Can. J. Math. Vol. **43** (6), 1991 pp. 1150–1160

VALUE GROUPS AND DISTRIBUTIVITY

H. H. BRUNGS AND J. GRÄTER

0. Let F be a skew field with a valuation (also called total) subring B, i.e. x in $F\setminus B$ implies x^{-1} in B. Such rings are useful not only in the investigation and construction of division algebras (see for example [5],[6],[12]) but also in geometry ([15]).

Associated with B is an invariant subring R of F and a value group G. We investigate the relationship between properties like the distributivity of R and properties like being lattice ordered of G.

In particular, we construct in Section 3 examples for B and F such that R is distributive but G is not lattice ordered and we need some results of algebraic number theory in the process. An example where R is not distributive and G is not lattice ordered is also provided. If B is commutative or invariant ([19]) then $R = B$ is itself a valuation ring and G is totally ordered.

1. Let B be a valuation subring of the skew field F and we define $W = \{kB \mid 0 \neq k \in F\} \cup \{\infty\}$ with ∞ as its largest element and $aB \leq bB$ if and only if $aB \supseteq bB$. The set W is totally ordered and the mapping v from F onto W with $v(a) = aB$ if $a \neq 0$, $v(0) = \infty$, satisfies the following conditions:

(1) $v(x) = \infty$ if and only if $x = 0$;

(2) $v(x+y) \geq \min\{v(x), v(y)\}$ for all x, y in F;

(3) $v(x) \leq v(y)$ implies $v(zx) \leq v(zy)$ for all x, y, z in F.

Conversely, given a skew field F, a totally ordered set W with largest element ∞ and a mapping v from F onto W satisfying (1), (2), (3) above—called a valuation on F—then $B_v = \{x \in F \mid v(x) \geq v(1)\}$ is a valuation subring of F.

Associated with such a valuation is the group G_v of order preserving bijections $\tilde{x}$ of W_v defined by $\tilde{x}(w) = \tilde{x}\big(v(k)\big) = v(xk)$ for x in $F^* = F\setminus\{0\}$, $w = v(k)$ in W, k in F.

The mappings $\tilde{x}$ are well defined because of condition (3) and the operation in G_v is given by $\tilde{x} \circ \tilde{y} = \widetilde{xy}$ for x, y in F^*.

The group

$$G_v = \{\tilde{x} \mid x \text{ in } F^*\}$$

is called the value group of the valuation v and it is partially ordered by $\tilde{x} \leq \tilde{y}$ if and only if $\tilde{x}(w) \leq \tilde{y}(w)$ for all w in W. We say that G_v is lattice ordered if the infimum and the supremum exist for any two elements $\tilde{x}, \tilde{y}$ in G_v in which case G_v is a distributive lattice.

The first author's work was supported in part by NSERC Canada.
Received by the editors April 9, 1990; revised May 2, 1991.
AMS subject classification: 16A14, 16A10, 0615, 12F10.

If the valuation subring B of F is invariant under all inner automorphisms of F, then one can define on the set $\{kB \mid k \text{ in } F^*\}$ an operation $aBbB = abB$ to obtain an ordered group Γ.

The valuation v, associated with B, onto $\Gamma \cup \{\infty\}$ satisfies conditions (1) and (2) above and

$$(3') \qquad v(xy) = v(x)v(y) \text{ for } x, y \text{ in } F.$$

Conversely, to every mapping v from F with (1), (2) and (3′) onto a set $\Gamma \cup \{\infty\}$ for an ordered group Γ there corresponds an invariant valuation subring B_v of F. In this case we have $\Gamma \cong G_v$ for an isomorphism φ of ordered groups given by $\varphi\big(v(x)\big) = \tilde{x}$ for x in F^*.

These are the valuations considered by Schilling in [18].

If F is finite dimensional over its center K and v a valuation of F then G_v is lattice ordered. The group G_v is totally ordered in this case only if B is invariant, i.e. v is a Schilling valuation ([5],[11]).

On the other hand, it is in general not necessary for B_v to be invariant in order for G_v to be totally ordered. This condition is equivalent with the existence of an invariant valuation ring B of F with $B \subseteq B_v \subset F$ ([13]) and defines subinvariant valuations. The not necessarily invariant ring B_v is then a localization of the invariant valuation ring B.

Another interesting class of valuations is given by the locally invariant valuations which can be defined by properties of G_v ([9],[1]) and which are exactly the valuations for which the general approximation theorem holds.

As before, let B_v be a valuation subring of the skew field F. We define:

$$R_v = \bigcap_{a \in F^*} aB_va^{-1}$$

and R_v is an invariant subring of F, i.e. $aR_va^{-1} = R_v$ for all $a \neq 0$ in F.

We denote by $H(R_v)$ the set of all cyclical R_v-submodules $\neq (0)$ of F, i.e. $H(R_v) = \{aR_v \mid a \in F^*\}$ and $H(R_v)$ is a group with $aR_vbR_v = abR_v$ as operation.

The group $H(R_v)$ is partially ordered with $aR_v \leq bR_v$ if and only if $aR_v \supseteq bR_v$ and the mapping η from G_v to $H(R_v)$ defined by $\eta(\tilde{x}) = xR_v$ is well defined and an order preserving group isomorphism; G_v is a lattice ordered group if and only if $H(R_v)$ is lattice ordered.

2. Let F be a skew field with valuation ring B_v and G_v, R_v and $H(R_v)$ as defined in Section 1.

The first result gives a condition for $H(R_v)$ to be lattice ordered.

THEOREM 2.1. *$H(R_v)$ is lattice ordered if and only if $aR_v \cap bR_v \in H(R_v)$ for a, b in F^*.*

PROOF. Let $H(R_v)$ be lattice ordered and $cR_v = \sup\{aR_v, bR_v\}$ be the supremum of aR_v and bR_v.

It follows that $cR_v \subseteq aR_v \cap bR_v$ and for every d in $aR_v \cap bR_v$ we have $dR_v \geq aR_v$, bR_v, hence, $dR_v \subseteq cR_v$ and $cR_v = aR_v \cap bR_v$ follows.

Conversely, assume that for any a, b in F^* there exists c in F^* with $aR_v \cap bR_v = cR_v$. Then obviously $cR_v = \sup\{aR_v, bR_v\}$. In addition, there exists an element d in F^* with $a^{-1}R_v \cap b^{-1}R_v = dR_v$ which implies $d^{-1}R_v = \inf\{aR_v, bR_v\}$, since $aR_v \subseteq tR_v$ if and only if $t^{-1}R_v \subseteq a^{-1}R_v$ for a, t in F^*.

COROLLARY. *If G_v is lattice ordered then F is the skew field of quotients of R_v.*

PROOF. For any k in F^* there exists an x in F^* with $kR_v \cap R_v = xR_v$ and x in R_v, $x = ka$, a in R_v, follows.

It was mentioned in Section 1 that G_v is lattice ordered if F is finite dimensional over its center. This follows from the fact that R_v is a Bezout domain in this case, i.e. every finitely generated ideal of R_v is a principal ideal. We consider one other condition on R_v. We say R_v is a distributive ring if the lattice of ideals of R_v is distributive, i.e. $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all ideals A, B, C of R_v.

THEOREM 2.2. *Let R_v be a distributive ring with F as skew field of fractions. Then the following statements are equivalent:*

(i) R_v is a Bezout ring;

(ii) G_v is lattice ordered;

(iii) For arbitrary elements $\tilde{x}, \tilde{y}$ in G_v there exists $\tilde{z}$ in G_v with $\tilde{z}(w) = \min\{\tilde{x}(w), \tilde{y}(w)\}$ for all w in W_v.

PROOF. If (iii) holds then $\tilde{z} = \inf\{\tilde{x}, \tilde{y}\}$. Further, we have $\sup\{\tilde{x}, \tilde{y}\} = (\inf\{\tilde{x}^{-1}, \tilde{y}^{-1}\})^{-1}$, i.e. G_v is lattice ordered.

To prove that (ii) implies (i) choose $a, b \neq 0$ in R_v. It must be proved that $I = aR_v + bR_v$ is a principal ideal. We define $I^{-1} = \{x \in F \mid xI \subseteq R_v\}$ and $I^{-1} = \{x \in F \mid Ix \subseteq R_v\}$ follows, since R_v is invariant.

We have $I^{-1} = a^{-1}R_v \cap b^{-1}R_v$ and (by [8], 2.3)

$$(a^{-1}R_v \cap b^{-1}R_v)(aR_v + bR_v) = I^{-1}I = R_v.$$

Using Theorem 2.1 there exists an element c in F with $a^{-1}R_v \cap b^{-1}R_v = cR_v$ and $cR_v(aR_v + bR_v) = R_v$, $aR_v + bR_v = c^{-1}R_v$ follows.

Next, we assume (i) and for x, y in F^* there exists a z in F^* with $xR_v + yR_v = zR_v$, since F is the skew field of fractions of R_v. It must be shown that $\tilde{z}(w) = \min\{\tilde{x}(w), \tilde{y}(w)\}$ for all w in W_v which is equivalent with $zkB_v = xkB_v + ykB_v$ for all k in F, which in turn follows from the above equation and the invariance of R_v.

3. In this section we construct valuation rings in skew power series rings which have value groups that are not lattice ordered and the associated invariant ring is distributive in one case (Example 2) and not distributive in another (Example 1).

Let K be a skew field with invariant valuation ring B and let G be a group of automorphisms of K. Then G can be considered as the homomorphic image of a free group Γ under a homomorphism φ and Γ as a free group can be totally ordered.

The skew field $D = K\big((\Gamma, \varphi)\big)$ of skew power series $d = \Sigma\gamma k_\gamma$, $\gamma \in \Gamma$, with well ordered support $\{\gamma \mid k_\gamma \neq 0\}$, $k_\gamma \in K$, exists where the multiplication is defined by $k\gamma = \gamma k^{\varphi(\gamma)}$ (see [16]). If we identify the elements γ of Γ with $\gamma 1$ in D then we have $\gamma^{-1}k\gamma = k^{\varphi(\gamma)}$ for k in K and γ in Γ.

The subring $B' = \{\Sigma\gamma k_\gamma \in D \mid \gamma \geq e,\ k_e \in B\}$ is a valuation ring of D where e is the identity in Γ. We show that B' is total by considering an element $d = \gamma_0(k_{\gamma_0} - \Sigma_{\gamma > e}\gamma k_\gamma)$ in D with $k_{\gamma_0} \neq 0$ and γ_0 the least element in the support of d.

We have $dk_{\gamma_0}^{-1}\gamma_0^{-1} = 1 - \Sigma\gamma_0\gamma k_\gamma k_{\gamma_0}^{-1}\gamma_0^{-1} = 1 - m$ and $(1-m)^{-1} = \Sigma_{i=0}^{\infty} m^i$ follows and hence

$$d^{-1} = \gamma_0^{-1}(k_{\gamma_0}^{-1})^{\varphi(\gamma_0^{-1})}(1 + m + m^2 + \ldots).$$

One concludes that either $\gamma_0 > e$ or $\gamma_0 = e$ and k_{γ_0} in B in which case d is in B' or $\gamma_0 = e$ and $k_{\gamma_0}^{-1}$ in B or $\gamma_0^{-1} > e$ in which case d^{-1} is in B'.

For σ in G we denote with B'_σ the following subring of D:

$$B'_\sigma = \{k_e + \sum_{\gamma > e}\gamma k_\gamma \mid k_e \text{ in } \sigma(B)\}$$

and it follows that the set $\{B'_\sigma \mid \sigma \in G\}$ is exactly the set of subrings of D conjugate to B' in D.

To see this we write $d \neq 0$ in D as before in the form $d = (1-m)\gamma_0 k_{\gamma_0}$ with $m \in M = \{\Sigma_{\gamma > e}\gamma k_\gamma \in D\}$. Then

$$\begin{aligned} dB'd^{-1} &= (1-m)\gamma_0 k_{\gamma_0}(B + M)k_{\gamma_0}^{-1}\gamma_0^{-1}\Big(\sum_0^\infty m^i\Big) \\ &= (1-m)[\gamma_0 B\gamma_0^{-1} + \gamma_0 M\gamma_0^{-1}]\Big(\sum_0^\infty m^i\Big) \\ &\subseteq \gamma_0 B\gamma_0^{-1} + M = \sigma^{-1}(B) + M \end{aligned}$$

if $\varphi(\gamma_0) = \sigma \in G$, where we also use the fact that B is invariant in K.

The same argument shows $d^{-1}\big(\sigma^{-1}(B) + M\big)d \subseteq B + M = B'$ and $\sigma^{-1}(B) + M = dB'd^{-1}$ follows.

If we write $R = \bigcap_{\sigma \in G}\sigma(B)$ and $R' = \bigcap_{d \in D^*} dB'd^{-1}$ we obtain

$$R' = \bigcap_{\sigma \in G} B'_\sigma = \{k_e + \sum_{\gamma > e}\gamma k_\gamma \in D \mid k_e \in R\} = R + M.$$

THEOREM 3.1. *The following conditions are equivalent:*

(i) For any a, b in K^* there exists c in K^* with $aR \cap bR = cR$.

(ii) For any a, b in D^* there exists c in D^* with $aR' \cap bR' = cR'$.

To prove that (i) implies (ii) we can assume that $a = 1$, i.e. $aR' = R'$. We write $b = \gamma_0(k_{\gamma_0} + m)$ with m in M, $k_{\gamma_0} \neq 0$ in K. For $\gamma_0 > e$ it follows that γ_0 is in M and $bR' \subseteq M \subseteq R'$.

For $\gamma_0 < e$ it follows that $\gamma_0^{-1} > e$, $\gamma_0^{-1} \in M$ and $R' = aR' \subseteq bR'$. For $\gamma_0 = e$ we have $bR' = k_eR' = k_eR + M$. Let d be in K^* with $R \cap k_eR = dR$ and it follows that $R' \cap bR' = (R + M) \cap (k_eR + M) = (R \cap k_eR) + M = dR + M = dR'$.

To prove that (ii) implies (i) let a, b be in K^* and $aR' = aR + M$, $bR' = bR + M$ and $aR' \cap bR' = (aR \cap bR) + M = cR'$ follows for some $c = k_e + m$, $m \in M$, $k_e \in K$.

It is enough to show that $k_e \neq 0$ since we have $aR \cap bR = k_eR$. This is trivial if $R = K$ and otherwise there exists $k \in K \backslash R$. Under the assumption $k_e = 0$ we have $c \in M$, $ckR' \subseteq M \subseteq aR' \cap bR' = cR'$ and therefore the contradiction $k \in R'$, $k \in R$.

The next two results are proved in similar fashion:

THEOREM 3.2. *The following conditions are equivalent:*
(i) For any a, b in K^* there exists c in K^* with $aR + bR = cR$.
(ii) For any a, b in D^* there exists c in D^* with $aR' + bR' = cR'$.

THEOREM 3.3. *The following conditions are equivalent:*
(i) $aR \cap (bR + cR) = (aR \cap bR) + (aR \cap cR)$ for any a, b, c in K.
(ii) $aR' \cap (bR' + cR') = (aR' \cap bR') + (aR' \cap cR')$ for any a, b, c in D.

These results show that the following properties: $H(R')$ is lattice ordered, R' is a Bezout ring, R' is distributive, follow from related properties of the R-module K.

In the following examples B is a valuation subring of a commutative field K that admits a group G of automorphisms such that $R = \cap \sigma(B)$, $\sigma \in G$, has the desired properties.

EXAMPLE 1. Let L be an algebraically closed field and $K = L(x)$, the function field in one indeterminate x over L. Let B be the valuation ring of K associated with the x-adic valuation on K.

We define for any ℓ in L the L-mapping φ_ℓ from K to K with $\varphi_\ell(x) = x - \ell$.

The mapping φ_ℓ is an automorphism of K and $\varphi_\ell(B)$ is the valuation ring of K associated with the $(x - \ell)$-adic valuation of K. Since L is algebraically closed, it follows that the set $\{(x - \ell) \mid \ell \in L\}$ is the set of all irreducible polynomials in $L[x]$ and we obtain:

$$\bigcap_{\ell \in L} \varphi_\ell(B) = L[x].$$

Finally, let φ_∞ be the L-automorphism of K with $\varphi_\infty(x) = \frac{1}{x}$ and let G be the subgroup of the automorphism group of K generated by φ_∞ and the elements φ_ℓ, $\ell \in L$. Then:

$$R = \bigcap_{\sigma \in G} \sigma(B) = L.$$

We have $1R \cap xR = \{0\}$; the condition (i) in Theorem 3.1 is not satisfied. It follows from Theorem 2.1 that the value group associated with B' is not lattice ordered where B' is the valuation ring of D constructed as above from K, B and G. From Theorem 3.3 it follows that R', the intersection of all valuation rings in D conjugate to B', is not distributive, since

K is a vectorspace over $R(=L)$ of dimension greater than one and hence the distributive law does not hold for the lattice of L-subspaces of K.

In Section 4 we will investigate Gauss-extensions of valuations in skew polynomial rings. As for invariant or subinvariant valuation rings or for valuation rings in division algebras finite dimensional over their centers the value group in all these cases is lattice ordered and R', the intersection of all conjugates of the valuation ring B', is distributive. In Example 1, neither is the value group of the valuation ring B' lattice ordered nor is R' distributive.

EXAMPLE 2. We construct a valuation ring B in a field K with a group G of automorphisms such that

(i) $R = \cap\sigma(B)$, $\sigma \in G$, is distributive with K as its field of quotients and

(ii) R is not a Bezout domain.

It then follows from the above construction and Theorems 3.2 and 3.3 that R' is distributive, but not a Bezout ring. Theorem 2.2 then shows that the value group associated with the valuation ring B' is not lattice ordered even though R' is distributive.

The construction of B and K with (i) and (ii) will be based on some preliminary results from algebraic number theory.

LEMMA 3.4. *Let S be a Dedekind domain with quotient field F and F' a separable field extension of F with S' the integral closure of S in F'. Let I be an ideal of S with $IS' = aS'$ for a in S'. Then $IS'' = aS''$ for the integral closure S'' of S in $F'' = F(a)$.*

We prove first that $IS' \cap S'' = IS''$ where $IS'' \subseteq IS' \cap S''$ is trivial. To prove the opposite inclusion it is enough to consider only the case $[F' : F''] < \infty$, since every element of IS' is contained in a finite extension of F''. The result follows immediately from [17] (1.A, p. 161) for the Dedekind rings S', S'' and the ideal IS'' of S''.

Now, we show $aS'' = aS' \cap S''$ where $aS'' \subseteq aS' \cap S''$ holds trivially. Assume $as = t \in S''$, $s \in S'$ and $s = a^{-1}t \in F'' \cap S' = S''$ follows. We conclude $IS'' = IS' \cap S'' = aS' \cap S'' = aS''$.

LEMMA 3.5. *Let S be a Dedekind domain with quotient field F and $I \neq (0)$ an ideal of S with order n as element of the class group of S. Let F' be a separable field extension of F and S' be the integral closure of S in F'. If $IS' = aS'$ for some a in S', then n divides $[F(a) : F]$.*

For a proof let S'' be the integral closure of S in $F'' = F(a)$. By Lemma 3.4 it follows that $IS'' = aS''$ and $N_{F''/F}(IS'') = I^{[F'':F]} = N_{F''/F}(a) \cdot S$ is a principal ideal in S and the statement of the lemma follows.

We now turn to the construction of the example. Let L be an algebraically closed field of characteristic $\neq 2, 3$ and let $L[t]$ be the polynomial ring over L in one indeterminate with $L(t)$ as field of quotients. Then $L[t]$ is a Dedekind ring. We define $F = L(t)(\sqrt{t^3+1})$ and let S be the integral closure of $L[t]$ in F.

It follows that $S = L[t, \sqrt{t^3+1}]$, since $t^3 + 1$ is square free (char $L \neq 3$) in $L[t]$ and S is a Dedekind domain.

Let $M = tL[t]$ and the minimal polynomial $x^2 - t^3 - 1$ of $\sqrt{t^3+1}$ over $L(t)$ splits into two distinct ($\operatorname{char} L \neq 2$) irreducible factors modulo M. Hence, $MS = M_1M_2$ for two distinct maximal ideals M_i of S.

We claim that M_1 is not a principal ideal in S. Otherwise, $M_1 = aS$ and $N_{F/L(t)}(aS) = N_{F/L(t)}(a)L[t] = N_{F/L(t)}(M_1) = M = tL[t]$. Hence, there exists $\ell \in L^*$ with $\ell t = N_{F/L(t)}(a)$. If $a = f(t)+g(t)\sqrt{t^3+1}, f(t), g(t) \in L[t]$, then $f(t) \neq 0 \neq g(t)$ and $N_{F/L(t)}(a) = f^2(t) - g^2(t)(t^3+1)$.

The degree of $(t^3+1)g^2(t)$ is odd and the degree of $f^2(t)$ is even and hence $1 = \deg(\ell t) = \deg\big(f^2(t) - (t^3+1)g^2(t)\big) = \max\{\deg f^2(t), \deg(t^3+1)g^2(t)\} \geq 3$ leads to a contradiction.

Next, it will be shown that the order of M_1 in the class group of S is three. Since we know that M_1 is not principal, it is sufficient to prove that M_1^3 is a principal ideal.

Let $\alpha = 1+\sqrt{t^3+1}$ and $\alpha' = 1-\sqrt{t^3+1}$. It follows that neither α nor α' is contained in any maximal ideal P of S with $P \cap L[t] = (t-\ell)L[t]$ for $\ell \neq 0$.

To see this assume α in P or α' in P, hence $\alpha\alpha' = 1 - t^3 - 1 = -t^3$ is in $P \cap L[t] = (t-\ell)L[t]$— a contradiction. Let σ be the $L(t)$-automorphism of F different from the identity. Then $\sigma(M_1) = M_2$ and $\sigma(\alpha') = \alpha$.

We have $\alpha\alpha' = -t^3 \in M \subseteq M_1$ and we can assume $\alpha \in M_1$. Since $\alpha + \alpha' = 2$, it follows that $\alpha' \notin M_1$ and $\alpha = \sigma(\alpha') \notin \sigma(M_1) = M_2$. We conclude that M_1 is the only maximal ideal of S containing αS and therefore $\alpha S = M_1^k$.

However,

$$\begin{aligned} M^3 &= t^3L[t] = N_{F/L(t)}(\alpha)L[t] \\ &= N_{F/L(t)}(\alpha S) = N_{F/L(t)}(M_1^k) \\ &= \big(N_{F/L(t)}(M_1)\big)^k = M^k, \end{aligned}$$

which shows that $k = 3$. We consider $F' = L(t)\big(\{\sqrt{(t-\ell)^3+1} \mid \ell \in L\}\big)$. This is an infinite Galois extension of $L(t)$ and we denote by Σ the Galois group of F' over $L(t)$.

If V is the valuation ring of $L(t)$ associated with the t-adic valuation and B an extension of V in F', then

$$R_B = \bigcap \sigma(B), \quad \sigma \in \Sigma,$$

is the integral closure of V in F'.

For every ℓ in L one can define the L-automorphism φ_t of $L(t)$ with $\varphi_\ell(t) = t - \ell$. Since L is algebraicall·' closed, we have $L[t] = \bigcap_{\ell\in L} \varphi_\ell(V)$. Every φ_ℓ can be extended to an automorphism of F' which we denote again by φ_ℓ and it follows that

$$\bigcap \varphi_\ell\big(\sigma(B)\big) = \bigcap \sigma\varphi_\ell(B), \quad \sigma \in \Sigma,$$

is the integral closure of $\varphi_\ell(V)$ the valuation ring of $L(t)$ associated with the $(t-\ell)$-adic valuation.

We obtain: $S' = \bigcap_{\ell\in L}\bigcap_{\sigma\in\Sigma} \varphi_\ell\big(\sigma(B)\big)$ is the integral closure of $L[t]$ in F'.

Let G be the subgroup of the automorphism group of F' generated by Σ and $\{\varphi_\ell \mid \ell \in L\}$. It follows that Σ is a normal subgroup of G and that

$$S' = \bigcap_{\gamma \in G} \gamma(B).$$

Every element γ in G can be written as $\gamma = \sigma\varphi_\ell = \varphi_\ell\sigma'$ for some $\ell \in L$ and some σ, $\sigma' \in \Sigma$.

Since F' is algebraic over $L(t)$ and $L[t]$ is distributive, it follows that S' is distributive (i.e. S' is a Prüfer domain).

It remains to show that S' is not a Bezout domain. Let M_1 be the maximal ideal in $S = L[t, \sqrt{t^3+1}]$ defined above. The ideal M_1 is finitely generated, since S is a Dedekind domain, and hence M_1S' is finitely generated in S'. The assumption $M_1S' = aS'$ for some a in S' implies that 3 divides $[F(a): F]$ by Lemma 3.5 and the result proved above says that 3 is the order of M_1 in the class group of S. However, $\left[F[a]: F\right]$ is a power of 2 by construction. The contradiction shows that S' is not a Bezout domain and the valuation ring B of the field $F' = K$ with $R = S'$ satisfy the required conditions.

4. In this section we consider the extension of valuations on a skew field K to an Ore extension of K. Let K be a skew field with an automorphism σ. The skew polynomial ring $K[x, \sigma]$ with $xa = \sigma(a)x$ defining the multiplication is a right and left Ore domain with a skew field $F = K(x, \sigma)$ of quotients. If v is a valuation of K with valuation ring B_v which satisfies $\sigma(B_v) = B_v$ then v can be extended to a valuation u from $K(x, \sigma)$ to W_v by defining $u\left(\sum_{i=1}^n a_ix^i\right) = \min\{v(a_i) \mid i = 1, \dots, n\}$, (see: [4],[14]). This extension u of v will be called Gauss-extension of v. The elements in $K(x, \sigma)$ are of the form $kt(x)s^{-1}(x)$ for some k in K and $t(x), s(x)$ in $K[x, \sigma]$ which are units in B_u. The set of the valuation rings in $K(x, \sigma)$ conjugated to B_u is therefore $\{kB_uk^{-1} \mid k \in K^*\}$ and kB_uk^{-1} is the valuation ring associated with the valuation u_k from $K(x, \sigma)$ to W_v defined by $u_k(y) = u(k^{-1}yk)$.

In particular,

$$\begin{aligned} u_k(a_0 + a_1x + \cdots + a_nx^n) &= u\left(k^{-1}a_0k + k^{-1}a_1\sigma(k)x + \cdots + k^{-1}a_n\sigma^n(k)x^n\right) \\ &= \min\left\{v(k^{-1}a_0k), v\left(k^{-1}a_1\sigma(k)\right), \dots, v\left(k^{-1}a_n\sigma^n(k)\right)\right\}. \end{aligned}$$

The next results establish a relationship between properties of $R_v = \bigcap_{k \in K^*} kB_vk^{-1}$ and properties of the ring $R_u = \bigcap_{s \in F^*} sB_us^{-1} = \bigcap_{k \in K^*} kB_uk^{-1}$.

THEOREM 4.1. *Let v be a valuation of the skew field K, σ an automorphism of K with $\sigma(B_v) = B_v$ and u the Gauss extension of v to $F = K(x, \sigma)$. Then R_u is a distributive ring if K is the skew field of quotients of R_v and R_v is distributive.*

PROOF. It must be shown that $A \cap (B + C) \subseteq (A \cap B) + (A \cap C)$ for any ideals A, B, C in R_u—the opposite inclusion holds trivially.

The following proof is similar to the proof of Theorem 1 in [10].

Let d' be an element in $A \cap (B + C)$, hence $a' = b' + c'$, $b' \in B$, $c' \in C$. There exist $a, b, c, d \in K[x, \sigma]$ with $d \neq 0$ and

$$d' = d^{-1}a,\ b' = d^{-1}b,\ c' = d^{-1}c \text{ and } a = \sum_{i=0}^{n} a_i x^i,\ b = \sum_{i=0}^{n} b_i x^i,\ c = \sum_{i=0}^{n} c_i x^i$$

and

$$a_i = b_i + c_i,\ a_i \in a_i R_v \cap (b_i R_v + c_i R_v) = (a_i R_v \cap b_i R_v) + (a_i R_v \cap c_i R_v)$$

follows for $i = 0, \ldots, n$. Hence, elements r_i, s_i exist in R_v with

$$a_i = b_i r_i + c_i s_i \text{ and } b_i r_i,\ c_i s_i \in a_i R_v.$$

Define the element

$$h = b_0(1 - r_0) + b_1(1 - r_1)x + \cdots + b_n(1 - r_n)x^n$$

and since $a = (b - h) + (h + c)$ it is sufficient to show that $b - h \in aR_u \cap bR_u$ and $h + c \in aR_u \cap cR_u$. This in turn follows if one can prove that $u_k(b - h) \geq u_k(a)$, $u_k(b - h) \geq u_k(b)$ and also $u_k(h + c) \geq u_k(a)$, $u_k(h + c) \geq u_k(c)$ for all $k \in K^*$ holds. We will show the first inequality.

$$u_k(b - h) = \min\{v(k^{-1}b_0 r_0 k),\ v(k^{-1}b_1 r_1 \sigma(k)), \ldots, v(k^{-1}b_n r_n \sigma^n(k))\}$$
$$u_k(a) = \min\{v(k^{-1}a_0 k),\ v(k^{-1}a_1 \sigma(k)), \ldots, v(k^{-1}a_n \sigma^n(k))\}.$$

We have: $b_i r_i \in a_i R_v$, hence, $k^{-1}b_i r_i \sigma^i(k) \in k^{-1}a_i R_v \sigma^i(k) = k^{-1}a_i \sigma^i(k) R_v$ and $v(k^{-1}b_i r_i \sigma^i(k)) \geq v(k^{-1}a_i \sigma^i(k))$.

THEOREM 4.2. *Let v be a valuation of the skew field K, σ an automorphism of K with $\sigma(B_v) = B_v$ and u the Gauss-extension of v to $K(x, \sigma)$. Assume that K is the skew field of quotients of R_v and that R_v is a Bezout domain. Then:*

(i) R_u is a Bezout domain with $F = K(x, \sigma)$ as its field of quotients.

(ii) G_u is lattice ordered.

PROOF. To show (ii) let $\tilde{a}, \tilde{b}$ be elements in G_u and there exist c, d, e in $K[x, \sigma]$ with $0 \neq e$ and $a = e^{-1}c$, $b = e^{-1}d$. As we observed earlier, it is enough to show that $\inf\{\tilde{c}, \tilde{d}\}$ exists in G_u.

Let $c = c_0 + c_1 x + \cdots + c_n x^n$ and $d = d_0 + d_1 x + \cdots + d_n x^n$. Since R_v is a Bezout domain, there exists for every $i = 0, \ldots, n$ an element f_i in K with $c_i R_v + d_i R_v = f_i R_v$ and for all k in K we obtain

$$c_i \sigma^i(k) R_v + d_i \sigma^i(k) R_v = f_i \sigma^i(k) R_v \text{ and}$$
$$c_i \sigma^i(k) B_v + d_i \sigma^i(k) B_v = f_i \sigma^i(k) B_v \text{ which implies}$$
$$v(f_i \sigma^i(k)) = \inf\{v(c_i \sigma^i(k)),\ v(d_i \sigma^i(k))\}.$$

It remains to show that $\inf\{\tilde{c},\tilde{d}\} = \tilde{f}$ for $f = f_0 + f_1x + \cdots + f_nx^n$. For all k in K we have:

$$\begin{aligned}
\tilde{f}\big(v(k)\big) &= \tilde{f}\big(u(k)\big) = u(fk) = u\Big(\sum_{i=1}^{n} f_i\sigma^i(k)x^i\Big) \\
&= \inf\big\{v(f_0k),\dots,v\big(f_n\sigma^n(k)\big)\big\} \\
&= \inf\Big\{\inf\{v(c_0k),v(d_0k)\},\dots,\inf\big\{v\big(c_n\sigma^n(k)\big),v\big(d_n\sigma^n(k)\big)\big\}\Big\} \\
&= \inf\Big\{\inf\big\{v(c_0k),\dots,v\big(c_n\sigma^n(k)\big)\big\},\ \inf\big\{v(d_0k),\dots,v\big(d_n\sigma^n(k)\big)\big\}\Big\} \\
&= \inf\{u(ck),u(dk)\} \\
&= \inf\big\{\tilde{c}\big(v(k)\big),\tilde{d}\big(u(k)\big)\big\},
\end{aligned}$$

and the equation $\tilde{f} = \inf\{\tilde{c},\tilde{d}\}$ follows.

To prove (i) one observes that by (ii) and the Corollary to Theorem 2.1, it follows that $K(x,\sigma)$ is the skew field of quotients of R_u. Theorem 4.1 shows that R_u is distributive and Theorem 2.2 can be applied to R_u and (i) follows from (ii).

5. Let B be a valuation ring of the skew field D, G the value group and $R = \bigcap_{d\in D^*} dBd^{-1}$ the intersection of all subrings in D conjugate to B. We were not able to answer the following two questions:

A. Is D the skew field of quotients of R?

B. Does there exist an example for B and D such that R with D as its skew field of quotients is not distributive, however, G is lattice ordered?

If one would want to construct such an example using the methods in Section 3, then the following question arises: does there exist a valuation ring B_0 of a commutative field K and a subgroup H of the automorphism group of K such that $R_0 = \bigcap \sigma(B_0)$, $\sigma \in H$, does not satisfy the distributivity condition in Theorem 3.3 for all elements a,b,c in K, but for $a,b \in K^*$ there exists $c \in K^*$ with $aR_0 \cap bR_0 = cR_0$?

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Department of Mathematics
University of Alberta
Edmonton, Alberta T6G 2G1

Institut für Analysis
Abt. für Topologie u. Grundlagen der Analysis
Technische Universität, Pockelsstrasse 14
D-3300 Braunschweig, Germany