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RESEARCH ARTICLE

On the nilpotent orbit theorem of complex variations of Hodge structure

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Abstract
We prove some results on the nilpotent orbit theorem for complex variations of Hodge structure.

Contents

0. Introduction

The nilpotent and SL_2-orbit theorems of Schmid have been fundamental in understanding the degeneration of Hodge structure, particularly in the context of integral variation of Hodge structures. However, their complete generalization to complex variations of Hodge structure remains unproven. This paper aims to the study of Schmid's nilpotent orbit theorem for complex variations of Hodge structure. The main result of this paper is the main component of the nilpotent orbit theorem.

Theorem A. *Let X be a complex manifold, and let $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor on X. Let $(V, \nabla, F^\bullet, Q)$ be a complex polarized variation of Hodge structure on $X \backslash D$. Then for any multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{R}^\ell$, $F^p_{\boldsymbol{\alpha}} := j_* F^p \cap V^{\mathrm{Del}}_{\boldsymbol{\alpha}}$ and $F^p_{\boldsymbol{\alpha}} / F^{p+1}_{\boldsymbol{\alpha}}$ are both locally free sheaves. Here, $V^{\mathrm{Del}}_{\boldsymbol{\alpha}}$ is the Deligne extension of the flat bundle (V, ∇) with the eigenvalues of the residue of ∇ over D_i lying in $[-\alpha_i, -\alpha_i + 1)$.*

We prove moreover that the grading $\oplus_{p+q=m}F_\alpha^p/F_\alpha^{p+1}$ is naturally identified with $\oplus_{p+q=m}\mathcal{P}_\alpha E^{p,q}$, where $\mathcal{P}_\alpha E^{p,q}$ is the prolongation of the Hodge bundles $E^{p,q} := F^p/F^{p+1}$ in terms of the norm growth of the Hodge metric (see §1.6 for the definition).

Based on Theorem A, we can generalize main parts of Schmid's nilpotent orbit theorem to complex polarized variation of Hodge structure.

Theorem B. *Let $(V, \nabla, F^\bullet, Q)$ be a complex polarized variation of Hodge structure on $(\Delta^*)^p \times \Delta^q$. Denote by $\Phi : \mathbb{H}^p \times \Delta^q \to \mathscr{D}$ its period mapping, where $\mathscr{D}$ is the period domain and $\mathbb{H} = \{z \in \mathbb{C} \mid \Re z < 0\}$. Let us denote by $2\pi i R_i$ is the logarithm of the monodromy operator associated to the counterclockwise generator of the fundamental group of the i-th copy of Δ^* in $(\Delta^*)^p$, whose eigenvalues lie in $(2\pi i(\alpha_i - 1), 2\pi i\alpha_i]$ for some $\boldsymbol{\alpha} \in \mathbb{R}^p$. Then for the holomorphic mapping $\Psi : (\Delta^*)^p \times \Delta^q \to \check{\mathscr{D}}$ induced by $\tilde{\Psi} := \exp(\sum_{i=1}^p z_i R_i) \circ \Phi(z, w)$,*

(i) *Ψ extends holomorphically to Δ^{p+q};*
(ii) *the holomorphic mapping*

$$\vartheta : \mathbb{H}^p \times \Delta^q \to \check{\mathscr{D}}$$
$$(z, w) \mapsto \exp\left(-\sum_{i=1}^p z_i R_i\right) \circ a(w)$$

is horizontal, where $a(w) := \Psi(0, w)$, and $\check{\mathscr{D}}$ is the compact dual of the period domain $\mathscr{D}$.
(iii) *In the one variable case, $\exp(-zR) \circ a$ lies in $\mathscr{D}$ when $\Re z \le -C$ for some $C > 0$. Moreover, we have the distance estimate*

$$d_{\mathscr{D}}(\exp(-zR) \circ a, \Phi(z)) \le C' |\Re z|^\beta e^{\delta \Re z} \quad \textit{for some} \quad C', \delta, \beta > 0$$

if $\Re z \le -C$.

When (V, ∇) has quasi-unipotent monodromies around D, Theorems A and B are contained in Schmid's nilpotent orbit theorem [10]. Under this monodromy assumption, he proved Theorem B.(iii) for the case of several variables.

Theorems A and B were also proved by Sabbah and Schnell [9] for the case of one variable in a different way. Their methods can be extended to prove the general case.

Our proof of Theorem A is based on Mochizuki's work on the prolongation of acceptable bundles [8] and methods in L^2-estimates. The proof of Theorems B.(ii) and B.(iii) essentially follows Schmid's method in [10].

We conclude the introduction by explaining applications of the main result in this paper. One application is related to the work [13] on the injectivity and vanishing theorem for $\mathbb{R}$-Hodge modules. We remark that in [13 , Lemma 2.10], nilpotent orbit theorem for real variation of Hodge structures was claimed. It seems to the author that the proof needs some amplification, and as such, Theorem A complements the proof presented therein.

Another potential further application is on the complex Hodge modules. It is well-established that Saito's theory of mixed Hodge modules fundamentally relies on the nilpotent orbit theorem. Currently, Sabbah and Schnell are developing the theory of mixed Hodge modules for complex Hodge structures. We remark that the nilpotent orbit theorem established in this paper should serve as a foundational component in their work.

1. Preliminary

1.1. Complex polarized variations of Hodge structure

Let us briefly recall the definition of *polarized Hodge structure*. We refer the readers to [12, 9] for more details. A Hodge structure of weight w on a complex vector space V is a decomposition

$$V = \bigoplus_{p+q=w} V^{p,q}$$

and a *polarization* is a Hermitian pairing $Q : V \otimes_{\mathbb{C}} \bar{V} \to \mathbb{C}$ such that the above decomposition is orthogonal with respect to Q and such that $(-1)^q Q$ is positive definite on the subspace $V^{p,q}$. For any $v \in V$, its *Hodge norm* is defined

$$|v|^2 = \sum_{p+q=w} (-1)^q Q\,(v^{p,q}, v^{p,q}),$$

where $v^{p,q}$ is the (p,q)-component of v in the Hodge decomposition $V = \bigoplus_{p+q=w} V^{p,q}$. Such a polarized Hodge structure induces the *Hodge filtration* $F^\bullet V$ of V defined by

$$F^p V = \bigoplus_{i \geq p} V^{i,w-i}.$$

As introduced by [12], a complex polarized variation of Hodge structure $(V = \bigoplus_{r+s=w} V^{r,s}, \nabla, Q)$ of weight w on a complex manifold U consists of the following data:

(a) a smooth vector bundle V with a Hodge decomposition $V = \bigoplus_{r+s=w} V^{r,s}$;
(b) a flat connection ∇ satisfies the Griffiths' transversality condition

$$\nabla : V^{r,s} \to A^{0,1}(V^{r+1,s+1}) \oplus A^{1,0}(V^{r,s}) \oplus A^{0,1}(V^{r,s}) \oplus A^{1,0}(V^{r-1,s+1}); \tag{1.1}$$

(c) a parallel Hermitian form Q which makes the Hodge decomposition orthogonal and which on $\mathrm{V}^{r,s}$ is positive definite if r is even and negative definite if r is odd.

We decompose $\nabla = \bar{\theta} + \partial + \bar{\partial} + \theta$ according to the above transversality condition in Equation (1.1). The most important component of the connection turns out to be the *Higgs field*, which is the linear operator

$$\theta : V^{r,s} \to A^{1,0}(V^{r-1,s+1}).$$

We decompose $\nabla = \nabla' + \nabla''$ into its (1,0)-component $\nabla' : A^0(V) \to A^{1,0}(V)$ and its $(0,1)$-component $\nabla'' : A^0(V) \to A^{0,1}(V)$. Then ∇'' gives V the structure of a holomorphic vector bundle, which we denote by the symbol $\mathcal{V}$, and ∇' defines an integrable holomorphic connection $\nabla' : \mathcal{V} \to \Omega_U^1 \otimes_{\mathscr{O}_U} \mathcal{V}$ on this bundle. The condition on ∇ in Equation (1.1) is saying that the Hodge filtration

$$F^p V = \bigoplus_{i \geq p} V^{i,w-i}$$

is a holomorphic subbundles of $\mathcal{V}$, and that the connection ∇' satisfies Griffiths' transversality relation

$$\nabla'(F^p \mathcal{V}) \subseteq \Omega_U^1 \otimes_{\mathscr{O}_U} F^{p-1}\mathcal{V}.$$

Note that $(V^{p,q}, \bar{\partial})$ defines a holomorphic vector bundle, denoted by $E^{p,q}$. From the above point of view, the Higgs field is simply the holomorphic operator

$$\theta : E^{p,q} \to \Omega_U^1 \otimes_{\mathscr{O}_U} E^{p-1,q+1}.$$

Denote by $(E, \theta) = (\oplus_{p+q=w} E^{p,q}, \theta)$, which is called the *system of Hodge bundles* relative to the $\mathbb{C}$-VHS $(V = \oplus_{p+q=w} V^{p,q}, \nabla, Q)$. Let us define $h_{p,q} := (-1)^p Q$, which is a Hermitian metric for $E^{p,q}$ by Item (c). For the Hermitian metric $h = \oplus_{p+q=w} h_{p,q}$ of E, the component $\bar{\theta} : V^{r,s} \to A^{0,1}(V^{r+1,s-1})$ is the adjoint of θ with respect to h. Since the primary objective of this paper is to establish the nilpotent orbit theorem for $\mathbb{C}$-VHS, we will adopt the notation $(V, \nabla, F^\bullet V, Q)$ to represent the $\mathbb{C}$-VHS, as the Hodge filtration $F^\bullet V$ plays a central role in the nilpotent orbit theorem.

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1.2. Deligne extension

Let X be a complex manifold, and let D be a simple normal crossing divisor on X. For the flat bundle (V, ∇) defined on $U := X \backslash D$, Deligne introduced a way to extend it across D. We recall this construction briefly and refer the readers to [10, 7, 9] for more details. For any point $x \in D$, we choose an admissible coordinate $(\Omega; z_1, \ldots, z_n)$ such that $\Omega \simeq \Delta^n$ and $D \cap \Omega = (z_1 \ldots z_p = 0)$ (see Definition 1.1 for the definition). Write $q = n - p$. The fundamental group $\pi_1\big((\Delta^*)^p \times \Delta^q\big)$ is generated by elements $\gamma_1, \ldots, \gamma_p$, where γ_j may be identified with the counterclockwise generator of the fundamental group of the j-th copy of Δ^* in $(\Delta^*)^p$. We denote by V^∇ the space of multivalued flat sections of (V, ∇), which is a finite-dimensional $\mathbb{C}$-vector space. Set T_j to be the monodromy transformation with respect to γ_j, which pairwise commute and are endomorphisms of V^∇; that is, for any multivalued section $v(t_1, \ldots, t_{p+q}) \in V^\nabla$, one has

$$v(t_1, \ldots, e^{2\pi i} t_j, \ldots, t_{p+q}) = (T_j v)(t_1, \ldots, t_{p+q})$$

and $[T_j, T_k] = 0$ for any $j, k = 1, \ldots, p$. Let us write $Sp(T_j)$ the set of eigenvalues of T_j, and for any $\lambda_j \in Sp(T_j)$, we denote by $\mathbb{E}(T_j, \lambda_j) \subset V^\nabla$ the corresponding eigenspace. We know that all $\lambda_j \in Sp(T_j)$ has norm 1 (see, e.g., [9]). Write $Sp := \prod_{i=1}^p Sp(T_j)$. For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)$, we define

$$\mathbb{E}_{\boldsymbol{\lambda}} := \cap_{j=1}^p \mathbb{E}(T_j, \lambda_j).$$

Since T_j pairwise commute, one has

$$V^\nabla = \oplus_{\boldsymbol{\lambda} \in Sp} \mathbb{E}_{\boldsymbol{\lambda}},$$

and $\mathbb{E}_{\boldsymbol{\lambda}}$ is an invariant subspace of T_j for any $\boldsymbol{\lambda} \in Sp$ and any j.

Let us fix a p-tuple $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p$. Then for $\boldsymbol{\lambda} \in Sp$, there exists a unique $\beta_i \in (\alpha_i - 1, \alpha_i]$ such that $\exp(2\pi i \beta_i) = \lambda_i$. Since $\lambda_i^{-1} T_i|_{\mathbb{E}_{\boldsymbol{\lambda}}}$ is unipotent, its logarithm can be defined as

$$\log(\lambda_i^{-1} T_i|_{\mathbb{E}_{\boldsymbol{\lambda}}}) := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(\lambda_i^{-1} T_i|_{\mathbb{E}_{\boldsymbol{\lambda}}} - I)^k}{k}.$$

We denote $N_i := \frac{\log(\lambda_i^{-1} T_i|_{\mathbb{E}_{\boldsymbol{\lambda}}})}{2\pi i}$. Then for any $v \in \mathbb{E}_{\boldsymbol{\lambda}}$, we define

$$\tilde{v}(t) := \exp\left(-\sum_{i=1}^p (\beta_i I + N_i) \cdot \log t_i\right) v(t) = \prod_{i=1}^p t_i^{-\beta_i} \exp\left(-\sum_{i=1}^p N_i \cdot \log t_i\right) v(t). \tag{1.2}$$

One can check that $\tilde{v}$ is single valued and that $\nabla^{0,1}\tilde{v} = 0$. We now fix a basis $v_1, \ldots, v_r$ of V^∇ such that each v_i belongs to some $\mathbb{E}_{\boldsymbol{\lambda}}$. Then the holomorphic sections $\tilde{v}_1, \ldots, \tilde{v}_r$ of $\mathcal{V}$ defines a prolongation of $\mathcal{V}$ over X which we denoted by $V_{\boldsymbol{\alpha}}^{\mathrm{Del}}$. One can check that this construction does not depend on our choice of the basis. This is called the *Deligne extension* of the flat bundle (V, ∇) with the eigenvalues of the residue of ∇ over D_i lying in $[-\alpha_i, -\alpha_i + 1)$. Note that it is defined for any flat bundle (V, ∇) (not necessarily complex variation of Hodge structure).

If (V, ∇) underlies a complex polarized variation of Hodge structure $(V, \nabla, F^\bullet V, Q)$, we define $F_{\boldsymbol{\alpha}}^p := j_* F^p \cap V_{\boldsymbol{\alpha}}^{\mathrm{Del}}$. It is called the extension of the Hodge filtration. It should be noted that, a priori, we do not know whether $F_{\boldsymbol{\alpha}}^p$ is locally free.

1.3. Acceptable bundles

Definition 1.1 (Admissible coordinate). Let X be a complex manifold, and let D be a simple normal crossing divisor. Let x be a point of X, and assume that $\{D_j\}_{j=1,\ldots,\ell}$ are the components of D containing

p. An *admissible coordinate* around x is the tuple $(\Omega; z_1, \ldots, z_n; \varphi)$ (or simply $(\Omega; z_1, \ldots, z_n)$ if no confusion arises) where

- Ω is an open subset of X containing x.
- There is a holomorphic isomorphism $\varphi : \Omega \to \Delta^n$ such that $\varphi(D_j) = (z_j = 0)$ for any $j = 1, \ldots, \ell$.

We shall write $\Omega^* := \Omega - D$, $\Omega(r) := \{z \in \Omega \mid |z_i| < r, \forall i = 1, \ldots, n\}$ and $\Omega^*(r) := \Omega(r) \cap \Omega^*$.

We define a (incomplete) Poincaré-type metric ω_P on $(\Delta^*)^\ell \times \Delta^{n-\ell}$ by

$$\omega_P = \sum_{j=1}^{\ell} \frac{\sqrt{-1}dz_j \wedge d\bar{z}_j}{|z_j|^2(\log |z_j|^2)^2} + \sum_{k=\ell+1}^{n} \sqrt{-1}dz_k \wedge d\bar{z}_k. \tag{1.3}$$

Note that

$$\omega_P = i\partial\bar{\partial} \log\left(\prod_{j=1}^{\ell}(-\log |z_j|^2)^{-1} \cdot \prod_{k=\ell+1}^{n} \exp(|z_k|^2)\right).$$

For any system of Hodge bundles (E, θ, h), we have the following crucial norm estimate for its Higgs field θ. The one-dimensional case is due to Simpson [11, Theorem 1] and the general case was proved by Mochizuki in [5, Proposition 4.1]. Its proof relies on a clever use of Ahlfors–Schwarz lemma.

Theorem 1.2. *Let (E, θ, h) be a system of Hodge bundle on $X\backslash D$. Then for any point $x \in D$, it has an admissible coordinate $(\Omega; z_1, \ldots, z_n)$ such that the norm $|\theta|_{h,\omega_P} \le C$ holds over Ω^* for some constant $C > 0$. Here, $|\theta|_{h,\omega_P}$ denotes the norm of θ with respect to h and ω_P.*

Here, we also recall the following definition in [7, Definition 2.7].

Definition 1.3 (Acceptable bundle). Let (E, h) be a Hermitian vector bundle over $X\backslash D$. We say that (E, h) is an acceptable at $p \in D$, if the following holds: There is an admissible coordinate $(\Omega; z_1, \ldots, z_n)$ around p such that the norm $|R(E, h)|_{h,\omega_P} \le C$ for some $C > 0$. Here, $R(E, h)$ is the Chern curvature of (E, h). When (E, h) is acceptable at any point p of D, it is called acceptable.

Hodge filtrations and Hodge bundles endowed with the Hodge metric are all acceptable.

Lemma 1.4. *Let $(V, \nabla, F^\bullet, Q)$ be a complex polarized variation of Hodge structure of weight m on $X\backslash D$. Let h (resp. $h_{p,q}$) be the Hermitian metric on V (resp. on $E^{p,q}$) introduced in §1.1. Consider the induced Hermitian metric $h_p := h|_{F^p}$ on the Hodge filtration F^p. Then both (F^p, h_p) and $(E^{p,q}, h_{p,q})$ are acceptable bundles.*

Proof. We write $\theta_{p,q} := \theta|_{E^{p,q}}$ and let $\theta^\dagger_{p,q} : E^{p-1,q+1} \to A^{0,1}(E^{p,q})$ be its adjoint with respect to $h_{p,q}$. For the Hermitian bundle (F^p, h_p), its curvature is

$$R_{h_p}(F^p) = -2\theta^\dagger_{m,0} \wedge \theta_{m,0} + 2\sum_{i=1}^{m-p}(-\theta^\dagger_{m-i,i} \wedge \theta_{m-i,i} - \theta_{m-i+1,i-1} \wedge \theta^\dagger_{m-i+1,i-1}) + \theta^\dagger_{p,m-p} \wedge \theta_{p,m-p}.$$

The curvature of the bundle $(E^{p,q}, h_{p,q})$ is

$$R_{h_{p,q}}(E_{p,q}) = -\theta^\dagger_{p,q} \wedge \theta_{p,q} - \theta_{p+1,q-1} \wedge \theta^\dagger_{p+1,q-1}.$$

By Theorem 1.2, for any point $x \in D$ there is an admissible coordinate $(\Omega; z_1, \ldots, z_n)$ around x such that the norm

$$\sum_{p+q=m} |\theta_{p,q}|_{h_{p,q},\omega_P} = |\theta|_{h,\omega_P} \le C$$

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holds over Ω^* for some constant $C > 0$. Since $\theta^\dagger_{p,q}$ is the adjoint of $\theta_{p,q}$ with respect to $h_{p,q}$, one has $|\theta^\dagger_{p,q}|_{h,\omega_P} \leq C$ for any p. It follows that $|R_{h_p}(F^p)|_{h_{p,q},\omega_P} \leq C'$ and $|R_{h_{p,q}}(E_{p,q})|_{h_{p,q},\omega_P} \leq C'$ for some $C' > 0$. Hence, (F^p, h_p) and $(E^{p,q}, h_{p,q})$ are both acceptable. □

1.4. Adapted to log order

We recall some notions in [7, §2.2.2]. Let X be a complex manifold, D be a simple normal crossing divisor on X and E be a holomorphic vector bundle on $X\backslash D$ such that $E|_{X\backslash D}$ is equipped with a Hermitian metric h. Let $\boldsymbol{v} = (v_1, \ldots, v_r)$ be a smooth frame of $E|_{X\backslash D}$. We obtain the $H(r)$-valued function $H(h, \boldsymbol{v})$ defined over $X\backslash D$, whose (i, j)-component is given by $h(v_i, v_j)$.

Let us consider the case $X = \mathbb{D}^n$ and $D = \sum_{i=1}^{\ell} D_i$ with $D_i = (z_i = 0)$. We have the coordinate $(z_1, \ldots, z_n)$. Let h, E and $\boldsymbol{v}$ be as above.

Definition 1.5. A smooth frame $\boldsymbol{v}$ on $X\backslash D$ is called *adapted up to log order*, if the following inequalities hold over $X\backslash D$:

$$C^{-1}\left(-\sum_{i=1}^{\ell} \log|z_i|\right)^{-M} \leq H(h, \boldsymbol{v}) \leq C\left(-\sum_{i=1}^{\ell} \log|z_i|\right)^{M}$$

for some positive numbers M and C.

1.5. Parabolic vector bundles

In this subsection, we recall the notions of parabolic (vector) bundles. For more details, we refer to [6]. Let X be a complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a reduced simple normal crossing divisor, $U = X\backslash D$ be the complement of D and $j : U \to X$ be the inclusion.

Definition 1.6. A parabolic bundle $\mathcal{P}_*E$ on (X, D) is a holomorphic vector bundle E on U, together with an $\mathbb{R}^\ell$-indexed filtration $\mathcal{P}_{\boldsymbol{\alpha}}E$ (*parabolic structure*) by locally free subsheaves of j_*E such that

(i) $\boldsymbol{\alpha} \in \mathbb{R}^\ell$ and $\mathcal{P}_{\boldsymbol{\alpha}}E|_U = E$.
(ii) $\mathcal{P}_{\boldsymbol{\alpha}}E \subset \mathcal{P}_{\boldsymbol{\beta}}E$ if $\alpha_i \leq \beta_i$ for all i.
(iii) For $1 \leq i \leq \ell$, $\mathcal{P}_{\boldsymbol{\alpha}+\boldsymbol{1}_i}E = \mathcal{P}_{\boldsymbol{\alpha}}E \otimes \mathcal{O}_X(D_i)$, where $\boldsymbol{1}_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the i-th component.
(iv) $\mathcal{P}_{\boldsymbol{\alpha}+\boldsymbol{\epsilon}}E = \mathcal{P}_{\boldsymbol{\alpha}}E$ for any vector $\boldsymbol{\epsilon} = (\epsilon, \ldots, \epsilon)$ with $0 < \epsilon \ll 1$.
(v) The set of *weights* $\{\boldsymbol{\alpha} \mid \mathcal{P}_{\boldsymbol{\alpha}}E/\mathcal{P}_{<\boldsymbol{\alpha}}E\} \neq 0$ is discrete in $\mathbb{R}^\ell$.

1.6. Prolongation via norm growth

Let X be a complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor, $U = X\backslash D$ be the complement of D and $j : U \to X$ be the inclusion. Let (E, h) be a Hermitian vector bundle on U. For any $\boldsymbol{\alpha} = (a_1, \ldots, a_\ell) \in \mathbb{R}^\ell$, we can prolong E over X by a sheaf of $\mathcal{O}_X$-module $\mathcal{P}_{\boldsymbol{\alpha}}E$ as follows:

$$\mathcal{P}_{\boldsymbol{\alpha}}E(U) = \left\{\sigma \in \Gamma(U\backslash D, E|_{U\backslash D}) \mid |\sigma|_h \lesssim \prod_{i=1}^{\ell} |z_i|^{-\alpha_i - \varepsilon} \text{ for all } \varepsilon > 0\right\}.$$

In [8, Theorem 21.3.1], Mochizuki proved that the prolongation of acceptable bundles defined above are parabolic bundles.

Theorem 1.7 (Mochizuki). *Let (E, h) be an acceptable bundle over $X\backslash D$. Then $\mathcal{P}_*E$ defined above is a parabolic bundle.*

1.7. *Period domain and period mapping*

In this subsection, we quickly review the definitions of period domain and period mapping. We refer the readers to [1, 4, 9] for more details.

Let $(V = \oplus_{p+q=m} V^{p,q}, Q)$ be a polarized complex Hodge structure of weight m defined in §1.1. Recall that the Hodge filtration is defined to be $F^p := \oplus_{i\geq p} V^{i,m-i}$. After fixing m and $\dim_{\mathbb{C}} F^p$, the set of all such filtration $F^\bullet$ is a complex flag manifold, which is denoted by $\check{\mathscr{D}}$. It is a closed submanifold of a product of Grassmannians and is thus a projective manifold. The subset $\mathscr{D}$ of all complex polarized Hodge structures are charcterized by

(a) $F^p = F^p \cap (F^{p+1})^\perp \oplus F^{p+1}$.
(b) $(-1)^p Q$ is positive definite over $F^p \cap (F^{p+1})^\perp$.

It is an open submanifold of $\check{\mathscr{D}}$. We usually write F instead of $F^\bullet$ to lighten the notation. Since the groups $\mathrm{GL}(V)$ and $G := U(V, Q)$ act transitively on $\check{\mathscr{D}}$ and $\mathscr{D}$, respectively, $\check{\mathscr{D}}$ and $\mathscr{D}$ are thus homogeneous spaces.

For any Hodge structure $F \in \check{\mathscr{D}}$, the holomorphic tangent space $T_{\check{\mathscr{D}},F}$ of $\check{\mathscr{D}}$ at F is identified with

$$\mathrm{End}(V)/\{A \in \mathrm{End}(V) \mid A(F^p) \subset F^p \text{ for all } p\}.$$

For any $A \in \mathrm{End}(V)$, we denote by $[A]_F$ its image in $T_{\check{\mathscr{D}},F}$.

A tangent vector $[A]_F$ in $T_{\check{\mathscr{D}},F}$ is called *horizontal* if $A(F^p) \subset F^{p-1}$ for all p. The subbundle of $T_{\check{\mathscr{D}}}$ consisting of horizontal vectors is denoted by $T^{-1,1}_{\check{\mathscr{D}}}$, and one can show that it is a holomorphic subbundle of $T_{\check{\mathscr{D}}}$. A holomorphic map $f : \Omega \to \check{\mathscr{D}}$ from a complex manifold Ω is called *horizontal* if $df : T_\Omega \to f^* T_{\check{\mathscr{D}}}$ factors through $f^* T^{-1,1}_{\check{\mathscr{D}}}$.

A complex (unpolarized) variation of Hodge structure $(V = \oplus_{p+q=m}, \nabla)$ over a complex manifold Ω induces a horizontal holomorphic map $\Phi : \tilde{\Omega} \to \check{\mathscr{D}}$ by the Griffiths' transversality, where $\tilde{\Omega}$ is the universal cover of Ω. Here, we choose the reference space of $\check{\mathscr{D}}$ to be the space of multivalued flat sections V^∇. Φ is called the period mapping associated to $(V, \nabla, F^\bullet)$. When this complex variation of Hodge structure is moreover polarized, Φ factors through $\mathscr{D}$.

2. Nilpotent orbit theorem

2.1. *Two results of L^2-estimate*

Set $X = \Delta^n$ and $D = (z_1 \cdots z_\ell = 0)$. We equip the complement $U := X \backslash D$ with the Poincaré metric ω_P defined in Equation (1.3). Write

$$X(r) := \{z \in X \mid |z_i| < r \text{ for } i = 1, \ldots, \ell\} \quad \text{and} \quad U(r) = X(r) \cap U. \tag{2.1}$$

Lemma 2.1. *Let (F, h_F) be a Hermitian vector bundle on U such that $|R_{h_F}(F)| \leq C\omega_P$ for some constant $C > 0$. Then for any section $\eta \in \mathscr{C}^\infty(U, \Lambda^{0,1} T^*_U \otimes F)$ such that $|\eta|_{h_F,\omega_P} \lesssim \prod_{j=1}^{\ell} |z_j|^\varepsilon$ for some $\varepsilon > 0$ and $\bar{\partial}\eta = 0$, there exists $\sigma \in \mathscr{C}^\infty(U, F)$ such that $\bar{\partial}\sigma = \eta$ and*

$$\int_U |\sigma|^2_{h_F} \prod_{j=1}^{\ell} (-\log |z_j|^2)^N dvol_{\omega_P} < \infty$$

for some $N \gg 1$.

Proof. For the line bundle K_U^{-1} endowed with the natural metric g induced by ω_P, it is acceptable. Hence, for the Hermitian vector bundle $(E, h) := (K_U^{-1} \otimes F, g \cdot h_F)$, it is also acceptable. It follows from [3, Lemma 1.10] that one can choose $N \gg 1$ such that

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$$iR_h(E) \geq_{Nak} -(N-1)\omega_P \otimes \mathrm{Id}_E,$$

where "$\geq_{Nak}$" stands for Nakano semipositive (see [2, Définition 2.2]). For the function

$$\varphi := \log\left(\prod_{j=1}^{\ell}(-\log|z_j|^2)^{-1} \cdot \prod_{k=\ell+1}^{n}\exp(|z_k|^2)\right), \tag{2.2}$$

one has $i\partial\bar{\partial}\log\varphi = \omega_P$. For any $k \in \mathbb{Z}$, we define a new metric $h(k) = h \cdot e^{-k\varphi}$ for E. Therefore,

$$iR_{h(N)}(E) = iR_h(E) + N\omega_P \otimes \mathrm{Id}_E \geq_{Nak} \omega_P \otimes \mathrm{Id}_E.$$

Note that $\mathscr{C}^\infty(U, \Lambda^{n,1}T_U^* \otimes E) = \mathscr{C}^\infty(U, \Lambda^{0,1}T_U^* \otimes F)$ with $|\eta|_{h,\omega_P} = |\eta|_{h_F,\omega_P}$. Since $|\eta|_{h_F,\omega_P} \lesssim \prod_{j=1}^{\ell}|z_j|^{\varepsilon}$, $|\eta|_{h(N),\omega_P} \leq C'$ for some $C' > 0$. Hence, $\|\eta\|_{h(N),\omega_P} < \infty$. Thanks to the Demailly–Hörmander L^2-estimate [2, Théorème 4.1 and Remarque 4.2], there exists $\sigma \in \mathscr{C}^\infty(U, K_U \otimes E) = \mathscr{C}^\infty(U, F)$ such that $\bar{\partial}\sigma = \eta$ and $\|\sigma\|_{h(N)} < \infty$. Here, we note that the smoothness of σ follows from the elliptic regularity of the Laplacian. The lemma is proved. □

Lemma 2.2. *Let (E, h) be a Hermitian vector bundle on U such that $|R_h(E)| \leq C\omega_P$ for some constant $C > 0$. Assume that $\sigma \in H^0(U, E)$ such that $\|\sigma\|_{h(N)} < \infty$ for some integer $N \geq 1$, where $h(N) := h \cdot e^{-N\varphi}$ with φ defined in Equation (2.2), then over $U(\frac{1}{2})$, $|\sigma|_h \lesssim \prod_{j=1}^{\ell}|z_j|^{-\varepsilon}$ for any $\varepsilon > 0$.*

Proof. Since $|R_h(E)| \leq C\omega_P$ for some constant $C > 0$, it follows from [3, Lemma 1.10] that $(E, h(-N'))$ is Griffiths' seminegative for some $N' \gg 1$, where $h(-N') := h \cdot e^{N'\varphi}$ with φ defined in Equation (2.2). One can show that $\log|\sigma|^2_{h(-N')}$ is a plurisubharmonic function. For any $z \in U^*(\frac{1}{2})$, one has

$$\begin{aligned}
\log|\sigma(z)|^2_{h(-N')} &\leqslant \frac{4^n}{\pi^n\prod_{i=1}^{\ell}|z_i|^2}\int_{\Omega_z}\log|\sigma(w)|^2_{h(-N')}d\mathrm{vol}_g \\
&\leqslant \log\left(\frac{4^n}{\pi^n\prod_{i=1}^{\ell}|z_i|^2}\cdot\int_{\Omega_z}|\sigma(w)|^2_{h(-N')}d\mathrm{vol}_g\right) \\
&\leqslant \log\left(C\int_{\Omega_z}\frac{1}{\prod_{i=1}^{\ell}|w_i|^2}|\sigma(w)|^2_{h(-N')}d\mathrm{vol}_g\right) \\
&\leqslant C_1 + \log\int_{\Omega_z}|\sigma(w)|^2_{h(-N')}\cdot\left|\prod_{i=1}^{\ell}(\log|w_i|^2)^2\right|d\mathrm{vol}_{\omega_P} \\
&\leqslant C_2 + \log\int_{\Omega_z}|\sigma(w)|^2_{h(N)}d\mathrm{vol}_{\omega_P} \\
&\leqslant C_2 + \log\|\sigma\|^2_{h(N)},
\end{aligned}$$

where $\Omega_z := \{w \in U^* \mid |w_i - z_i| \leq \frac{|z_i|}{2} \text{ for } i \leq \ell; |w_i - z_i| \leq \frac{1}{2} \text{ for } i > \ell\}$ and g is the Euclidean metric. C_1, C_2 are two positive constants which do not depend on $z \in U^*(\frac{1}{2})$. The first inequality is due to the mean value inequality, and the second one follows from the Jensen inequality. It follows that

$$\begin{aligned}
|\sigma(z)|_h &= |\sigma(z)|_{h(-N')}\cdot\left(\prod_{j=1}^{\ell}(-\log|z_j|^2)^{\frac{N'}{2}}\cdot\prod_{k=\ell+1}^{n}\exp(|z_k|^2)^{-\frac{N'}{2}}\right) \\
&\leq \exp\left(\frac{C_2}{2}\right)\cdot\|\sigma\|_{h(N)}\cdot\left(\prod_{j=1}^{\ell}(-\log|z_j|^2)^{\frac{N'}{2}}\cdot\prod_{k=\ell+1}^{n}\exp(|z_k|^2)^{-\frac{N'}{2}}\right) \lesssim \prod_{i=1}^{\ell}|z_i|^{-\varepsilon}
\end{aligned}$$

for any $\varepsilon > 0$. □

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2.2. Proof of Theorem A

We first prove that the Deligne extension of the flat bundle underlying a complex variation of Hodge structure coincides with the prolongation defined in §1.6.

Proposition 2.3. *Let X be a complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor. For a complex variation of Hodge structure $(V, \nabla, F^\bullet, h)$ defined on $U := X \backslash D$, one has $V_{\boldsymbol{\alpha}}^{\mathrm{Del}} = \mathcal{P}_{\boldsymbol{\alpha}} V$ for any multi-index $\boldsymbol{\alpha} \in \mathbb{R}^{\ell}$, where $\mathcal{P}_{\boldsymbol{\alpha}} V$ is the prolongation of V defined in §1.6.*

Proof. Step 1: We prove that $V_{\boldsymbol{\alpha}}^{\mathrm{Del}} \subset \mathcal{P}_{\boldsymbol{\alpha}} V$. We will use the notation in §1.2. Since this is a local problem, we can assume that $X = \Delta^n$ and $D = (t_1 \cdots t_p = 0)$. By the construction of $V_{\boldsymbol{\alpha}}^{\mathrm{Del}}$, one can take a basis $v_1, \ldots, v_r$ of V^∇ with each $v_i \in \mathbb{E}_{\boldsymbol{\lambda}(v_i)}$ for some $\boldsymbol{\lambda}(v_i) \in Sp$ such that $\{\tilde{v}_1, \ldots, \tilde{v}_r\}$ defined in Equation (1.2) forms a basis of $V_{\boldsymbol{\alpha}}^{\mathrm{Del}}$. It thus suffices to estimate the norm

$$\tilde{v}(t) := \exp\left(-\sum_{i=1}^{p} (\beta_i I + N_i) \cdot \log t_i\right) v(t) = \prod_{i=1}^{p} t_i^{-\beta_i} \exp\left(-\sum_{i=1}^{p} N_i \cdot \log t_i\right) v(t)$$

for any $\boldsymbol{\lambda}$ and $v \in \mathbb{E}_{\boldsymbol{\lambda}}$.

By the weaker norm estimate in [7, Lemma 9.31] for general harmonic bundles, there exists a frame $v_1, \ldots, v_r$ of V^∇ with each $v_i \in \mathbb{E}_{\boldsymbol{\lambda}(v_i)}$ and $\{a_{ij}\}_{i=1,\ldots,r; j=1,\ldots,p} \subset \mathbb{R}$ such that if we put $v_i' := v_i \cdot \prod_{j=1}^{p} |t_j|^{-a_{ij}}$, then for the multivalued smooth sections $\boldsymbol{v}' = (v_1', \ldots, v_r')$, over a given sector of U one has the norm estimate

$$\left(\prod_{i=1}^{p} |\log|t_i||\right)^{-M} \lesssim H(h, \boldsymbol{v}') \lesssim \left(\prod_{i=1}^{p} |\log|t_i||\right)^{M} \tag{2.3}$$

for some $M > 0$. Here, h is the Hodge metric and $H(h, \boldsymbol{v}')$ is the $H(r)$-valued function defined in §1.4.

Fix some $\{t_1, \ldots, t_p\} \subset \Delta^*$. For each $\ell = 1, \ldots, p$, over any sector of Δ^*, we have

$$|\log|t||)^{-M} \lesssim |v_i|^2(t_1, \ldots, t_{\ell-1}, t, t_{\ell+1}, \ldots, t_n) \lesssim |\log|t||^M$$

by the Hodge norm estimate in one-dimensional case in [9, 11]. Together with Equation (2.3), it implies that $a_{ij} = 0$ for all $i = 1, \ldots, r; j = 1, \ldots, \ell$. Therefore, we conclude that over a given sector of U one has the norm estimate

$$\left(\prod_{i=1}^{p} |\log|t_i||\right)^{-M} \lesssim H(h, \boldsymbol{v}) \lesssim \left(\prod_{i=1}^{p} |\log|t_i||\right)^{M}. \tag{2.4}$$

Then for any multivalued flat section $v \in V^\nabla$, over any given sector of U one has

$$\left(\prod_{i=1}^{p} |\log|t_i||\right)^{-M} \lesssim |v(t)|_h \lesssim \left(\prod_{i=1}^{p} |\log|t_i||\right)^{M}$$

for some $M > 0$. Since all N_i are nilpotent and pairwise commute,

$$\exp\left(-\sum_{i=1}^{p} N_i \cdot \log t_i\right) v(t) = \sum_{i=1}^{p} \sum_{k=0}^{N} \frac{1}{k!} (\log t_i)^k (N_i^k v)(t)$$

for some integer $N > 0$. Note that if $v \in \mathbb{E}_{\boldsymbol{\lambda}}$, we also have $N_i^k v \in \mathbb{E}_{\boldsymbol{\lambda}}$ for any $k \geq 0$. Since one can cover $X \backslash D$ by finitely many sectors, this proves the norm estimate

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$$|\exp\left(-\sum_{i=1}^{p} N_i \cdot \log t_i\right) v(t)|_h \lesssim \left(\prod_{i=1}^{p} |\log |t_i||\right)^{M'}$$

for some $M' > 0$. Hence,

$$|\tilde{v}(t)|_h \lesssim \prod_{i=1}^{p} |t_i|^{-\beta_i-\varepsilon} \lesssim \prod_{i=1}^{p} |t_i|^{-\alpha_i-\varepsilon}$$

for any $\varepsilon > 0$. This proves the inclusion $V_\alpha^{\mathrm{Del}} \subset \mathcal{P}_\alpha V$ by the definition of $\mathcal{P}_\alpha V$ in §1.6.

Step 2: We prove that $\mathcal{P}_\alpha V \subset V_\alpha^{\mathrm{Del}}$. First, we note that the decomposition $V^\nabla = \oplus_{\lambda \in Sp} \mathbb{E}_\lambda$ induces a decomposition of the flat bundle (V, ∇) into

$$(V, \nabla) = \oplus_{\lambda \in Sp} (V(\boldsymbol{\lambda}), \nabla|_{V(\boldsymbol{\lambda})}), \tag{2.5}$$

where $(V(\boldsymbol{\lambda}), \nabla|_{V(\boldsymbol{\lambda})})$ is the flat subbundle induced by $\mathbb{E}_{\boldsymbol{\lambda}}$. We fix a basis $(v_1, \ldots, v_r) \in V^\nabla$ such that $v_i \in \mathbb{E}_{\boldsymbol{\lambda}(v_i)}$ for some $\boldsymbol{\lambda}(v_i) \in Sp$. This means that such basis is compatible with the above decomposition (2.5); namely, v_j is a mutivalued flat section of $(V(\boldsymbol{\lambda}(v_j)), \nabla|_{V(\boldsymbol{\lambda}(v_j))})$. Consider the dual bundle V^* of V, and it can endowed with the natural connection ∇ defined by

$$(\nabla\mu)v = d(\mu(v)) - \mu(\nabla(v))$$

for μ and v sections in V^* and V, respectively. (V^*, ∇) is thus also a flat bundle. Moreover, $(V^*)^\nabla$ is the dual space of (V^∇). Consider the dual basis $(v_1^*, \ldots, v_r^*)$ of $(v_1, \ldots, v_r)$. Since

$$(\nabla v_i^*)v_j = d(v_i^*(v_j)) - v_i^*(\nabla v_j) = 0,$$

one has $v_i^* \in (V^*)^\nabla$. Recall that T_j is the monodromy transformation of (V, ∇) with respect to γ_j defined by

$$v(t_1, \ldots, e^{2\pi i} t_j, \ldots, t_{p+q}) = (T_j v)(t_1, \ldots, t_{p+q})$$

for any $v \in V^\nabla$. Let us denote by $\tilde{T}_j$ the monodromy transformation of (V, ∇) with respect to γ_j defined by

$$\mu(t_1, \ldots, e^{2\pi i} t_j, \ldots, t_{p+q}) = (\tilde{T}_j \mu)(t_1, \ldots, t_{p+q})$$

for any $\mu \in (V^*)^\nabla$. Then for any $v \in V^\nabla$ and any $\mu \in (V^*)^\nabla$ one has

$$\begin{aligned}\mu(t)(v(t)) &= \mu(t_1, \ldots, e^{2\pi i} t_j, \ldots, t_{p+q})(v(t_1, \ldots, e^{2\pi i} t_j, \ldots, t_{p+q})) \\ &= (\tilde{T}_j \mu(t))(T_j v(t)) = (\tilde{T}_j \mu)(T_j v) = (T_j^* \tilde{T}_j \mu)(v),\end{aligned}$$

where $T_j^* : (V^*)^\nabla \to (V^*)^\nabla$ is the adjoint of T_j. Hence

$$\tilde{T}_j = (T_j^*)^{-1}. \tag{2.6}$$

It follows that $Sp(\tilde{T}_j) = \{\lambda^{-1}\}_{\lambda \in Sp(T_i)}$. Set $\mathbb{E}(\tilde{T}_j, \lambda_j) \subset (V^*)^\nabla$ to be the corresponding eigenspace of $\lambda_j \in Sp(\tilde{T}_j)$. We know that all $\lambda_j \in Sp(\tilde{T}_j)$ have norm 1 since (V^*, ∇) admits a complex variation of Hodge structure. For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p) \in Sp$, we define

$$\tilde{\mathbb{E}}_{\boldsymbol{\lambda}} := \cap_{j=1}^{p} \mathbb{E}(\tilde{T}_j, \lambda_j^{-1}) \subset (V^*)^\nabla.$$

Since $T_j's$ are pairwise commute, one has

$$(V^*)^{\nabla} = \oplus_{\lambda\in Sp}\tilde{\mathbb{E}}_\lambda,$$

and $\tilde{E}_\lambda$ is an invariant subspace of $\tilde{T}_j$ for any $\lambda \in Sp$ and any j.

By Lemma 2.4 below, one can show that for any $\mu \in \tilde{\mathbb{E}}_{\lambda'}$ and $v \in \mathbb{E}_\lambda$, $\mu(v) = 0$ if $\lambda \neq \lambda'$, which implies that $v_j^* \in \tilde{\mathbb{E}}_{\lambda(v_j)}$.

For $\lambda \in Sp$, there exists a unique $\beta_i \in (\alpha_i - 1, \alpha_i]$ such that $\exp(2\pi i\beta_i) = \lambda_i$. Denote $N_i := \frac{\log(\lambda_i^{-1}T_i|_{\mathbb{E}_\lambda})}{2\pi i}$. Recall that for any $v \in \mathbb{E}_\lambda$, we define

$$\tilde{v}(t) := \exp\left(-\sum_{i=1}^{p}(\beta_i I + N_i)\cdot \log t_i\right)v(t) = \prod_{i=1}^{p} t_i^{-\beta_i}\exp\left(-\sum_{i=1}^{p} N_i\cdot\log t_i\right)v(t).$$

Likewise, since $\lambda_i\tilde{T}_i|_{\tilde{\mathbb{E}}_\lambda}$ is unipotent, its logarithm can be defined as

$$\log(\lambda_i\tilde{T}_i|_{\tilde{\mathbb{E}}_\lambda}) := \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(\lambda_i\tilde{T}_i|_{\tilde{\mathbb{E}}_\lambda} - I)^k}{k}.$$

Write $\tilde{N}_i := \frac{\log(\lambda_i\tilde{T}_i|_{\tilde{\mathbb{E}}_\lambda})}{2\pi i}$. Then for any $\mu \in \tilde{\mathbb{E}}_\lambda$, we define

$$\tilde{\mu}(t) := \exp\left(-\sum_{i=1}^{p}(-\beta_i I + \tilde{N}_i)\cdot\log t_i\right)\mu(t) = \prod_{i=1}^{p} t_i^{\beta_i}\exp\left(-\sum_{i=1}^{p}\tilde{N}_i\cdot\log t_i\right)\mu(t). \tag{2.7}$$

Since $\tilde{T}_i = (T_i^*)^{-1}$, one has $\tilde{N}_j = -N_j^*$. Therefore,

$$\begin{aligned}\tilde{\mu}(t)(\tilde{v}(t)) &= \exp\left(-\sum_{i=1}^{p}\tilde{N}_i\cdot\log t_i\right)\mu(t)\left(\exp\left(-\sum_{i=1}^{p}N_i\cdot\log t_i\right)v(t)\right)\\ &= \exp\left(\sum_{i=1}^{p}N_i^*\cdot\log t_i\right)\mu(t)\left(\exp\left(-\sum_{i=1}^{p}N_i\cdot\log t_i\right)v(t)\right)\\ &= \mu(v) = \text{constant}.\end{aligned}$$

This implies that $\tilde{v}_i^*(t)(\tilde{v}_j(t)) = v_i^*(v_j) \equiv \delta_{ij}$ if $\lambda = \lambda'$, where $\tilde{v}_i^*$ is defined in Equation (2.7) in terms of $v_i^* \in \tilde{\mathbb{E}}_{\lambda(v_i)}$.

If $\mu \in \tilde{\mathbb{E}}_\lambda$ and $v \in \mathbb{E}_{\lambda'}$ with $\lambda \neq \lambda'$, the above construction shows that $\tilde{\mu}$ and $\tilde{v}$ are holomorphic sections of $V^*(\lambda)$ and $V(\lambda')$. Here, $V(\lambda')$ is the invariant flat subbundle of (V,∇) defined in Equation (2.5), and $V^*(\lambda)$ is defined to be the invariant flat subbundle of (V^*,∇) generated by $\tilde{E}_\lambda$. Hence, $\tilde{v}_i^*$ and $\tilde{v}_j$ are holomorphic sections of $V^*(\lambda(v_i))$ and $V(\lambda(v_j))$, respectively. This shows that $\tilde{v}_i^*(t)(\tilde{v}_j(t)) \equiv 0$ for $\lambda \neq \lambda'$ by Lemma 2.4.

In conclusion, we prove that $\tilde{v}_1^*, \ldots, \tilde{v}_r^*$ is the dual frame of $\tilde{v}_1, \ldots, \tilde{v}_r$.

Recall that $v_j \in \mathbb{E}_{\lambda(v_j)}$ for some $\lambda(v_j) \in Sp$. There exists a unique $\beta(v_j)_i \in (\alpha_i - 1, \alpha_i]$ such that $\exp(2\pi i\beta(v_j)_i) = \lambda(v_j)_i$. Define a smooth section $\tilde{v}_j' = \tilde{v}_j \cdot \prod_{i=1}^{p}|t_i|^{\beta(v_j)_i}$. By the norm estimate in the first step, for all $\tilde{v}_j'$ one has the norm estimate

$$|\tilde{v}_j'|_h \lesssim \left(\prod_{i=1}^{p}|\log|t_i||\right)^M$$

for some $M > 0$. It follows that

$$H(h; \tilde{v}'_1, \dots, \tilde{v}'_r) \lesssim \left(\prod_{i=1}^{p} |\log |t_i|| \right)^M .$$

Here, $H(h; \tilde{v}'_1, \dots, \tilde{v}'_r)$ is a $r \times r$-matrix function whose (i, j)-component is $h(\tilde{v}'_i, \tilde{v}'_j)$. On the other hand, we put $\mu'_i = \tilde{v}^*_i \cdot \prod_{j=1}^{p} |t_j|^{-\beta(v_i)_j}$. Since complex polarized variation of Hodge structure is functorial by taking dual, (V^*, ∇) admits a complex polarized variation of Hodge structure whose Hodge metric is the dual metric h^* of the Hodge metric h for $(V, \nabla, F^\bullet, Q)$. In the same manner, we obtain

$$|\mu'_i|_{h^*} \lesssim \left(\prod_{i=1}^{p} |\log |t_i|| \right)^{M'}$$

for every μ'_i and some $M' > 0$. This implies that

$$H(h^*; \mu'_1, \dots, \mu'_r) \lesssim \left(\prod_{i=1}^{p} |\log |t_i|| \right)^{M'} .$$

By our construction, $\mu'_1, \dots, \mu'_r$ is the dual of the smooth frame $\tilde{v}'_1, \dots, \tilde{v}'_r$. It follows that

$$\left(\prod_{i=1}^{p} |\log |t_i|| \right)^{-M'} \lesssim H(h^*; \mu'_1, \dots, \mu'_r)^{-1} = H(h; \tilde{v}'_1, \dots, \tilde{v}'_r).$$

Hence,

$$\left(\prod_{i=1}^{p} |\log |t_i|| \right)^{-M'} \lesssim H(h; \tilde{v}'_1, \dots, \tilde{v}'_r) \lesssim \left(\prod_{i=1}^{p} |\log |t_i|| \right)^{M} . \tag{2.8}$$

Now, we are ready to prove the inclusion $\mathcal{P}_\alpha V \subset V_\alpha^{\mathrm{Del}}$. For any $s \in \mathcal{P}_\alpha V(U)$, it can be written as $s = \sum_{i=1}^{r} f_i \tilde{v}_i$, where f_i is a holomorphic function on U. By Equation (2.8) one has

$$\sum_{i=1}^{r} |f_i|^2 \cdot \prod_{j=1}^{p} |t_j|^{-2\beta(v_i)_j} \cdot \left(\prod_{k=1}^{p} |\log |t_k|| \right)^{-2M'} \lesssim |s|_h^2 \lesssim \prod_{i=1}^{p} |t_j|^{-2\alpha_j - \varepsilon}$$

for any $\varepsilon > 0$. Since $\beta(v_i)_j \in (\alpha_j - 1, \alpha_j]$, one can choose $\delta > 0$ such that $\beta(v_i)_j - \alpha + 1 > \delta$ for all v_i and every $j = 1, \dots, p$. The above inequality implies that for every f_i,

$$|f_i| \lesssim \prod_{i=1}^{p} |t_j|^{-1+\delta}.$$

Hence, all f_i extend to holomorphic functions over X. This proves that $s \in V_\alpha^{\mathrm{Del}}(X)$ since $\tilde{v}_1, \dots, \tilde{v}_r$ is a holomorphic basis of V_α^{Del} by the definition of Deligne extension in §1.2. The inclusion $\mathcal{P}_\alpha V \subset V_\alpha^{\mathrm{Del}}$ is proved. We complete the proof of the proposition. □

We leave the proof of the following lemma of linear algebra to the reader.

Lemma 2.4. *Let $T : V \to V$ be an isomorphism of a finite-dimensional $\mathbb{C}$-vector space V. Decompose $V = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_k}$ into eigenspaces of T, where λ_i is a eigenvalue of T and V_{λ_i} is the corresponding eigenspace. Denote by V^* the dual vector space. Then for the isomorphism $(T^*)^{-1} : V^* \to V^*$, its* *eigenvalues are $\lambda_1^{-1}, \ldots, \lambda_k^{-1}$ and its eigenspace decomposition is $V^* = V^*_{\lambda_1^{-1}} \oplus \ldots \oplus V^*_{\lambda_k^{-1}}$, where $V^*_{\lambda_j^{-1}}$ is the corresponding eigenspace of λ_j^{-1}. Moreover, one has $\mu(v) = 0$ if $\mu \in V^*_{\lambda_i^{-1}}$ and $v \in V_{\lambda_j}$ with $i \neq j$.*

Theorem 2.5. *Let X be a complex manifold, and let $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor on X. Let $(V, \nabla, F^\bullet, Q)$ be a complex polarized variation of Hodge structure of weight m on $X \backslash D$. Let $\mathcal{P}_* F^p$ and $\mathcal{P}_* E_{p,m-p}$ be the induced filtered bundle of Hermitian bundles (F^p, h_p) and $(E_{p,m-p}, h_{p,m-p})$ defined in §1.6. Then for every p, $\mathcal{P}_* F^p$ and $\mathcal{P}_* E_{p,m-p}$ are parabolic bundles and for each multi-index $\alpha \in \mathbb{R}^\ell$, there is a natural exact sequence*

$$0 \to \mathcal{P}_\alpha F^{p+1} \to \mathcal{P}_\alpha F^p \to \mathcal{P}_\alpha E_{p,m-p} \to 0. \tag{2.9}$$

Proof. By Lemma 1.4, (F^p, h_p) and $(E_{p,m-p}, h_{p,m-p})$ are acceptable bundles for every p. It follows from Theorem 1.7 that $\mathcal{P}_* F^p$ and $\mathcal{P}_* E_{p,m-p}$ defined in §1.6 are parabolic ones.

We first show that we can define

$$0 \to \mathcal{P}_\alpha F^{p+1} \to \mathcal{P}_\alpha F^p \xrightarrow{q} \mathcal{P}_\alpha E_{p,m-p}, \tag{2.10}$$

which is exact. Note that we have the following exact sequence

$$0 \to F^{p+1} \to F^p \xrightarrow{q} E_{p,m-p} \to 0 \tag{2.11}$$

on $X \backslash D$ by the definition of $\mathbb{C}$-VHS. Pick any $x \in D$ and any admissible coordinate $(\Omega; z_1, \ldots, z_n)$ centering at x such that $D \cap \Omega = (z_1 \cdots z_k = 0)$. By the prolongation via norm growth defined in §1.6, any section $s \in \mathcal{P}_\alpha F^{p+1}(\Omega)$ satisfies that $s \in F^{p+1}(\Omega \backslash D)$ and $|s|_{h^{p+1}} \lesssim \prod_{i=1}^{k} |z_i|^{-\alpha_i - \varepsilon}$ for any $\varepsilon > 0$. Since h_{p+1} is the induced metric of h_p on F^{p+1}, it follows that

$$|s|_{h_{p+1}} = |s|_{h_p} \lesssim \prod_{i=1}^{k} |z_i|^{-\alpha_i - \varepsilon} \tag{2.12}$$

for any $\varepsilon > 0$. Hence, the inclusion $F^{p+1} \subset F^p$ also results in the inclusion $\mathcal{P}_\alpha F^{p+1} \subset \mathcal{P}_\alpha F^p$. We proved that Equation (2.10) is exact in the left.

Note that the metric $h_{p,m-p}$ on $E_{p,m-p}$ is the quotient metric of h_p. It follows that for any section $s \in \mathcal{P}_\alpha F^p(\Omega)$, we have

$$|q(s)|_{h_{p,m-p}} \leq |s|_{h_p} \lesssim \prod_{i=1}^{k} |z_i|^{-\alpha_i - \varepsilon}$$

for any $\varepsilon > 0$. Hence, the quotient $q : F^p \to E_{p,m-p}$ induces the morphism $\mathcal{P}_\alpha F^p \to \mathcal{P}_\alpha E_{p,m-p}$ and thus Equation (2.10) can be defined. Next, we will show that Equation (2.10) is exact in the middle.

Take any section $s \in \mathcal{P}_\alpha F^p(\Omega)$ such that $q(s) = 0$. Thanks to the exactness in Equation (2.11), we have $s \in F^{p+1}(\Omega \backslash D)$. By Equation (2.12), we conclude that $s \in \mathcal{P}_\alpha F^{p+1}(\Omega)$. This implies the exactness of Equation (2.10).

In what follows, we will prove that Equation (2.10) is exact on the right. It suffices to prove that for any point $x \in D$ and any section $s \in \mathcal{P}_\alpha E_{p,m-p}(\Omega)$, where Ω is a neighborhood of x, there is a section $\tilde{s} \in \mathcal{P}_\alpha F^p(\Omega')$ for some smaller neighborhood Ω' of x such that $q(\tilde{s}) = s|_{\Omega'}$. We shall construct such $\tilde{s}$ by utilizing the previous results on L^2-estimate in Lemmas 2.1 and 2.2.

Since this is a local problem, we can assume that $X = \Delta^n$, $D = (z_1 \cdots z_\ell = 0)$ and x is the origin. We equip the complement $U := X \backslash D$ with the Poincaré metric ω_P. Let $X(r)$ and $U(r)$ be defined as

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in Equation (2.1). By the semicontinuity of the parabolic bundle in Definition 1.6.(iv), we can choose $\boldsymbol{\beta} \in \mathbb{R}^\ell$ such that $\beta_i > \alpha_i$ and

$$\mathcal{P}_{\boldsymbol{\beta}} F^p = \mathcal{P}_{\boldsymbol{\alpha}} F^p. \tag{2.13}$$

Pick any $s \in \mathcal{P}_{\boldsymbol{\alpha}} E_{p,m-p}(X)$. Then $s \in H^0(U, E_{p,m-p})$ with $|s|_{h_{p,m-p}} \lesssim \prod_{i=1}^{\ell} |z_i|^{-\alpha_i-\varepsilon}$ for any $\varepsilon > 0$. We will construct a section $\tilde{s} \in H^0(U(r), F^p)$ for some $0 < r < 1$ such that $q(\tilde{s}) = s|_{U(r)}$ and $|\tilde{s}|_{h_p} \lesssim \prod_{i=1}^{\ell} |z_i|^{-\beta_i-\varepsilon}$ for any $\varepsilon > 0$. Note that there is a canonical smooth isomorphism (and isometry)

$$\Phi : (F^p, h_p) \to (F^{p+1}, h_{p+1}) \oplus (E_{p,m-p}, h_{p,m-p})$$

such that the holomorphic structure of F^p via Φ is defined by

$$\begin{bmatrix} \bar{\partial}_{F^{p+1}} & \theta^\dagger_{p+1,m-p-1} \\ 0 & \bar{\partial}_{E_{p,m-p}}, \end{bmatrix},$$

where $\theta^\dagger_{p+1,m-p-1}$ is the adjoint of $\theta_{p+1,m-p-1}$ with respect to $h_{p+1,m-p-1}$. If $q(\tilde{s}) = s$, then $\Phi(\tilde{s}) = [\sigma, s]$ for some $\sigma \in \mathscr{C}^\infty(U, F^{p+1})$ such that

$$\begin{bmatrix} \bar{\partial}_{F^{p+1}} & \theta^\dagger_{p+1,m-p-1} \\ 0 & \bar{\partial}_{E_{p,m-p}} \end{bmatrix} \begin{bmatrix} \sigma \\ s \end{bmatrix} = 0.$$

Hence, $\bar{\partial}_{F^{p+1}} \sigma = -\theta^\dagger_{p+1,m-p-1} s$. We will solve this $\bar{\partial}$-equation with proper norm estimate.

By Theorem 1.2, after we replace U by $U(r)$ for some $0 < r < 1$, we have $|\theta_{p+1,m-p-1}|_{h,\omega_P} \leq C$ over U. This implies that $|\theta^\dagger_{p+1,m-p-1}|_{h,\omega_P} \leq C$ over U. Hence,

$$|\theta^\dagger_{p+1,m-p-1} s|_{h_{p+1},\omega_P} \leq |\theta^\dagger_{p+1,m-p-1}|_{h,\omega_P} \cdot |s|_{h_{p,m-p}} \lesssim \prod_{i=1}^{\ell} |z_i|^{-\alpha_i-\varepsilon}$$

for any $\varepsilon > 0$. We now introduce a new metric for F^p defined by

$$h_p(\boldsymbol{\beta}) := h_p \cdot \prod_{i=1}^{\ell} |z_i|^{\beta_i}.$$

Since $\beta_i > \alpha_i$ for each i, we have

$$|\theta^\dagger_{p+1,m-p-1} s|_{h_{p+1}(\boldsymbol{\beta}),\omega_P} \lesssim \prod_{j=1}^{\ell} |z_j|^{\delta}$$

for some $\delta > 0$. Note that $\theta^\dagger_{p+1,m-p-1} s \in A^{0,1}(E_{p+1,m-p-1})$. We have

$$\begin{aligned} \bar{\partial}_{F^{p+1}} (\theta^\dagger_{p+1,m-p-1} s) &= (\bar{\partial}_{E_{p+1,m-p-1}} + \theta^\dagger_{p,m-p})(\theta^\dagger_{p+1,m-p-1} s) \\ &= \bar{\partial}_{E_{p+1,m-p-1}} (\theta^\dagger_{p+1,m-p-1} s) \\ &= (D_h^{0,1} \theta^\dagger_{p+1,m-p-1}) s - \theta^\dagger_{p+1,m-p-1} (\bar{\partial}_{E_{p,m-p}} s) = 0, \end{aligned}$$

where the second equality follows from $\theta^\dagger_{p+1,m-p-1} \wedge \theta^\dagger_{m-p} = 0$, and the last one follows from $D_h^{0,1}(\theta^\dagger) = 0$. Here, D_h is the Chern connection for the Hodge bundle $(E = \oplus_{p+q=m} E_{p,q}, h)$. Since $(F^{p+1}, h_{p+1}(\boldsymbol{\beta}))$ is also acceptable by Lemma 1.4, we can invoke Lemma 2.1 to find some $\sigma \in \mathscr{C}^\infty(U, F^{p+1})$ such that

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$$\bar{\partial}_{F^{p+1}}(\sigma) = -\theta^{\dagger}_{p+1,m-p-1}s$$

and

$$\int_U |\sigma|^2_{h_{p+1}(\boldsymbol{\beta},N)} d\mathrm{vol}_{\omega_P} < \infty$$

for some $N \gg 1$. Here, $h_{p+1}(\boldsymbol{\beta}, N)$ is a new metric for F^{p+1} define by

$$h_{p+1}(\boldsymbol{\beta}, N) = h_{p+1} \cdot \prod_{i=1}^{\ell} |z_i|^{\beta_i} \cdot e^{-N\varphi}$$

with φ defined in Equation (2.2). Since $|s|_{h_{p,m-p}} \lesssim \prod_{i=1}^{\ell} |z_i|^{-\alpha_i-\varepsilon}$ for any $\varepsilon > 0$, it follows that $|s|_{h_{p,m-p}(\boldsymbol{\beta},N)} < C$ for some constant $C > 0$, where we define

$$h_{p,m-p}(\boldsymbol{\beta}, N) = h_{p,m-p} \cdot \prod_{i=1}^{\ell} |z_i|^{\beta_i} \cdot e^{-N\varphi}.$$

Thus, the section $\tilde{s} := \Phi^{-1}([\sigma, s])$ is a holomorphic section of F^p such that

$$\int_U |\tilde{s}|^2_{h_p(\boldsymbol{\beta},N)} d\mathrm{vol}_{\omega_P} = \int_U |\sigma|^2_{h_{p+1}(\boldsymbol{\beta},N)} d\mathrm{vol}_{\omega_P} + \int_U |s|^2_{h_{p,m-p}(\boldsymbol{\beta},N)} d\mathrm{vol}_{\omega_P} < \infty.$$

Since $(F^p, h_p(\boldsymbol{\beta}))$ is also acceptable by Lemma 1.4, thanks to Lemma 2.2, over some $U(r)$ for $0 < r < 1$ we have $|\tilde{s}|_{h_p(\boldsymbol{\beta})} \lesssim \prod_{j=1}^{\ell} |z_j|^{-\varepsilon}$ for any $\varepsilon > 0$. Therefore, $|\tilde{s}|_{h_p} \lesssim \prod_{j=1}^{\ell} |z_j|^{-\beta_j-\varepsilon}$ for any $\varepsilon > 0$. It follows that $\tilde{s} \in \mathcal{P}_{\boldsymbol{\beta}} F^p(X(r))$. By Equation (2.13), we conclude that $\tilde{s} \in \mathcal{P}_{\boldsymbol{\alpha}} F^p(X(r'))$ for some $0 < r' < 1$. This implies the right exactness of Equation (2.9) as $q(\tilde{s}) = s$. The theorem is proved. □

Let us prove Theorem A.

Proof of Theorem A. Thanks to Proposition 2.3, we have $V_{\boldsymbol{\alpha}}^{\mathrm{Del}} = \mathcal{P}_{\boldsymbol{\alpha}} V$. By Lemma 1.4, (F^p, h_p) and $(E_{p,m-p}, h_{p,m-p})$ are acceptable bundles for any p. It follows from Theorem 1.7 that the induced filtered bundle $\mathcal{P}_* F^p$ and $\mathcal{P}_* E_{p,m-p}$ defined in §1.6 are parabolic ones. In particular, $\mathcal{P}_{\boldsymbol{\alpha}} F^p$ and $\mathcal{P}_{\boldsymbol{\alpha}} E_{p,m-p}$ are locally free sheaves. Denote by $j : X\backslash D \to X$ the inclusion map. Note that

$$\mathcal{P}_{\boldsymbol{\alpha}} F^p = j_*(F^p) \cap \mathcal{P}_{\boldsymbol{\alpha}} V \overset{\text{Proposition 2.3}}{=} j_* F^p \cap V_{\boldsymbol{\alpha}}^{\mathrm{Del}} =: F_{\boldsymbol{\alpha}}^p. \tag{2.14}$$

Hence, the exact sequence (2.9) in Theorem 2.5 implies the following one

$$0 \to F_{\boldsymbol{\alpha}}^{p+1} \to F_{\boldsymbol{\alpha}}^p \to \mathcal{P}_{\boldsymbol{\alpha}} E_{p,m-p} \to 0.$$

In particular, $F_{\boldsymbol{\alpha}}^p / F_{\boldsymbol{\alpha}}^{p+1}$ is isomorphic to $\mathcal{P}_{\boldsymbol{\alpha}} E_{p,m-p}$, which is thus locally free. The theorem is proved. □

2.3. On the nilpotent orbit theorem

In this subsection, we apply Theorem A to prove Theorem B following closely Schmid's original method [10, p. 288-289]. We will use the notations and conventions in §1.7.

Let $(V, \nabla, F^\bullet, Q)$ be a complex polarized variation of Hodge structure on $(\Delta^*)^p \times \Delta^q$. Denote by $\Phi : \mathbb{H}^p \times \Delta^q \to \mathscr{D}$ its period mapping, where we set

$$\begin{aligned} \mathbb{H}^p \times \Delta^q &\to \Delta^n \\ (z, w) &\mapsto (e^{z_1}, \ldots, e^{z_p}, w) \end{aligned}$$

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to be the uniformizing map. Let T_j be the monodromy transformation defined in §1.2. For some fixed $\boldsymbol{\alpha} \in \mathbb{R}^p$, there exist $S_i, N_i \in \mathrm{End}(V^\nabla)$ such that

- $T_i = \exp(2\pi i(S_i + N_i))$;
- $[S_i, S_j] = 0$, $[S_i, N_j] = 0$, and $[N_i, N_j] = 0$;
- S_i is semisimple whose eigenvalues lying in $(\alpha_i - 1, \alpha_i]$ and N_i is nilpotent.

Let us define

$$\tilde{\Psi}(z, w) := \exp\left(\sum_{i=1}^{p}(S_i + N_i)z_i\right)\Phi(z, w),$$

which satisfies $\tilde{\Psi}(z_1, \ldots, z_i + 2\pi i, \ldots, z_p, w) = \tilde{\Psi}(z, w)$ for $i = 1, \ldots, p$. It thus descends to a single-valued map $\Psi : (\Delta^*)^p \times \Delta^q \to \check{\mathscr{D}}$ such that $\Psi(e^{z_1}, \ldots, e^{z_p}, w) = \tilde{\Psi}(z, w)$.

Lemma 2.6. *The twisted holomorphic map Ψ extends holomorphically to Δ^{p+q}.*

Proof. We use the notations in §1.2. We fix a basis $v_1, \ldots, v_r$ of V^∇ such that each v_i belongs to some $\mathbb{E}_\lambda$. Then the sections $\tilde{v}_1, \ldots, \tilde{v}_r$ defined in Equation (1.2) induces a trivialization

$$V^\nabla \otimes_{\mathbb{C}} \Delta^{p+q} \simeq V^{\mathrm{Del}}_{\boldsymbol{\alpha}},$$

where $V^{\mathrm{Del}}_{\boldsymbol{\alpha}}$ is the Deligne extension. Under such trivialization, the Hodge filtration $F^\bullet_{\boldsymbol{\alpha}}(t_1, \ldots, t_{p+1})$ becomes $\Psi(t_1, \ldots, t_{p+1})$. Thanks to Theorem A, the Hodge filtration $F^\bullet_{\boldsymbol{\alpha}}$ extends to locally free sheaves over Δ^{p+q} such that $F^p_{\boldsymbol{\alpha}}/F^{p+1}_{\boldsymbol{\alpha}}$ is also locally free. Therefore, Ψ extends holomorphic maps over Δ^{p+q}. □

This lemma thus proves Theorem B.(i). We write $a(w) := \Psi(0, w)$. In general, it does not lie in $\mathscr{D}$. The following well-known result follows from the fact that $\mathrm{GL}(V^\nabla)$ acts transitively on $\check{\mathscr{D}}$.

Lemma 2.7. *For any $g \in \mathrm{GL}(V^\nabla)$, consider the left translation $L_g : \check{\mathscr{D}} \to \check{\mathscr{D}}$ with $L_g(F) := gF$. Then*

$$(L_g)_* : T^{-1,1}_{\check{\mathscr{D}},F} \xrightarrow{\sim} T^{-1,1}_{\check{\mathscr{D}},gF}.$$

Recall that for any $A \in \mathrm{End}(V^\nabla)$ and any $F \in \check{\mathscr{D}}$, we denote by $[A]_F$ the image of A under the natural map $\mathrm{End}(V^\nabla) \to T_{\check{\mathscr{D}},F}$.

Lemma 2.8. *For each $i = 1, \ldots, p$, $[S_i + N_i]_{a(w)} \subset T^{-1,1}_{\check{\mathscr{D}},a(w)}$.*

Proof. Since

$$\tilde{\Psi}_*\left(\frac{\partial}{\partial z_i}\right)(z, w) = [S_i + N_i]_{\tilde{\Psi}(z,w)} + (L_{\exp(\sum_{i=1}^p (S_i+N_i)z_i)})_*\Phi_*\left(\frac{\partial}{\partial z_i}\right)(z, w),$$

$\Phi_*(\frac{\partial}{\partial z_i})$ is horizontal since Φ is a horizontal mapping by §1.7. By Lemma 2.7, $(L_{\exp(\sum_{i=1}^p (S_i+N_i)z_i)})_*\Phi_*(\frac{\partial}{\partial z_i})(z, w)$ is horizontal. On the other hand,

$$\tilde{\Psi}_*\left(\frac{\partial}{\partial z_i}\right)(z, w) = \Psi_*\left(\frac{\partial}{\partial t_i}\right)(e^{z_1}, \ldots, e^{z_p}, w) \cdot e^{z_i}$$

which tends to zero if $\Re z_i \to -\infty$ and $\Re z_j \leq C$ for other j. By the continuity, this implies that

$$[S_i + N_i]_{a(w)} \in T^{-1,1}_{\check{\mathscr{D}},a(w)}.$$ □

We are ready to prove Theorem B.(ii).

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Lemma 2.9. *The holomorphic mapping*

$$\vartheta : \mathbb{H}^p \times \Delta^q \to \check{\mathscr{D}}$$
$$(z, w) \mapsto \exp\left(-\sum_{i=1}^{p} z_i(S_i + N_i)\right) \circ a(w)$$

is horizontal.

Proof. Note that $\vartheta_*(\frac{\partial}{\partial z_i}) = [S_i + N_i]_{\vartheta(z,w)}$. Since $[S_i, N_i] = 0$, one has

$$(L_{\exp(\sum_{i=1}^{p}(S_i+N_i)z_i)})_*\big([S_i + N_i]_{\vartheta(z,w)}\big) = \Big[\mathrm{Ad}_{\exp(\sum_{i=1}^{p}(S_i+N_i)z_i)}(S_i + N_i)\Big]_{a(w)} = [S_i + N_i]_{a(w)}.$$

It then follows from Lemmas 2.7 and 2.8 that $[S_i + N_i]_{\vartheta(z,w)} \in T^{-1,1}_{\check{\mathscr{D}},\vartheta(z,w)}$. We conclude that $\vartheta_*(\frac{\partial}{\partial z_i})$ is horizontal.

On the other hand, one has

$$\vartheta_*\left(\frac{\partial}{\partial w_i}\right) = (L_{\exp(-\sum_{i=1}^{p}(S_i+N_i)z_i)})_* a_*\left(\frac{\partial}{\partial w_i}\right),$$

and

$$\Psi_*\left(\frac{\partial}{\partial w_i}\right)(e^z, w) = \tilde{\Psi}_*\left(\frac{\partial}{\partial w_i}\right)(z, w) := (L_{\exp(\sum_{i=1}^{p}(S_i+N_i)z_i)})_*\Phi_*\left(\frac{\partial}{\partial w_i}\right)(z, w).$$

Since $\Phi_*(\frac{\partial}{\partial w_i})$ is horizontal, by Lemma 2.7,

$$\tilde{\Psi}_*\left(\frac{\partial}{\partial w_i}\right)(z, w) := (L_{\exp(\sum_{i=1}^{p}(S_i+N_i)z_i)})_*\Phi_*\left(\frac{\partial}{\partial w_i}\right)(z, w)$$

is horizontal, and thus $\Psi_*(\frac{\partial}{\partial w_i})(e^z, w)$ is horizontal. Letting $\Re z_i \to -\infty$ for $i = 1, \ldots, p$, we conclude that

$$\Psi_*\left(\frac{\partial}{\partial w_i}\right)(0, w) = a_*\left(\frac{\partial}{\partial w_i}\right)$$

is also horizontal. We apply Lemma 2.7 again to conclude that $\vartheta_*(\frac{\partial}{\partial w_i})$ is horizontal. In conclusion, ϑ is a horizontal mapping. We proved Theorem B.(ii). □

The rest of the paper is devoted to the proof of Theorem B.(iii). We will only deal with the case of one variable. We first start a lemma in linear algebra whose proof is direct (cf. [9, §7.5] for a detailed proof).

Lemma 2.10. *Let* $S \in \mathrm{End}(V^\nabla)$ *be semisimple with real eigenvalues. Then there exists a constant* $C > 0$ *such that*

$$\|\mathrm{Ad}\, e^{xS}\| \le C \exp((\lambda_{\max} - \lambda_{\min}) \cdot |x|) \quad \textit{for all} \quad x \in \mathbb{R},$$

where $\lambda_{\max}$ *and* $\lambda_{\min}$ *are the largest and smallest eigenvalue of S. Let* $N \in \mathrm{End}(V^\nabla)$ *be nilpotent. Then*

$$\|\mathrm{Ad}\, e^{xN}\| \le C'|x|^m$$

for some $C', m > 0$.

Here, we fix a reference polarized Hodge structure $o \in \mathscr{D}$ *which induces metrics for* V^∇ *and* $\mathrm{End}(V^\nabla)$. $\|\mathrm{Ad}\, e^{xS}\|$ *is the operator norm with respect to such metric of* $\mathrm{End}(V^\nabla)$.

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The following two lemmas are due to Schmid [10, Lemmas 8.12 & 8.19]. They are stated for period domains of real Hodge structures. However, their proofs can be generalized to period domains of complex polarized Hodge structures verbatim, and we thus omit their proofs here.

Lemma 2.11. *If $g \in \mathrm{GL}(V^\nabla)$, then for some natural distance $d_{\check{\mathscr{D}}}$ of $\check{\mathscr{D}}$, we have*

$$d_{\check{\mathscr{D}}}(ga, gb) \leq \|\mathrm{Ad}\, g\| d_{\check{\mathscr{D}}}(a, b)$$

for any points $a, b \in \check{\mathscr{D}}$.

Lemma 2.12. *Let $\Phi : \mathbb{H} \to \mathscr{D}$ be the period map associated to a complex polarized variation of Hodge structure on Δ^*. Fix $\alpha, k > 0$ and a reference point $o \in \mathscr{D}$. Choose $g(z) \in G = U(V^\nabla, Q)$ such that $g(z) \cdot o = \Phi(z)$. Then there exist $C, \beta > 0$ such that if $|\Im z| \leq k$, one has*

$$\|\mathrm{Ad}\, g(z)^{-1}\| \leq C|\Re z|^\beta$$

for $\Re z < -\alpha$. Here, $\|\mathrm{Ad}\, g(z)^{-1}\|$ is the operator norm defined in Lemma 2.10.

By [9, §4.1], we know that $G := U(V^\nabla, Q)$ acts transitively on the period domain $\mathscr{D}$, and $\mathscr{D}$ admits a natural G-invariant distance $d_{\mathscr{D}}$.

Proof of Theorem B.(iii). Let $T \in \mathrm{GL}(V^\nabla)$ be the monodromy operator associated to the counter-clockwise generator of $\pi_1(\Delta^*)$. Note that $T \in G := U(V^\nabla, Q)$. Recall that there exist commuting $S, N \in \mathrm{GL}(V^\nabla)$ such that

- $\exp(2\pi i(S + N)) = T$;
- S is semisimple with eigenvalues lying in $(\alpha - 1, \alpha]$;
- N is nilpotent.

Denote by $a = \Psi(0)$. Then for $|t|$ small enough, one has

$$d_{\check{\mathscr{D}}}(a, \Psi(t)) < C|t| \quad \text{for some } \ C > 0,$$

which is equivalent to that

$$d_{\check{\mathscr{D}}}(a, \Psi(e^z)) < Ce^x \tag{2.15}$$

when $x \leq -M$ for some $M > 0$. Here, we write $z = x + iy$. Assume now $|y| \leq 2\pi$ and $x \leq -M$. Then

$$\begin{aligned} d_{\check{\mathscr{D}}}(\exp(-(S+N)z)a, \Phi(z)) &\leq \|\mathrm{Ad}\, \exp((S+N)z)\| \cdot d_{\check{\mathscr{D}}}(a, \Psi(e^z)) \\ &\leq \|\mathrm{Ad}\, \exp(Nx)\| \cdot \|\mathrm{Ad}\, \exp(i(S+N)y)\| \cdot \|\mathrm{Ad}\, \exp(Sx)\| \cdot d_{\check{\mathscr{D}}}(a, \Psi(e^z)) \\ &\leq C_1 \|\mathrm{Ad}\, \exp(Nx)\| \cdot \|\mathrm{Ad}\, \exp(Sx)\| \cdot d_{\check{\mathscr{D}}}(a, \Psi(e^z)) \\ &\leq C_2 |x|^m \cdot \exp((\lambda_{\max} - \lambda_{\min}) \cdot |x|) \cdot d_{\check{\mathscr{D}}}(a, \Psi(e^z)) \\ &\leq C_3 |x|^m \cdot \exp((\lambda_{\max} - \lambda_{\min}) \cdot |x|) \cdot e^x \leq C_3 |x|^m e^{\delta x}. \end{aligned} \tag{2.16}$$

The first inequality is due to Lemma 2.11, the third one holds since $|y| \leq 2\pi$, the fourth one follows from Lemma 2.10, and the fifth one follows from Equation (2.15). Here, $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalue of S. Therefore, $\lambda_{\max} - \lambda_{\min} < 1$, and thus the last inequality can be achieved for some $\delta > 0$. Here, $C_1, \ldots, C_3 > 0$ are some positive constants.

Fix a reference point $o \in \mathscr{D}$, and let $g(z) \in G$ such that $g(z) \cdot o = \Phi(z)$. By Lemmas 2.11 and 2.12 together with Equation (2.16), one gets

$$\begin{aligned} d_{\check{\mathscr{D}}}(g(z)^{-1} \exp(-(S+N)z)a, o) &\leq \|\mathrm{Ad}\, g(z)^{-1}\| \cdot d_{\check{\mathscr{D}}}(\exp(-(S+N)z)a, \Phi(z)) \\ &\leq C_4 |x|^{m+\beta} e^{\delta x} \end{aligned} \tag{2.17}$$

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if $|y| \le 2\pi$ and $x < -M_2$ for some $M_2 \ge M$ and $C_4, \beta > 0$. Pick a small neighborhood U of o in $\mathscr{D}$ such that the distance functions $d_{\mathscr{D}}$ and $d_{\check{\mathscr{D}}}$ are mutually bounded over U. By Equation (2.17) when $|y| \le 2\pi$, $x \le -M_3$ for some $M_3 \ge M_2$, $g(z)^{-1}\exp(-(S+N)z)a$ will be entirely contained in U. Note that $g(z) \in G$, it follows that $\exp(-(S+N)z)a \in \mathscr{D}$ if $|y| \le 2\pi$ and $x \le -M_3$. When $|y| > 2\pi$ and $x \le -M_3$, we find some integer ℓ such that $|y - 2\pi\ell| \le 2\pi$. Then $\exp(-(S+N)(z-2\pi i\ell))a \in \mathscr{D}$. Since $\exp(-(S+N)z)a = T^{-\ell}\exp(-(S+N)(z-2\pi i\ell))a$ and $T \in G$, it follows that $\exp(-(S+N)z)a \in \mathscr{D}$. In conclusion, $\exp(-(S+N)z)a \in \mathscr{D}$ if $x \le -M_3$. We prove the first claim in Theorem B.(iii).

Recall that the distance functions $d_{\mathscr{D}}$ and $d_{\check{\mathscr{D}}}$ are mutually bounded over U. By Equation (2.17) again for some $C_5 > 0$ we have

$$d_{\mathscr{D}}(g(z)^{-1}\exp(-(S+N)z)a, o) \le C_5|x|^{m+\beta}e^{\delta x}$$

for $|y| \le 2\pi, x \le -M_3$. Since the action of $g(z)$ is $d_{\mathscr{D}}$-distance invariant, we obtain the distance estimate

$$d_{\mathscr{D}}(\exp(-(S+N)z)a, \Phi(z)) \le C_5|x|^{m+\beta}e^{\delta x}$$

for $|y| \le 2\pi, x \le -M_3$. When $|y| > 2\pi$ and $x \le -M_3$, one picks some integer ℓ such that $|y - 2\pi\ell| \le 2\pi$. Then

$$d_{\mathscr{D}}(\exp(-(S+N)(z-2\pi i\ell))a, \Phi(z-2\pi i\ell)) \le C_5|x|^{m+\beta}e^{\delta x}.$$

In other words,

$$d_{\mathscr{D}}(T^{\ell}\exp(-(S+N)z)a, T^{\ell}\Phi(z)) \le C_5|x|^{m+\beta}e^{\delta x}.$$

As T is also $d_{\mathscr{D}}$-distance invariant, it follows that

$$d_{\mathscr{D}}(\exp(-(S+N)z)a, \Phi(z)) \le C_5|x|^{m+\beta}e^{\delta x}.$$

for $x \le -M_3$. The distance estimate is obtained. □

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