Computation of an Astronomical Running Fix

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1. INTRODUCTION. It was with considerable interest that I read the paper by Arturo and Raffaele Chiesa[1] because this is a subject on which I have done some work myself. I have, however, been absorbed more by the problem of the astronomical running fix, where the second observation – probably of the same heavenly body as the first – is taken after the observer has travelled an appreciable distance along a rhumb line. There are certain theoretical implications in the problem of transferring a position circle which were first pointed out to me in a convincing intuitive argument by my friend and colleague T. R. Beggs, but which Chiesa and Chiesa do not mention. In their paper, when they discuss the application of their method to the problem of the astronomical running fix, they say that it is simply a case of transferring the geographical position of the heavenly body from the time of the first observation along the course and distance run by the observer between the observations, and then using this position as the centre of the transferred position circle. The mathematics does, however, I believe, show that the problem of transferring the position circle is more complicated than that, and that the transferred locus is no longer a circle but suffers a distortion.

Although, in practice, the effects observed will be almost negligible in most cases, and less anyway than any error that might be introduced by the estimation of rhumb line course and distance run between the observations, nevertheless I think that it might be of interest to some if the details of the analysis are shown.

2. MATHEMATICAL EXPRESSION FOR THE POSITION LOCI. Let us suppose that we have obtained a position circle centred at the position (D_1, G_1) on the surface of the Earth considered to be a sphere. The position is the geographical position of a heavenly body whose declination is D_1 and where $G_1 = 2\pi - GHA$. The angles are expressed in radians and are bounded by:

$$-\tfrac{1}{2}\pi \leqslant D_1 \leqslant \tfrac{1}{2}\pi \quad \text{(North positive)}$$

$$0 \leqslant G_1 \leqslant 2\pi \quad \text{(East positive)}.$$

The equation of the position circle on the surface of the spherical Earth in terms of latitude, ϕ, and longitude, λ, of the variable point (ϕ, λ) as the point moves around the circle is:

$$\text{Cos} \,(\lambda - G_1) = (\cos z_1 - \sin \phi \sin D_1)/(\cos \phi \cos D_1). \tag{1}$$

This is, in fact, the familiar spherical cosine formula, and z_1 is the zenith distance – the arc radius of the circle. We also have, in radians:

$$-\tfrac{1}{2}\pi \leqslant \phi \leqslant \tfrac{1}{2}\pi \quad \text{(North positive)}$$

$$0 \leqslant \lambda \leqslant 2\pi \quad \text{(East positive)}.$$

If we now move every point of the position circle (1) through a distance d (in nautical

444

miles) along a rhumb line on a course α, then the general point (ϕ, λ) on the position circle transfers to the general point (Φ, Λ). The coordinates are related by the equations:

$$\Phi = \phi + (d/a) \cos \alpha \tag{2}$$

$$\Lambda = \lambda + \tan \alpha \int_{\phi}^{\Phi} \sec u \, du \tag{3}$$

where Φ is the latitude, and Λ is the longitude of a point on the transferred locus, and 'a' is the radius of the Earth in nautical miles.

After transposing equations (2) and (3) to give ϕ and λ in terms of Φ and Λ and substituting these values into (1), we find the equation of the transferred position locus:

$$\text{Cos } (\Lambda - k - G_1) = \frac{\cos z_1 - \sin (\Phi - (d/a) \cos \alpha) \sin D_1}{\cos (\Phi - (d/a) \cos \alpha) \cos D_1} \tag{4}$$

where

$$k = \tan \alpha \int_{\phi}^{\Phi} \sec u \, du$$

k is the difference of longitude expressed in radians.

Now the rhumb line is a curve on the surface of a sphere which spirals towards an end limit point at each pole and is of finite length. The rhumb line, therefore, which passes through the point P with coordinates (ϕ_p, λ_p) on the position circle defined by equation (1) on course α, is of finite length to the pole so that, if the distance, d, steamed by the observer, is greater than the distance along the rhumb line on the course α from P to the pole, then the point P has no image point on the transferred locus defined by equation (4). A point (ϕ, λ) on the position circle will therefore have an image (Φ, Λ) on the transferred locus, provided that:

$$|\phi + (d/a) \cos \alpha| \leqslant \tfrac{1}{2}\pi.$$

Figure 1 is a sketch of the projection of a rhumb line in the stereographic plane. The curve is an equi-angular spiral whose equation, in polar coordinates is

$$r = r_0 \, e^{-(\cot \alpha)(\lambda - \lambda_0)}$$

where $r = 2a \tan (\tfrac{1}{4}\pi - \tfrac{1}{2}\phi)$, α is the course of the rhumb line, and (r_0, λ_0) is an initial position on the curve.

Figure 2 is a sketch, also in the stereographic plane, of the way in which the projection of a position circle, which passes close to the pole but which does not enclose it, is distorted when each point on the circle is transferred the same distance along an equi-angular spiral. In the figure, C is the position circle and C′ is the transferred locus. It will be seen that, in the lower latitudes, the transferred locus closely resembles the original position circle in shape but, nearer a pole, the transferred locus shows much more distortion. (A position circle on the surface of a sphere is projected into a circle on the stereographic plane but the centre of the projected circle does not always coincide with the projection of the centre of the position circle.)

When, at the completion of the run along the rhumb line, we find a second position circle, centred at the point (D_2, G_2), the geographical position of the heavenly body at the time of this second observation, then the equation of this second position circle is:

$$\text{Cos } (\Lambda - G_2) = (\cos z_2 - \sin \Phi \sin D_2)/(\cos \Phi \cos D_2) \tag{5}$$
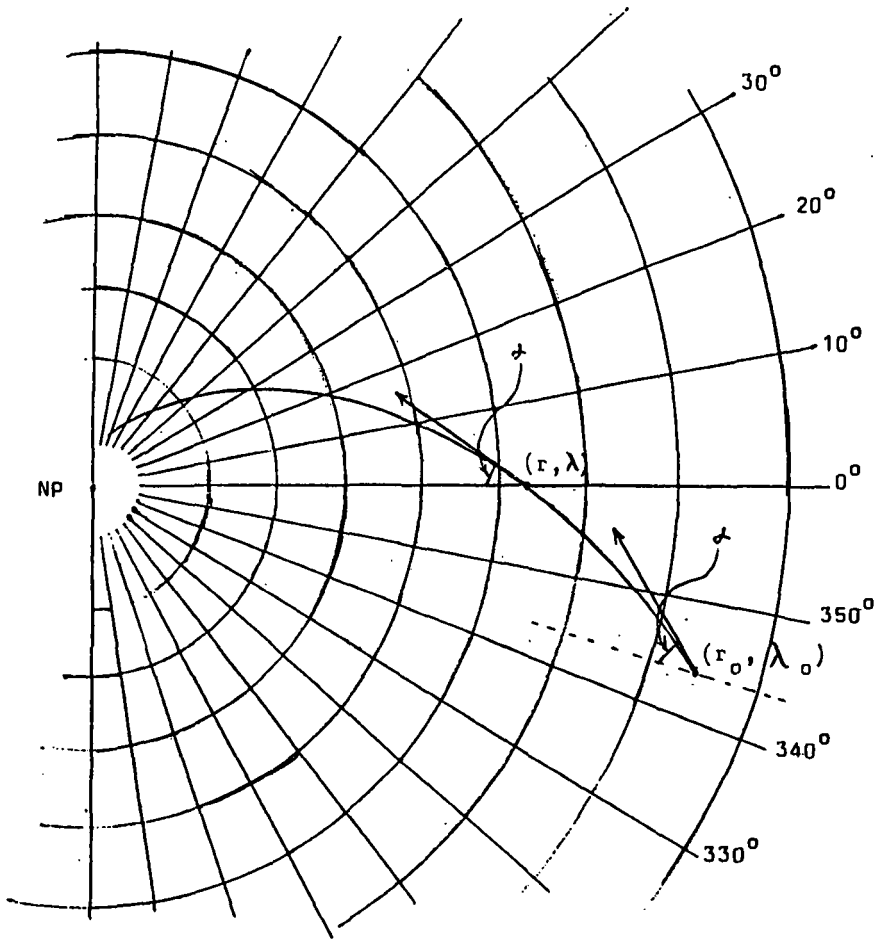
Fig. 1. Projection of a rhumb line on the polar stereographic plane

where z_2 is the zenith distance of the heavenly body at the time of this second observation.

3. COMPUTING THE OBSERVED POSITION. The observed position at the time of the second observation is therefore the selected solution of the pair of simultaneous equations (4) and (5). Since a direct method of doing this would seem to be out of the question, this would be best solved using a numerical method such as Newton's Method. We know that there are two solutions satisfying the pair of equations, each solution corresponding to a point of intersection of the loci, and that the Newton Method will find them one at a time. We must, then, find some information which will guide the iterative procedure towards the observed position first. This is most easily achieved by using the DR position as the first approximation.

4. CONCLUSION. This method of finding the observed position is only suitable for calculation by computer. Indeed, in a system designed to find the observed position from astronomical observations using the computer, such a method, finding the intersection of the position loci, could well be the most direct and efficient. Simultaneous and running fixes can be solved equally well by the one computer program.

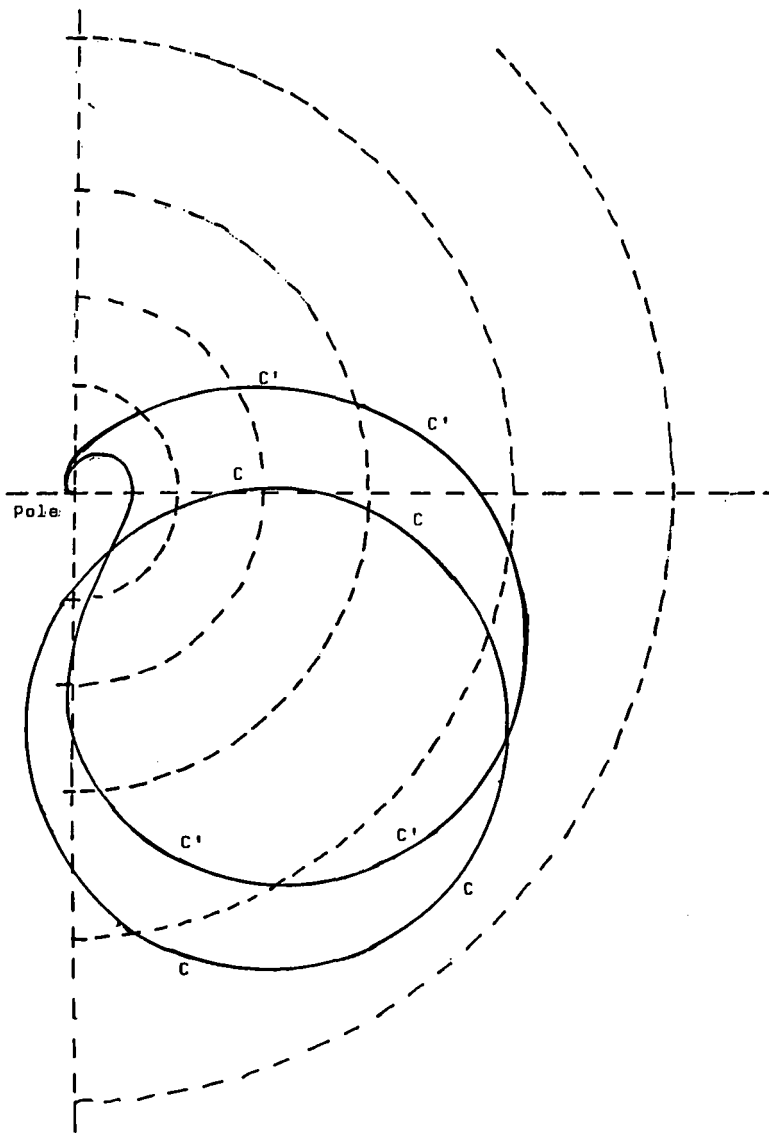In most cases, and in seagoing navigation particularly, where activity is usually

Fig. 2. Distortion of transferred position circle

restricted to what are termed 'navigable latitudes', a representation of the whole of the transferred position locus is not necessary to find its intersection with another position circle. Hence, that portion of the position circle represented by equation (1), which is relevant to the observer and on which he might be expected to lie, will almost certainly have a complete image on the locus represented by equation (4). It must be admitted that the comments made here are of little practical value, since the error in the estimate of the course and distance made good by the observer between the observations may well exceed any correction made by using the above computation. Nevertheless, it is always better to use approximations having knowledge of a theory which we believe to be correct.

The above analysis is based on the assumption that the Earth is a sphere. Any computation would, of course, make corrections for the fact that the Earth is a spheroid and the rhumb line distances would be calculated accordingly.

REFERENCE

[1] Chiesa, A. and Chiesa, R. (1990). A mathematical method of obtaining an astronomical vessel position. This *Journal*, **43**, 125.

KEY WORDS

1. Astro. 2. Computers.