such that $K_1(\gamma_1) = K_2(\gamma_2) = L(r, s)$ and $K_1(\gamma_1) \cup K_2(\gamma_2) = L(p, q)$, where $\Delta(\gamma_i, \mu) = 2|t_i|$ $(i = 1, 2)$. Moreover, since $\Delta(\gamma_i, \mu) = 2|t_i| \geq 2$, by the cyclic surgery theorem of Culler *et al.* [1], the exterior spaces $E(K_1)$ and $E(K_2)$ are Seifert fibred manifolds. However, as in [7, Lemma 4] this implies that K_1 and K_2 are fibres for some Seifert fibrations of $L(r, s)$. Moreover, it is well-known that any fibre for a Seifert fibration of a lens space is isotopic to some torus knot (see, for example, [7]). By Corollary 5, it follows that K_1 and K_2 are trivial in the original $L(r, s)$, completing the proof of Proposition 6.

We are ready to prove Theorem 1.

Proof of Theorem 1. Let K_T be a torus knot K_1 in $K_2(\gamma_2) = L(r, s)$ as in the last statement of Proposition 6. Since K_2 is trivial, K_T in $K_2(\gamma_2) = L(r, s)$ may be considered as a full-twisted K_1 along some disc spanned by K_2 in the original $L(r, s)$. Note that a full-twist operation for a circle does not change the homotopy class in $L(r, s)$. Hence, K_T is homotopically trivial in $K_2(\gamma_2) = L(r, s)$. Suppose that K_T is isotopic to $C_{m,n}$. Then it follows that $n = 0$ or $n = r$. Applying Theorem 4 to $K_1(\gamma_1) \cup K_2(\gamma_2) = L(p, q)$ and $d = 2t_1$, we have $L(p, q) = L(2t_1 a n'^2 + r(2t_1 m' n' \pm 1), 2t_1 b n'^2 + s(2t_1 m' n' \pm 1))$ for coprime m', n'. Since $C_{m,n}$ is isotopic to $C_{m',n'}$, we have $n' = 0$ or $n' = r \neq 0$. If $n' = 0$, then the statement of Theorem 1 apparently holds. So put $n' = r \neq 0$. Then since $p = 2t_1 a r^2 + r(2t_1 m' r \pm 1) = r(2t_1 a r + 2t_1 m' r \pm 1)$ and $2t_1 a r + 2t_1 m' r \pm 1 \neq 0$, p is factorized by r, verifying (i). Further suppose that $p = r$, then $r = r(2t_1 a r + 2t_1 m' r \pm 1)$. Therefore, $2t_1 a r + 2t_1 m' r \pm 1 = 1$ and, thus, $2t_1 r(a + m') = 0$ or 2. Since $t_1 \neq 0$, it follows that $r(a + m')$ is 0 or 1. If $r(a + m') = ra + m'n' = 0$, then $L(p, q) = L(r, s)$ because in this case $C_{m',n'}$ is the trivial knot in $L(r, s)$. If $r(a + m') = 1$, then $r = 1$ and $L(p, q) = L(r, s) = L(1, 0)$, verifying (ii). This completes the proof of Theorem 1.

Remark 7. We make the following remarks.

(i) It is still a problem to decide on the exact pairs of (p, q) and (r, s) such that $S(p, q) \xrightarrow{2} S(r, s)$.

(ii) At the time of writing, the author was not aware of any example of a nontrivial knot or link L such that a trivial knot or link is 2-adjacent to L.

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