ON THE AVERAGE NUMBER OF TREES IN CERTAIN MAPS

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1. Introduction. For a formal definition of "map" the reader is referred to (**7**, §2). The maps in this paper are rooted by specifying an orientation for one of the edges. This also specifies a root vertex, the negative end of the root, and a root face, the face on the left of the root edge. Counting is, as usual, defined on isomorphism classes.

Regular maps of even valence have been enumerated in a recent paper by Tutte. In this paper we determine the average number of trees in such maps, and include similar results for regular trivalent maps, that is, maps with three edges incident on every vertex. In the development for the latter, a formula for the number of trivalent maps with $2t$ vertices is produced.

2. Almost regular maps of even valence. An almost regular rooted map is a map in which all vertices with the possible exception of the root vertex have the same valence. Maps of even valence are in 1–1 correspondence with rooted bicubic maps as defined in (**7**, §4). Indeed if one contracts the faces of root colour in a bicubic map of (**7**), one obtains a regular map of even valence, and the operation can be reversed to recover the bicubic map. Since the number of rooted bicubic maps in which the root face has valence $2t$, and there are just q_s other faces of root colour with valence $2s$ $(s = 1, 2, 3 \ldots)$, is shown to be

$$\frac{(n-1)!}{(k-1)!\,(n-k+2)!}\,\frac{(2t)!}{t!\,(t-1)!}\prod_{s=1}^{\infty}\left\{\frac{(2s-1)!}{s!\,(s-1)!}\right\}^{q_s}, \tag{2.1}$$

where $2n$ is the total number of vertices, and k is the number of faces of the root colour, thus the number of rooted maps with v vertices, each non-root vertex of valence $2s$ and root vertex of valence $2t$, is

$$\frac{[(v-1)s+t-1]!\,(2t)!}{(v-1)!\,[(v-1)s+t-v+2]!\,t!\,(t-1)!}\left\{\frac{(2s-1)!}{s!\,(s-1)!}\right\}^{v-1}. \tag{2.2}$$

Let us denote the class of such maps by $\mathbf{M}(s, t, v)$ for future reference.

3. Trivalent maps. Unfortunately no formula corresponding to (2.2) is known for regular maps of odd valence. However, we shall develop a formula in the case of maps in which every vertex is trivalent. The enumeration will be done in terms of the dual maps as defined in (**7**). In general both these maps will be singular, that is loops are permitted. Thus one edge may be incident with multiplicity two on a vertex. Also a face may not be bounded by a simple

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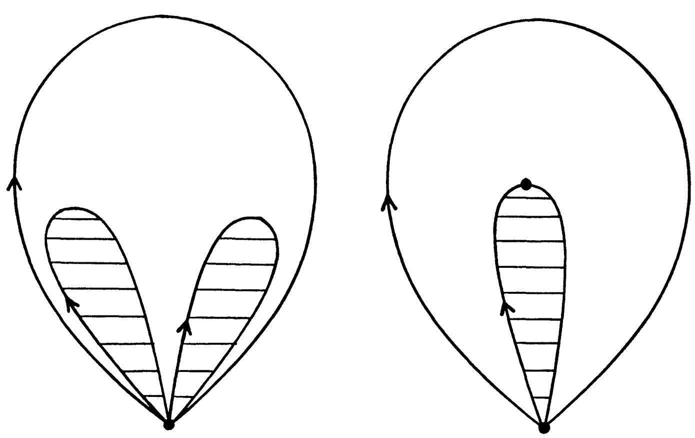
closed curve, but, by the topology of the plane, may be incident with multiplicity two on certain edges (called isthmuses) of its boundary. Thus in the case of loops each loop is counted twice when determining the number of edges incident with a vertex, and in the case of isthmuses each is counted twice when determining the number of edges incident with a face. Let us consider maps in which every face is incident with three (not necessarily distinct) edges. We shall call such maps triangular, noting that these maps differ from those triangular maps that occur in other papers, such as (**4**, **5**, **6**, and **7**), where only non-singular maps are permitted. The root face, edge, and vertex in the original are mapped onto the root vertex, edge, and face respectively under duality.

Let l_v be the number of such maps with v non-root faces in which the root face is exceptional inasmuch as it is bounded by a loop, and m_v be the number of triangular maps with v non-root faces and a root face which is a (non-singular) digon.

Let us define formal power series

$$L(x) = \sum_{v=1}^{\infty} l_v x^v, \qquad M(x) = \sum_{v=1}^{\infty} m_v x^v.$$

Removing the root from a map whose root face is bounded solely by a loop produces either a pair of maps, both of which have loop roots (Fig. 1(a)), or a



(a) (b)

FIGURE 1

triangularized digon (Fig. 1(b)), and the construction can be reversed. Thus

$$L(x) = x[L^2(x) + M(x)], \tag{3.2}$$

that is,

$$xL(x) = [1 - \sqrt{(1 - 4x^2M(x))}]/2, \tag{3.3}$$

where the radical represents the series with constant term 1. Let d_n represent the number of *triangulations* of a triangle in which there are n vertices not

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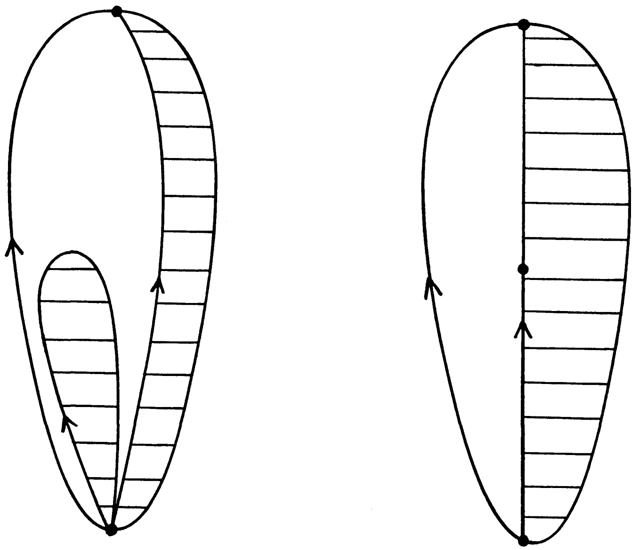
incident with the root face (for the definition of triangulation cf. **1**). Let us define the formal power series

$$D^*(x) = x \sum_{n=0}^{\infty} d_n x^n. \tag{3.4}$$

Let us refer to a *triangulation* of an n-gon as a *triangulated* n-gon and a *triangularization* of an n-gon as a *triangularized* n-gon.

Removing the root from a triangularized digon produces either a triangularized loop digon pair (Fig. 2(a)), or a triangularized non-singular triangle



(a) (b)

FIGURE 2

(Fig. 2(b)). The reduced entities may be rooted as in the figures. The maps of loop–digon pair variety are enumerated by $2xL(x)M(x)$. It can be shown that any triangularizations of a non-singular triangle can be obtained by replacing edges in a triangulated triangle by triangularized digons. Thus such triangularized triangles are enumerated by

$$x \sum_{n=0}^{\infty} d_n x^{2n+1}[M(x)]^{3n+3} = D^*[x^2M^3(x)], \tag{3.5}$$

and therefore

$$M(x) = 1 + 2xL(x)M(x) + D^*[x^2M^3(x)],$$

where 1 is included to enumerate the degenerate case of a single edge. Employing (3.3) yields

$$D^*[x^2M^3(x)] - M(x)\sqrt{[1 - 4x^2M(x)]} + 1 = 0. \tag{3.6}$$

As shown in **(1)**, if

$$x = u(1-u)^3,$$

(3.7) $$D^*(x) = u(1-2u).$$

Define $r(x)$ by

(3.8) $$r(x) = u[x^2M^3(x)].$$

Then

(3.9) $$x^2M^3(x) = r(1-r)^3,$$

(3.10) $$D[x^2M^3(x)] = r(1-2r),$$

and (3.6) yields

(3.11) $$M(x) = (1-r)(1+8r)^{1/2};$$

thus

(3.12) $$r = x^2(1+8r)^{3/2},$$

(3.13) $$xL(x) = \frac{1}{2}\left(1 - \frac{1+2r}{\sqrt{(1+8r)}}\right).$$

Applying Lagrange's power-series theorem **(8)** to (3.12) and (3.13), we obtain

(3.14) $$l_{2t+1} = \frac{(3t/2)_{t-1}\, 2^{3t}}{(t+1)!},$$

where $(x)_r = x(x-1)\ldots(x-r+1)$ for $r > 0$, $(x)_0 = 1$, $(x)_{-1} = 1/(x+1)$. Dualizing, one obtains the number of rooted trivalent maps "on a stick," that is, trivalent maps in which the root vertex is monovalent. By removing the root edge and rooting the homeomorphically reduced graph by an appropriate convention, one obtains a regular rooted trivalent map. The number of such maps with $2t$ vertices is again l_{2t+1}.

4. Plane trees and sequences. The enumeration in the following section is based on the fact that one may list the vertices of a plane tree as they are encountered as one proceeds in a fixed direction around the tree beginning at the root vertex and proceeding along the root edge (see Fig. 3). Let us assume

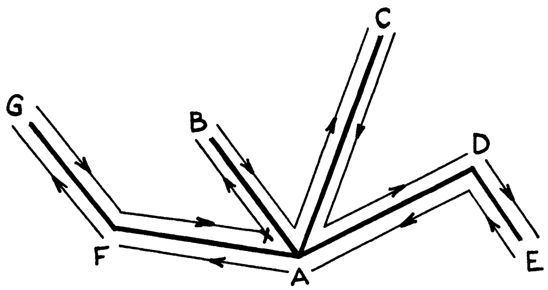


FIGURE 3. sequence $ABACADEDAFGFA$.

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that in such a sequence the root vertex is included as the final member of the sequence. (All trees and sequences considered are finite.) We list necessary and sufficient conditions that a sequence of characters correspond thus to a tree. The sequence $A_k, A_{k+1}, \ldots, A_s$ is said to be a *segment* (with end point x) of the sequence $A_1, A_2, \ldots, A_k, A_{k+1}, \ldots, A_s, \ldots, A_t$, if $A_k = A_s = x$ and $A_i \neq x$, $i = 1, 2, \ldots, k-1$ and $i = s+1, \ldots, t$, where $1 \leqslant k \leqslant s \leqslant t$.

A sequence is dendritic if (1) no consecutive symbols are identical and (2) no element is in both a segment and its complement.

A sequence S^c is said to be a contraction of S if every segment of the sequence S is replaced by its end point x. The operation is well defined for dendritic sequences. A monovalent of a sequence is an element of that sequence that occurs precisely once. A sequence $A_1, A_2, \ldots, A_{2t+1}$ is *symmetric* about A_{t+1} if $A_{t+1+i} = A_{t+1-i}, i = 1, 2, \ldots, t$.

A dendritic sequence S is *arboreal* if for every monovalent m of S the sequence $P^c m Q^c$ is symmetric about m, where $S = PmQ$. (Again we note that the operation is well defined since the sequence is dendritic.)

An elementary induction proves that a sequence is the vertex listing of a rooted plane tree if and only if it is arboreal, and that if the tree has v vertices, the corresponding sequence contains $2v - 1$ symbols. Having made these observations we return to the problem of finding the average number of trees in regular maps. To do so we shall take each map of the class with which we are dealing and further root it by distinguishing in it a spanning tree. By counting all such maps (called tree-rooted maps) that can be formed by taking all possible choices of spanning subtree, and dividing by the number of maps in the original class, we obtain an average value for the number of trees per map.

5. Tree-rooted maps. A rooted map is said to be *tree-rooted* if a spanning tree is distinguished as a root tree in the map. A tree-rooted map in which every face (possibly singular) is an $(r+2)$-gon, with the possible exception of the root-face which is a t-gon, and in which the total number of vertices is v, is said to be of type $[r, t, v]$.

Let $\mathbf{P}(r, n)$ denote the class of all rooted plane polygons with $n+1$ vertices, whose non-root-face is partitioned into $(r+2)$-gons by non-intersecting diagonals. Let us define an (r, t, v)*-tree-perm* as a sequence $(P_1, P_2, \ldots, P_t\colon T)$, where $P_i \in \mathbf{P}(r, n_i)$ and

$$\sum_{i=1}^{t} n_i = 2v - 2,$$

and T is a rooted plane tree with v vertices. We assume that $r \geqslant 1$.

LEMMA 1. *(r, t, v)-tree-perms are in 1–1 correspondence with tree-rooted maps of type $[r, t, v]$.*

Before beginning the proof of the lemma, it is first necessary to root the root-tree in each map, that is to orient one of the edges in the root-tree of the map by a suitable convention. For example, let us consider the non-loop edges radiating from the root vertex w of the original map as being listed in order of occurrence as one goes around the vertex such that one crosses the negative end of the root edge from left to right in some neighbourhood of w sufficiently small that the direction is uniquely defined. (That such a neighbourhood exists is proved in the forthcoming book on graph theory by W. T. Tutte.) That not all edges at the vertex are loops is guaranteed by the fact that in the case we are considering $r \geqslant 1$, that is, no non-root face is bounded solely by a loop; thus interior to each loop must be a non-loop edge. Since the root-tree is spanning, there will be some edge of the root-tree T which occurs as the first edge of T in the listing of edges at the root vertex as described above. This edge, oriented away from w, serves to root the spanning tree. Let us call the rooted root-tree T^*.

Proof of Lemma 1. Choose any tree-rooted map M of type $[r, t, v]$. Let us label the edges of the root-face $E = E_1, E_2, \ldots, E_t$ as they occur in cyclic order about the root-face in the orientation induced by the root edge. These edges are to be oriented consistent with the root-face. Let us trace around the root-tree T^* as in Section 4, beginning along the left side of the root edge of T^* (see Fig. 4). In the tree T^* there is a unique arc A_i joining the ends of $E_i, i = 1, 2, \ldots, t$.

Let us construct the tree-perm to be associated with the above tree-rooted map as follows. For the tree component select the root-tree T^*. If the arc A_i described above is the edge E_i, take the ith entry in the partitioned polygon portion of the vector to be the degenerate polygon P_i consisting of an oriented edge. If $A_i \neq E_i$, then the closure of $A_i \cup E_i$ bounds two residual domains.

Let R_i be the residual domain of $A_i \cup E_i$ which does not include the root-face. There is a path P^*_i joining the ends of E_i, obtained by listing vertices and edges encountered as one traces the residual portion T_i of T^*, beginning with one end of E and keeping T' on the left, never crossing T.

Construct a rooted plane polygon whose vertices and edges correspond in sequence to those of the circular path $E_i \cup P^*_i$ and orient it according to the orientation on E_i. The interior of this polygon can be partitioned into $(r + 2)$-gons by means of the sequence of faces along the path P^*_i in the "natural" way. The edges in the partition can be shown to be diagonals. Call this polygon P_i. Now form the tree-perm $(P_1, P_2, \ldots, P_t: T^*)$. In this, T^* has v vertices, and considering the class of P_i as being $\mathbf{P}(r, n_i)$, one observes that

$$\sum_{i=1}^{t} n_i = 2n - 2,$$

because the sequence of vertices listed in going around the tree contains $2v - 2$ characters if the root vertex is not included at the end, since the ends of each E_i are counted twice. The construction can be reversed. Indeed, if the first member of the perm is a single edge, the root of the tree is the root of the map. Otherwise the non-root exterior edge incident with the root vertex of the

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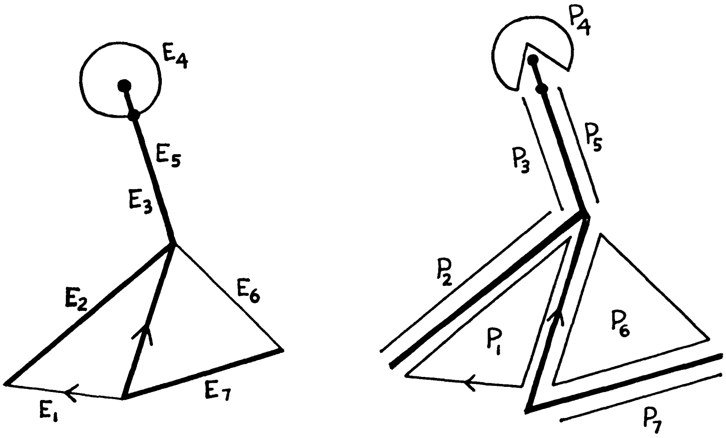
first member of the tree-perm is identified with the left side of the root edge of the tree. The construction continues as suggested by Fig. 4, the root edges of the tree-perm members bounding the root-face of the map being constructed.



(a) (b)

FIGURE 4

LEMMA 2. *The number of (r, t, v)-tree perms with a specified rooted tree component is*

$$\frac{t(k[r+1]+t-1)!}{k!\,(kr+t)!},$$

where $2v = kr + t + 2$.

Proof. It is observed in **(2)** that the number of non-isomorphic members of $\mathbf{P}(r, n)$ is the coefficient of x^n in that power series solution of

$$y = x + y^{r+1} \tag{5.1}$$

that is analytic at $x = 0$. Thus the number of tree-perms in question is the coefficient of x^{2v-2} in the expansion of y^t, where y is defined by (5.1). By Lagrange's theorem, if $D = d/dx$,

$$y^t = x^t + \sum_{k=1}^{\infty} \frac{t}{k!} D^{k-1}\{x^{k(r+1)+t-1}\} = \sum_{k=0}^{\infty} \frac{t(k[r+1]+t-1)!}{k!\,(kr+t)!} x^{kr+t}. \tag{5.2}$$

Hence the lemma follows.

COROLLARY. *The number of tree-rooted maps of type* $[r, t, v]$ *is*

$$\frac{t(2v+k-3)!}{(v-1)!\,v!\,k!} \tag{5.3}$$

where $$2v = kr + t + 2.$$

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This follows from the well-known result **(3)** that there are

$$\frac{(2v-2)!}{v!\,(v-1)!} \tag{5.4}$$

rooted plane trees with v vertices. Standard arguments with the Euler polyhedron formula show that the parameter k in the preceding formula is the number of non-root-faces in the $[r, t, v]$-map. Sincethere is a 1–1 correspondence between trees in a map and trees in the dual, the preceding result may be restated as follows.

The number of tree-rooted maps in which there are v vertices, the non-root vertices of valence u, and the root vertex of valence t is

$$\frac{t(2x+v-4)!}{(x)!\,(x-1)!\,(v-1)!}, \tag{5.5}$$

where

$$x = \tfrac{1}{2}[(v-1)(u-2)+t+2]. \tag{5.6}$$

6. Ratios. We may now deduce the statistical average of trees per map in $\mathbf{M}(s, t, v)$ by dividing the number of tree-rooted maps in $\mathbf{M}(s, t, v)$ by the number of maps in $\mathbf{M}(s, t, v)$. Thus the statistical average of trees per map in $\mathbf{M}(s, t, v)$ is, for $s \geqslant 2$,

$$\frac{[v(2s-1)+2(t-s)]!\,t!\,(t-1)!}{[(v-1)(s-1)+t]!\,[(v-1)s+t-1]!\,(2t-1)!}\left\{\frac{s!\,(s-1)!}{(2s-1)!}\right\}^{v-1}. \tag{6.1}$$

Clearly for $t = s$, that is, for regular maps, this average is

$$\frac{[v(2s-1)]!}{[v(s-1)+1]!\,(vs-1)!}\left\{\frac{s!\,(s-1)!}{(2s-1)!}\right\}^{v}. \tag{6.2}$$

The formula is evidently valid for $s = 1$. Further, the average number of trees per rooted trivalent map with $2t$ vertices is

$$\frac{3(4t)!}{(t+2)!\,(2t-1)!\,(3t/2)_{t-1}\,2^{3t}}. \tag{6.3}$$

7. Asymptotic formulae. Asymptotic approximations will be included for results concerning regular maps which occur in this paper. Letting $s = t$ in (2.2), the number of rooted regular maps with v vertices of valence $2s$ is

$$\frac{(2s)\,(vs-1)!}{(v-1)!\,[v(s-1)+2]!}\left\{\frac{(2s-1)}{s!\,(s-1)!}\right\}^{v}, \tag{7.1}$$

which, for fixed $s \neq 1$, as $v \to \infty$, is approximately

$$\sqrt{\frac{2s}{\pi}}\,\frac{s^{vs}[v(s-1)]^{-(5/2)}}{(s-1)^{v(s-1)}}\left\{\frac{(2s-1)!}{s!\,(s-1)!}\right\}^{v}. \tag{7.2}$$

The average number of trees for such a map is approximated asymptotically by

$$\frac{v^{-(1/2)}}{\sqrt{2\pi}} \frac{(2s-1)^{v(2s-1)+\frac{1}{2}}}{s^{sv-1/2}(s-1)^{v(s-1)+3/2}} \left\{\frac{s!\,(s-1)!}{(2s-1)!}\right\}^{v}. \tag{7.3}$$

The number of rooted trivalent maps with $2t$ vertices is asymptotically

$$\frac{1}{\sqrt{2\pi}} 3^{(3t+1)/2} 2^{2t+1} t^{-(5/2)}, \tag{7.4}$$

and the average number of trees in such a map is estimated by

$$\frac{1}{\sqrt{\pi t}} 2^{4t} 3^{-\frac{1}{2}(3t-1)}. \tag{7.5}$$

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