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PRECONDITIONING COLLOCATION METHOD USING QUADRATIC SPLINES WITH APPLICATIONS TO 2nd-ORDER SEPARABLE ELLIPTIC EQUATIONS

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Abstract

In this paper we propose a P_1 finite element preconditioning using the so-called 'hat-function', to a collocation scheme constructed by quadratic splines for a 2nd-order separable elliptic operator and we show that the resulting preconditioning system of equations is well conditioned with the condition number independent of the number of unknowns.

1. Introduction

Let Ω be the unit square $[0, 1] \times [0, 1]$ and consider a uniformly elliptic operator given by

$$L_a u = -[u_{xx} + u_{yy}] + a_1(x, y)u_x + a_2(x, y)u_y + a_0(x, y)u, \quad (x, y) \in \operatorname{Int}\Omega \tag{1.1}$$

with homogeneous boundary condition.

Let $A_{N,M}$ be a family of quadratic spline collocation discretizations based on Gaussian points which arise from a variational or weak representation of the operator L_a (see Section 5 and 6). Now consider the system of linear equations

$$\hat{A}_{N,M} U = F \tag{1.2}$$

which arise in the numerical solution of the boundary value problem

$$L_a u = f \tag{1.3}$$

using these collocation discretizations and the interpolating biquadratic basis for $S_{\pi,2}$ [see Section 2]. In this paper the proposed preconditioning of (1.2) we are interested in is given by

$$\tilde{\beta}_{N,M}^{-1} W_{N,M} \hat{A}_{N,M} U = \tilde{\beta}_{N,M}^{-1} W_{N,M} F, \tag{1.4}$$

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where $W_{N,M}$ is the diagonal matrix of the quadrative weights associated with the Gaussian quadrature and where the matrix $\tilde{\beta}_{N,M}$ is the stiffness matrix, constructed by the piecewise bilinear basis for $S_{\pi,1}$ [see Section 2], of any symmetric positive operator of the form

$$L_d u = -[u_{xx} + u_{yy}] + du \quad \text{in } \Omega, \tag{1.5}$$

where d is a nonnegative constant with homogeneous boundary condition. In this paper we will give an analysis of the $\tilde{\beta}_{N,M}$-singular values of

$$L_{N,M} = \tilde{\beta}_{N,M}^{-1} W_{N,M} \hat{A}_{N,M}. \tag{1.6}$$

The result is contained in the following theorem.

THEOREM. *Assume $min(N, M) \geq N_0$ for some integer N_0. Then there are two positive constants $0 < \alpha < \beta$, independent of N and M, such that we have, for any vector U,*

$$\alpha \leq \frac{(\tilde{\beta}_{N,M} L_{N,M} U, L_{N,M} U)_{l_2}}{(\tilde{\beta}_{N,M} U, U)_{l_2}} \leq \beta.$$

This result is important for the successful application of conjugate gradient methods for the solution of the algebraic system (1.4). The case using cubic splines and Gaussian points was analyzed in [14] and similar problems were discussed in [5], [16] and [18] for the case where there is only one finite element space and $\hat{A}_{N,M}$ and $\hat{\beta}_{N,M}$ are finite element discretizations. The collocation method using quadratic interpolatory splines was used for the numerical solution of two point boundary value problems in [15] and for linear second order elliptic partial differential equations in [6]. Some solvers for quadratic spline collocation equations were developed in [7]. Our goal is to develop the finite element preconditioning of (1.5) using piecewise linear shape functions to a collocation scheme for a uniformly elliptic operator (1.1) using quadratic interpolatory splines. When implementing the theory presented in Sections 5, 6, and 7, a spline tool-box may be used. Such a software is available, for example, in the MATLAB package [3]. Through repeated calls to the routines augknt, spapi, fnval, and fnder, the symmetrized collocation matrix $\tilde{B}_{N,M}$ associated with (1.1) is built and the so called "hat-function" is used to construct the stiffness matrix $\tilde{\beta}_{N,M}$ associated with (1.5) (see Theorem 5). Finally, the spline tool box can be used to construct the linear system (1.6), which is solved using the conjugate gradient method. The arguments here follow the line of the arguments in [14] and [17]. However in this work we need some basic estimates using the quadratic interpolatory spline basis. Some preliminary ideas, notations, etc. are presented in Section 2 and the properties of the interpolatory basis functions are studied in Section 3. Some basic one-dimensional estimates are shown in Section 4. In Sections 5 and 6 we present some basic preconditioning results and finally in Section 7, we extend the results in Section 5 and 6 to the general case.

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2. Preliminaries

Let $I = [0, 1]$ be the unit interval and let $\Omega = I \times I$ be the unit square. Let $\Delta_x = \{x_i\}_{i=0}^N$ and $\Delta_y = \{y_i\}_{i=0}^M$ be two strict partitions of I with $x_0 = 0$, $x_N = 1$, $y_0 = 0$ and $y_M = 1$. Let $I_i = [x_{i-1}, x_i]$ or $[y_{i-1}, y_i]$ where i is a positive integer and let $\Omega_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$ form a partition $\pi : \Delta_x \times \Delta_y$ of Ω. Let $h = x_i - x_{i-1}$ and $s = y_i - y_{i-1}$. Define for $k \geq 1$

$$S_{\Delta,k} = \{f \in C^{k-1}[0, 1], f|_{I_i} \in \mathbf{P}_k, f(0) = f(1) = 0\}.$$

Let $S_{\pi,k}$ be the set of all functions $f(x, y) \in C^{k-1}(\Omega)$ satisfying

(1) $f|_{\Omega_{ij}}$ is a polynomial in x of degree k or less and is a polynomial in y of degree k or less,

(2) $f(x, y) = 0$ for $(x, y) \in \partial\Omega$.

In this paper, we will particulary make use of $k = 1$ or 2. Let $\{\xi_i\}_{i=1}^N$ be the set of all Gaussian points on I occuring from the first Legendre polynomial, that is, $\xi_i = \frac{h}{2} + x_{i-1}$ (or $\frac{s}{2} + y_{i-1}$). Recall a quadratic spline on $\mathbf{R}$ as

$$\psi(x) = \begin{cases} \psi^l(x) := x^2 & \text{if } 0 \leq x \leq 1 \\ \psi^m(x) := -3 + 6x - 2x^2 & \text{if } 1 \leq x \leq 2 \\ \psi^r(x) := 9 - 6x + x^2 & \text{if } 2 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Using a linear transformation, define a normalized quadratic spline as $\psi_i(x) = \frac{2}{3}\psi(\frac{x}{h} - i + 2)$, where $i = 0, 1, 2, \cdots, N + 1$. Define the basis functions $\{\varphi_i\}_{i=1}^N$ for $S_{\Delta,2}$ using $\{\psi_i(x)\}_{i=1}^N$ as follows.

$$\begin{aligned} \varphi_1(x) &= \psi_1(x) - \psi_0(x), \qquad \varphi_N(x) = \psi_N(x) - \psi_{N+1}(x) \\ \varphi_k(x) &= \psi_k(x), \quad k = 2, \cdots, N - 1. \end{aligned} \tag{2.2a}$$

Set for $i = 1, 2, \cdots, N$

$$\psi_i^l(x) = \psi_i|_{I_{i-1}}, \qquad \psi_i^m(x) = \psi_i|_{I_i} \quad \text{and} \quad \psi_i^r(x) = \psi_i|_{I_{i+1}}. \tag{2.2b}$$

Define on I

$$\psi^{ol}(x) = x^2, \quad \psi^{om}(x) = -2x^2 + 2x + 1 \quad \text{and} \quad \psi^{or}(x) = x^2 - 2x + 1. \tag{2.2c}$$

Let us define the interpolatory basis functions $\{\phi_i\}_{i=1}^N$ for $S_{\Delta,2}$ such that with $\xi_0 = 0$ and $\xi_{N+1} = 1$,

$$\phi_i(\xi_j) = \delta_{ij}, \quad j = 0, 1, \cdots, N + 1. \tag{2.3}$$

For the basis functions $S_{\Delta,1}$, we will use the so-called "hat function " $\{\theta_i\}_{i=1}^N$ (see [13]) satisfying $\theta_i(\xi_j) = \delta_{ij}$, $j = 0, 1, \cdots, N+1$. Let $(\cdot, \cdot)_T$ be the usual L^2 inner product whose corresponding norm is $\|\cdot\|_0 = \sqrt{(\cdot, \cdot)_T}$, where T is I, I_i, Ω or Ω_{ij}. For simplicity we will use $(\cdot, \cdot)$ for $(\cdot, \cdot)_T$. Let $H^1(I)$, $H_0^1(I)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ be the usual Sobolev spaces and $\|\cdot\|_1$ the usual Sobolev norm. Since we use one Gaussian point on each subinterval I_i, it is the root of the first Legendre polynomial defined on I_i. Therefore on I_i, $i = 1, \cdots, N$, the Gaussian point is $\xi_i = x_{i-1} + \frac{h}{2}$ and its corresponding weight is $w_i = h$. Define

$$\langle u, v\rangle_N = \sum_{i=1}^{N} h \cdot u(\xi_i)v(\xi_i), \tag{2.4a}$$

$$\langle u, v\rangle_{N,M} = \sum_{i=1}^{N}\sum_{j=1}^{M} h \cdot s \cdot u(\xi_i, \eta_j)v(\xi_i, \eta_j). \tag{2.4b}$$

There are many occasions when we want to express the fact that two families of positive quantities $\{a_N\}$, $\{b_N\}$, [or $\{a_{N,M}\}$, $\{b_{N,M}\}$] are uniformly equivalent in the sense that there are two positive constants, (α, β), independent of N, [or (N, M)] such that for all N

$$0 < \alpha a_N < b_N < \beta a_N.$$

For this we will write $a_N \sim b_N$.

3. Properties of the interpolatory basis functions

In this section we will analyze the interpolatory basis functions to get a lower bound on $\langle u, v\rangle_N$ in terms of $\|u\|_0^2$ for $u \in S_{\Delta,2}$. The basic idea comes from [1],[2].

LEMMA 1. *Let $\{\phi_i\}_{i=1}^N$ be the interpolatory basis for $S_{\Delta,2}$. Then on I_j for $i \neq j$, we have*

$$\begin{pmatrix}\phi_i(x_j)\\ \phi_i'(x_j)\end{pmatrix} = \begin{pmatrix}-3 & -h\\ -8/h & -3\end{pmatrix}\begin{pmatrix}\phi_i(x_{j-1})\\ \phi_i'(x_{j-1})\end{pmatrix}. \tag{3.1}$$

PROOF. Since $\phi_i(\xi_j) = 0$ where $\xi_j = \frac{x_{j-1}+x_j}{2}$, $\phi_i(x) = (x - \xi_j)(Ax + B)$ for $x \in [x_{j-1}, x_j]$. Then

$$A = -\frac{4}{h^2}\phi_i(x_{j-1}) - \frac{2}{h}\phi_i'(x_{j-1})$$

and

$$B = \frac{2}{h}x_{j-1}\phi_i'(x_{j-1}) + \frac{2}{h^2}(3x_{j-1} - x_j)\phi_i(x_{j-1}).$$

Therefore (3.1) holds.

COROLLARY 1. *Under the same assumption as in Lemma 1, we have*

$$\phi_i(x_j)\phi_i'(x_j) \geq 0 \quad for \quad j < i, \qquad \phi_i(x_j)\phi_i'(x_j) \leq 0 \quad for \quad j > i.$$

PROOF. The proof proceeds by induction on j using (3.1).

COROLLARY 2. *Under the same assumption as in Lemma 1, we have*

(1) $\quad |\phi_i(x_{j-1})| \leq \frac{1}{3}|\phi_i(x_j)| \quad for \quad j < i,$

(2) $\quad |\phi_i(x_j)| \leq \frac{1}{3}|\phi_i(x_{j-1})| \quad for \quad j > i.$

PROOF. The inequalities come immediately from Corollary 1 using (3.1).

LEMMA 2. *Let $\{\phi_i\}_{i=1}^N$ be the interpolatory basis functions for $S_{\Delta,2}$. Then on I_i we have*

$$\begin{pmatrix} \phi_i(x_i) \\ \phi_i'(x_i) \end{pmatrix} = \begin{pmatrix} -3 & -h \\ -8/h & -3 \end{pmatrix} \begin{pmatrix} \phi_i(x_{i-1}) \\ \phi_i'(x_{i-1}) \end{pmatrix} + \begin{pmatrix} 4 \\ 8/h \end{pmatrix}. \tag{3.2}$$

PROOF. The proof is similar to that of Lemma 1.

For convenience, let $D(h) = \begin{pmatrix} -3 & -h \\ -8/h & -3 \end{pmatrix}$. Then the eigenvalues of $D(h)$ are

$$\lambda_1 = -3 + \sqrt{8} \quad \text{and} \quad \lambda_2 = -3 - \sqrt{8}. \tag{3.3}$$

LEMMA 3. *For the matrix $D(h)$, there exist matrices T_h and S_h such that*

$$T_h^{-1} D(h) T_h = S_h^{-1} D(-h) S_h = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

PROOF. We exhibit the matrices T_h and S_h.

$$T_h = \begin{pmatrix} -\frac{h}{\sqrt{8}} & \frac{h}{\sqrt{8}} \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad S_h = T_{-h}.$$

LEMMA 4. *Let $\{\phi_i\}_{i=1}^N$ be the interpolatory basis for $S_{\Delta,2}$. Let $\alpha_j^i = \phi_i(x_j)$ and $\beta_j^i = \phi_i'(x_j)$ for $i = 1, 2, \cdots, N$ and $j = 0, 1, \cdots, N$. Then*

(1) *for $0 \leq k \leq i-1$,*

$$\begin{pmatrix} \alpha_k^i \\ \beta_k^i \end{pmatrix} = T_h \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} T_h^{-1} \begin{pmatrix} \alpha_0^i \\ \beta_0^i \end{pmatrix}, \tag{3.4a}$$

(2) *for* $i \le k \le N$,

$$\begin{pmatrix} \alpha_k^i \\ \beta_k^i \end{pmatrix} = S_h \begin{pmatrix} \lambda_1^{N-k} & 0 \\ 0 & \lambda_2^{N-k} \end{pmatrix} S_h^{-1} \begin{pmatrix} \alpha_N^i \\ \beta_N^i \end{pmatrix}. \tag{3.4b}$$

PROOF. For $k < i$, from Lemma 1,

$$\begin{aligned} \begin{pmatrix} \alpha_k^i \\ \beta_k^i \end{pmatrix} &= D(h) \begin{pmatrix} \alpha_{k-1}^i \\ \beta_{k-1}^i \end{pmatrix} = D^k(h) \begin{pmatrix} \alpha_0^i \\ \beta_0^i \end{pmatrix} \\ &= T_h \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} T_h^{-1} \begin{pmatrix} \alpha_0^i \\ \beta_0^i \end{pmatrix}. \end{aligned}$$

Therefore (1) is proved. In a similar way, we have (3.4b).

LEMMA 5. *Under the same assumption and notations as in Lemma 4, there exist two positive constants C_1 and C_2 independent of h, i and N such that*

$$0 < \alpha_{i-1}^i < C_1 \quad \textit{and} \quad 0 < \alpha_i^i < C_2.$$

PROOF. First choose a basis function ϕ_i. Define for $k \in \mathbf{N}$,

$$p_k = \lambda_1^k - \lambda_2^k \quad \text{and} \quad q_k = \lambda_1^k + \lambda_2^k. \tag{3.5}$$

By (3.4a) with $k = i-1$ and $\alpha_0^i = 0$,

$$\begin{pmatrix} \alpha_{i-1}^i \\ \beta_{i-1}^i \end{pmatrix} = T_h \begin{pmatrix} \lambda_1^{i-1} & 0 \\ 0 & \lambda_2^{i-1} \end{pmatrix} T_h^{-1} \begin{pmatrix} 0 \\ \beta_0^i \end{pmatrix} = \begin{pmatrix} \frac{-h}{4\sqrt{2}} p_{i-1} \beta_0^i \\ \frac{1}{2} q_{i-1} \beta_0^i \end{pmatrix}. \tag{3.6}$$

Hence by (3.2)

$$\begin{pmatrix} \alpha_i^i \\ \beta_i^i \end{pmatrix} = D(h) \begin{pmatrix} \alpha_{i-1}^i \\ \beta_{i-1}^i \end{pmatrix} + \begin{pmatrix} 4 \\ 8/h \end{pmatrix} = \begin{pmatrix} (\frac{3}{4\sqrt{2}} p_{i-1} - \frac{1}{2} q_{i-1}) h \beta_0^i + 4 \\ (\sqrt{2} p_{i-1} - \frac{3}{2} q_{i-1}) \beta_0^i + \frac{8}{h} \end{pmatrix}. \tag{3.7}$$

By (3.4b) with $k = i$ and $\alpha_N^i = 0$,

$$\begin{pmatrix} \alpha_i^i \\ \beta_i^i \end{pmatrix} = S_h \begin{pmatrix} \lambda_1^{N-i} & 0 \\ 0 & \lambda_2^{N-i} \end{pmatrix} S_h^{-1} \begin{pmatrix} 0 \\ \beta_N^i \end{pmatrix} = \begin{pmatrix} \frac{h}{4\sqrt{2}} p_{N-i} & \beta_N^i \\ \frac{1}{2} q_{N-i} & \beta_N^i \end{pmatrix}. \tag{3.8}$$

Solving (3.7) and (3.8) for β_0^i and β_N^i, we have

$$\beta_0^i = \frac{16}{h} \frac{\sqrt{2} p_{N-i} - 2 q_{N-i}}{Den}, \quad \beta_N^i = \frac{16}{h} \frac{(-\sqrt{2} p_{i-1} + 2 q_{i-1})}{Den}, \tag{3.9}$$

where

$$Den = (3\sqrt{2}p_{i-1} - 4q_{i-1})q_{N-i} - (4p_{i-1} - 3\sqrt{2}q_{i-1})p_{N-i}$$
$$= 6\sqrt{2}p_{N-1} - 8q_{N-1} \neq 0 \quad \text{by} \quad (3.5).$$

From (3.6) and (3.8),

$$\alpha_{i-1}^i = \frac{-h}{4\sqrt{2}}p_{i-1}\beta_0^i \quad \text{and} \quad \alpha_i^i = \frac{h}{4\sqrt{2}}p_{N-i}\beta_N^i. \tag{3.10}$$

Note that from (3.3)

$$\frac{\sqrt{8}}{3} \leq -\frac{p_k}{q_k} \leq 1 \quad \text{for} \quad k \in N. \tag{3.11}$$

Then, using (3.9), (3.10) and (3.11), we see that there exist two positive constants satisfying the conclusions.

REMARK. The existence and uniqueness of the interpolatory basis functions come from Lemma 4 and 5. That is to say, for ϕ_i, we determine uniquely β_0^i and β_N^i (see (3.9)). Then, by (3.4), all α_k^i and β_k^i $(k = 0, 1, \dots, N)$ are determined uniquely.

THEOREM 1. *For the interpolatory basis functions* $\{\phi_i\}_{i=1}^N$, $\{\phi_i(x_k)\}_{k=0}^N$ *are uniformly bounded for* i *and* N.

PROOF. For any i, by Lemma 5, $|\phi_i(x_{i-1})| \leq c_1$, $|\phi_i(x_i)| \leq c_2$. Then by Corollary 2, the values at the knots decay exponentially toward the boundary. Therefore the proof is completed.

4. Bounds

In this section we will prove that $\sqrt{\langle u, u\rangle_N}$ is bounded by the L_2-norm of u. For the lower bound we will use the uniform bounds property and the vanishing property of the interpolatory basis functions and for the upper bound we will use the basis functions $\{\psi_i\}_{i=1}^N$.

LEMMA 6. *Let* $\{\phi_i\}_{i=1}^N$ *be the interpolatory basis functions. Then there exists a constant* C, *independent of* h, *such that for* $x \in I_j$ *where* $j = 1, 2, \cdots N$,

$$|\phi_i(x)| \leq \max\{|\phi_i(x_{j-1})|, |\phi_i(x_j)|\}C.$$

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PROOF. Choose a basis function ϕ_i. Since $\phi_i(\xi_j) = \delta_{i,j}$, $\phi_i(x) = (x - \xi_j)(Ax + B)$ on I_j $(j \neq i)$, where A and B are defined in the proof of Lemma 1. Putting $x = x_{i-1} + ah$, $(0 \leq a \leq 1)$, we have

$$\begin{aligned}\phi_i(x) &= (2 + 4a)\phi_i(x_{j-1}) + 2ah\phi_i'(x_{j-1})\\ &= (2 - 2a)\phi_i(x_{j-1}) - 2a\phi_i(x_j) \quad \text{by} \quad (3.1).\end{aligned}$$

Therefore the conclusion holds on I_j. On I_i, using (3.2) and a similar argument, we can verify the same conclusion.

PROPOSITION 1. *Let G_N be the matrix defined by $G_N(i, j) = \int_I \phi_i\phi_j dx$ and let λ_m be a maximal eigenvalue of G_N. Then there exists a positive constant C independent of h and N such that*

$$\lambda_m \leq C \cdot h.$$

PROOF. First, using Lemma 6 and Corollary 2, for $x \in I_k$,

$$|\phi_i(x)| \leq c_1\left(\frac{1}{3}\right)^{i-k} \quad \text{if} \quad k \leq i - 1, \tag{4.1a}$$

$$|\phi_i(x)| \leq c_2\left(\frac{1}{3}\right)^{k-i} \quad \text{if} \quad k \geq i + 1. \tag{4.1b}$$

Now without loss of generality, assume $i \leq j$. On I_k, for $k \leq i - 1$, by (4.1a)

$$|\phi_i(x)\phi_j(x)| \leq c_1^2\left(\frac{1}{3}\right)^{i-k}\left(\frac{1}{3}\right)^{j-k} = c_1^2\left(\frac{1}{3}\right)^{j-i}\left(\frac{1}{3}\right)^{2(i-k)}, \tag{4.2a}$$

and for $k > j$, by (4.1b)

$$|\phi_i(x)\phi_j(x)| \leq c_2^2\left(\frac{1}{3}\right)^{k-i}\left(\frac{1}{3}\right)^{k-j} = c_2^2\left(\frac{1}{3}\right)^{j-i}\left(\frac{1}{3}\right)^{2(k-j)}, \tag{4.2b}$$

and for $i \leq k \leq j$, by (4.1)

$$|\phi_i(x)\phi_j(x)| \leq c_1 \cdot c_2\left(\frac{1}{3}\right)^{k-i}\left(\frac{1}{3}\right)^{j-k} = c_1c_2\left(\frac{1}{3}\right)^{j-i}. \tag{4.2c}$$

Then by (4.2)

$$\begin{aligned}\left|\int_I \phi_i\phi_j dx\right| &\leq \sum_{k=1}^{i-1}\int_{I_k}|\phi_i\phi_j|dx + \sum_{k=i}^{j}\int_{I_k}|\phi_i\phi_j|dx + \sum_{k=j+1}^{N}\int_{I_k}|\phi_i\phi_j|dx\\ &\leq c_3 \cdot h \cdot \left(\frac{1}{3}\right)^{j-i}\left(\frac{1}{4} + j - i\right),\end{aligned}$$

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where c_3 depends only on c_1 and c_2. Hence for either $i > j$ or $i < j$,

$$\left|\int_I \phi_i \phi_j dx\right| \le c_3 \cdot h \cdot \left(\frac{1}{3}\right)^{|j-i|}\left(\frac{1}{4} + |j-i|\right).$$

Then, fixing j,

$$\sum_{i=1}^{N}\left|\int_I \phi_i \phi_j dx\right| \le c_3 \cdot h \cdot \sum_{i=1}^{N}\left(\frac{1}{3}\right)^{|j-i|}\left(\frac{1}{4} + |j-i|\right) = C_N h,$$

where

$$C_N = c_3 \cdot \sum_{s=1}^{N}\left(\frac{1}{3}\right)^{s}\left(\frac{1}{4} + s\right) \quad \text{converges as} \quad N \to \infty.$$

Hence by Geršgorin's theorem, we have the conclusion.

Now we can get the lower bound of $\langle u, u\rangle_N$ by $\|u\|_2^2$.

THEOREM 2. *Let $u \in S_{\Delta,2}$. There exists a positive constant c such that*

$$c\|u\|_0^2 \le \langle u, u\rangle_N.$$

PROOF. Let $u = \sum_{i=1}^{N} u_i \phi_i$. Since $\langle \phi_i, \phi_j\rangle_N = h\delta_{ij}$, we have

$$\langle u, u\rangle_N = \sum_{j=1}^{N} u_i u_j \langle \phi_i, \phi_j\rangle_N = h\sum_{i=1}^{N} u_i^2.$$

With $U = (u_1, \cdots, u_N)^{\mathrm{T}}$, by Proposition 1, we have

$$\int_I u^2 dx = \sum_{i,j=1}^{N} u_i u_j \int_I \phi_i \phi_j dx = U^{\mathrm{T}} G_N U \le c \cdot h \sum_{i=1}^{N} u_i^2 = c \cdot \langle u, u\rangle_N,$$

where G_N is defined as in Proposition 1.

PROPOSITION 2. *For $u \in S_{\Delta,2}$ there is a constant C independent of h, i and N such that, for $i = 1, 2, \cdots, N$,*

$$\langle u, u\rangle_i \le C\|u\|_i^2,$$

where

$$\|u\|_i^2 = (u, u)_{L_2},\ \langle u, u\rangle_i = u^2(\xi_i)h \quad \text{on} \quad I_i.$$

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PROOF. Let us take the basis functions $\{\psi_i\}_{i=1}^N$ for $S_{\Delta,2}$. Assume that u is not identically 0. Since $u \in S_{\Delta,2}$, $u = \sum_{i=1}^N d_i\psi_i$, where each d_i is real. Hence its restriction to I_i is $u = d_{i-1}\psi_{i-1}^r + d_i\psi_i^m + d_{i+1}\psi_{i+1}^l$. Define $\tilde{u}$ on I as $\tilde{u} = (d_{i-1}\sqrt{h})\psi^{or} + (d_i\sqrt{h})\psi^{om} + (d_{i+1}\sqrt{h})\psi^{ol}$. Using linear transformations, we have

$$\int_{I_i} u^2 dx = c\int_I \tilde{u}^2 dx, \quad \langle u, u\rangle_i = c\langle \tilde{u}, \tilde{u}\rangle_N^0, \tag{4.3}$$

where $\langle \tilde{u}, \tilde{u}\rangle_N^0 = [\tilde{u}(\frac{1}{2})]^2$.

Let $d(h) = (d_{i-1}\sqrt{h}, d_i\sqrt{h}, d_{i+1}\sqrt{h})$ and $|d(h)|^2 = d_{i-1}^2 h + d_i^2 h + d_{i+1}^2 h$. Define, with $|a| = a_1^2 + a_2^2 + a_3^2 = 1$,

$$v = \tilde{u}/d(h) = a_1\psi^{or} + a_2\psi^{om} + a_3\psi^{ol}. \tag{4.4}$$

Then by (4.3) and (4.4)

$$\sup_{u\in S_{\Delta,2}} \frac{\langle u, u\rangle_i}{\|u\|_i^2} = \sup_{\tilde{u}} \frac{\langle \tilde{u}, \tilde{u}\rangle_N^0}{\|\tilde{u}\|_0^2} = \sup_{v} \frac{\langle v, v\rangle_N^0}{\|v\|_0^2} = \sup_{|a|=1} \frac{F(a_1, a_2, a_3)}{G(a_1, a_2, a_3)}.$$

Since F and G are continuous on $\{a : |a| = 1\}$, F has a maximum M and G has a minimum m which is nonzero. Therefore

$$\sup_{u\in S_{\Delta,2}} \frac{\langle u, u\rangle_i}{\|u\|_i^2} \le \frac{M}{m} = C \quad \text{for all} \quad i.$$

Hence we have the conclusion.

Therefore the upper bound of $\langle u, u\rangle_N$ comes immediately from Proposition 2. We put this as a theorem here.

THEOREM 3. *Let $u \in S_{\Delta,2}$. There is a positive constant C independent of h and N such that*

$$\langle u, u\rangle_N \le C\|u\|_0^2.$$

PROOF. By Proposition 2, we have

$$\langle u, u\rangle_N = \sum_{i=1}^N \langle u, u\rangle_i \le C\sum_{i=1}^N \|u\|_i^2 = C\|u\|_0^2.$$

5. Basic 1D case

Consider a positive definite self-adjoint differential operator L_d defined on $C^2[0, 1]$ and a differential equation given by

$$L_d u = -u''(x) + du(x) = f(x) \quad \text{on} \quad I, \tag{5.1}$$

where $f \in C[0, 1]$ and d is a nonnegative constant with homogeneous Dirichlet boundary conditions. Let $V = H_0^1(\Omega)$. Define $b : V \times V \to \mathbf{R}$ by

$$b(u, v) = \int_0^1 u'v' + duv\, dx,$$

whose associated norm is $\|u\|_d$ which is equivalent to $\|u\|_1$.

Define $b_N(f, g)$ on $S_{\Delta,2} \times S_{\Delta,2}$ by

$$b_N(f, g) = \langle -f'', g\rangle_N + d\langle f, g\rangle_N \tag{5.3}$$

and define $b_{1,N}(u, v)$ on $S_{\Delta,1} \times S_{\Delta,1}$ by

$$b_{1,N}(u, v) = b(u, v). \tag{5.4}$$

These bilinear forms induce operators B_N and β_N;

$$B_N : S_{\Delta,2} \to S_{\Delta,2} \quad \text{by} \quad \langle B_N f, g\rangle_N = b_N(f, g), \tag{5.5}$$

$$\beta_N : S_{\Delta,1} \to S_{\Delta,1} \quad \text{by} \quad (\beta_N u, v)_{l_2} = b_{1,N}(u, v). \tag{5.6}$$

Let $\hat{B}_N$ be the matrix representation of the operator B_N with the basis $\{\phi_i\}$ and $\tilde{\beta}_N$ the matrix representation of the operator β_N with the basis $\{\theta_i\}$. Let I_N be the one-dimensional quadratic spline interpolation operator

$$I_N : S_{\Delta,1} \to S_{\Delta,2} \tag{5.7a}$$

given by

$$(I_N u)(\xi_i) = u(\xi_i), \quad i = 1, 2, \cdots, N, \quad \text{that is} \quad (I_N u)(x) = \sum_{i=1}^{N} u(\xi_i)\phi_i(x). \tag{5.7b}$$

Using the well-known approximation result of [1], [4] or [11], we have the following approximation result on $S_{\Delta,2}$.

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LEMMA 7. *$S_{\Delta,2}$ possesses the property that for any $u \in H^1(I)$ there exists a $v \in S_{\Delta,2}$ and a constant C independent of h and u such that*

$$\|u - v\|_0 + h\|(u - v)'\|_0 \le C \cdot h\|u'\|_0. \tag{5.8}$$

LEMMA 8. *For $u \in S_{\Delta,2}$ there is a positive constant C independent of h and N such that*

$$\langle u, u\rangle_N^{\frac{1}{2}} \le C \cdot \left[\|u\|_0 + h\|u'\|_0\right].$$

PROOF. This comes from Theorem 3 (or see Lemma 5.1 in [14]).

LEMMA 9. *Let $u \in S_{\Delta,1}$ Then there exists a constant C independent of h and N such that*

$$\|(I_N u)'\|_0 \le C\|u'\|_0.$$

PROOF. Since $(I_N u)' = (I_N u - u)' + u'$, it is enough to show that $\|(I_N u - u)'\|_0 \le c\|u'\|_0$ for some constant c. Since $u \in H_0^1(I)$, by Lemma 7, there is a $v \in S_{\Delta,2}$ satisfying (5.8). Then for such a v we have

$$\begin{aligned}\|(u - I_N u)'\|_0 &\le \|(u - v)'\|_0 + \|(v - I_N u)'\|_0 \\ &\le c\left(\|u'\|_0 + \frac{c}{h}\|v - I_N u\|_0\right).\end{aligned}$$

In the last inequality we used Lemma 7 and the inverse inequality [8]. Then using Theorem 2, (5.7) and Lemma 8 we have, with the chosen v,

$$\|(u - I_N u)'\|_0 \le c\left[\|u'\|_0 + \frac{c}{h}(\|v - u\|_0 + h\|(v - u)'\|_0)\right]. \tag{5.9}$$

Therefore we have the conclusion by applying Lemma 7.

LEMMA 10. *Let $u \in S_{\Delta,1}$. Then we have*

$$\|u'\|_0^2 \sim \langle -(I_N u)'', I_N u\rangle_N.$$

PROOF. Let $\xi_0 = 0$ and $\xi_{N+1} = 1$. Then, by the boundary conditions and (5.7b), we have $(I_N u)(\xi_i) = u(\xi_i)$ for $i = 0, \dots, N + 1$. Using the Fundamental Theorem of Calculus and the Schwartz inequality, we have

$$|u(\xi_i) - u(\xi_{i-1})|^2 = |(I_N u)(\xi_i) - (I_N u)(\xi_{i-1})|^2 \le |\xi_i - \xi_{i-1}| \int_{\xi_{i-1}}^{\xi_i} |(I_N u)'(t)|^2\, dt. \tag{5.10}$$

Therefore, by (5.10),

$$\|u'\|_0^2 = \sum_{i=1}^{N+1} \int_{\xi_{i-1}}^{\xi_i} |u'(t)|^2\, dt \le \sum_{i=1}^{N+1} \int_{\xi_{i-1}}^{\xi_i} |(I_N u)'(t)|^2\, dt = \|(I_N u)'\|_0^2.$$

Finally Lemma 3.3 in [10] implies

$$\|u'\|_0^2 \le \langle -(I_N u)'', I_N u\rangle_N. \tag{5.11}$$

On the other hand, Lemma 3.3 in [10] and Lemma 9 imply that

$$\langle -(I_N u)'', I_N u\rangle_N \le c\, \|u'\|_0^2. \tag{5.12}$$

Hence (5.11) and (5.12) imply the conclusion.

LEMMA 11. *Let* $u = \sum_{i=1}^N \alpha_i \theta_i \in S_{\Delta,1}$ *and* $v = \sum_{i=1}^N \alpha_i \phi_i \in S_{\Delta,2}$. *Then we have*

$$\|u\|_0^2 \sim \langle v, v\rangle_N.$$

PROOF. Note that with $\alpha = (\alpha_1, \cdots, \alpha_N)^{\mathsf T}$, $\|u\|_0^2 = h\alpha^{\mathsf T} A_N \alpha$, where $A_N(i,j) = \frac{1}{h}\int_I \theta_i \theta_j dx$ and $\langle v, v\rangle_N = h\alpha^{\mathsf T}\alpha$. Then the conclusion comes from Geršgorin's Theorem [12] applied to the matrix A_N.

Now we have a main result about the 1-dimensional case.

THEOREM 4. *For* $\alpha = (\alpha_1, \cdots, \alpha_N)^{\mathsf T}$ *we have*

$$(W_N \hat{B}_N \alpha, \alpha) \sim (\tilde{\beta}_N \alpha, \alpha).$$

PROOF. Let $u \in S_{\Delta,1}$. Then

$$u(x) = \sum_{i=1}^N \alpha_i \theta_i(x) \quad \text{and} \quad (I_N u)(x) = \sum_{i=1}^N \alpha_i \phi_i(x) \in S_{\Delta,2}.$$

Thus the vector α represents both u and $I_N u$. By Lemmas 10 and 11,

$$\langle -(I_N u)'', I_N u\rangle_N \sim \|u'\|_0^2 \quad \text{and} \quad \langle I_N u, I_N u\rangle_N \sim \|u\|_0^2,$$

and we have

$$b_N(I_N u, I_N u) \sim \|u\|_1^2. \tag{5.13}$$

Since, by definition,

$$(W_N \hat{B}_N \alpha, \alpha) = b_N(I_N u, I_N u) \quad \text{and} \quad (\tilde{\beta}_N \alpha, \alpha) = b(u, u) \sim \|u\|_1^2,$$

the conclusion holds by (5.13).

6. Basic 2D case

Consider the elliptic differential operator L_d defined on Ω and a differential equation given by

$$L_d u = -[u_{xx} + u_{yy}] + du = f, \tag{6.1}$$

where d is a nonnegative constant with homogeneous Dirichlet boundary condition. In this section we will discuss the preconditioned matrix

$$Q_{N,M} = \tilde{\beta}_{N,M}^{-1} W_{N,M} \hat{B}_{N,M}, \tag{6.2}$$

where $\tilde{\beta}_{N,M}$ is the stiffness matrix of the finite element method of L_d and $\hat{B}_{N,M}$ is the matrix representation of the collocation discretization of L_d. Let $V = H_0^1(\Omega)$.

Define $b : V \times V \to \mathbf{R}$ by

$$b(u, v) = (\nabla u, \nabla v)_\Omega + d(u, v)_\Omega, \tag{6.3}$$

where the associated norm is $\|u\|_d$ which is equivalent to $\|u\|_1$. Define $b_{N,M}(f, g)$ on $S_{\pi,2} \times S_{\pi,2}$ by

$$b_{N,M}(f, g) = \langle -\pi f, g\rangle_{N,M} + d\langle f, g\rangle_{N,M} \quad \text{for} \quad f, g \in S_{\pi,2}, \tag{6.4a}$$

which defines the collocation discretization operator corresponding to L_d

$$B_{N,M} : S_{\pi,2} \to S_{\pi,2} \tag{6.4b}$$

by

$$b_{N,M}(f, g) = \langle B_{N,M} f, g\rangle_{N,M} \quad \text{for} \quad f, g \in S_{\pi,2}. \tag{6.4c}$$

The finite element discretization operator $\beta_{N,M}$ of the operator L_d is defined by

$$\beta_{N,M} : S_{\pi,1} \to S_{\pi,1} \tag{6.5a}$$

and

$$b(u, v) = (\beta_{N,M} u, v)_{l_2} \quad \text{for} \quad u, v \in S_{\Delta,1}. \tag{6.5b}$$

Let $\hat{B}_{N,M}$ and $\tilde{\beta}_{N,M}$ be the matrix representation of the operator $B_{N,M}$ and $\tilde{\beta}_{N,M}$ respectively. Let $I_{N,M}$ be the two-dimensional quadratic spline interpolation operator

$$I_{N,M} : S_{\pi,1} \to S_{\pi,2} \tag{6.6a}$$

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defined by

$$(I_{N,M}u)(\xi_i, \eta_j) = u(\xi_i, \eta_j), \quad i(j) = 1, \cdots, N(M), \tag{6.6b}$$

that is,

$$(I_{N,M}u)(x, y) = \sum_{i=1}^{N} \sum_{j=1}^{M} u(\xi_i, \eta_j)\phi_i(x)\phi_j(y), \tag{6.6c}$$

where (ξ_i, η_j) are the Gaussian points on $I_i \times I_j$.
Let us order the Gaussian points $\{(\xi_i, \eta_j)\}_{i=1}^{N}{}_{j=1}^{M}$ by vertical lines. Then we list the Gaussian points as $P_1, P_2, \cdots, P_{NM}$. Put

$$(\xi_i, \eta_i) = P_\mu \quad \text{where} \quad \mu = j + (i-1)M. \tag{6.7}$$

We order the basis functions in $S_{\pi,2}$ and $S_{\pi,1}$ in the same order. Using the tensor product, we can define the biquadratic basis functions and bilinear functions for $S_{\pi,2}$ and $S_{\pi,1}$ respectively as

$$\Phi_\mu(x, y) = \phi_i(x)\phi_j(y) \in S_{\pi,2}, \quad \Theta_\mu(x, y) = \theta_i(x)\theta_j(y) \in S_{\pi,1}. \tag{6.8}$$

Let us decompose the operator L_d by

$$L_d = L_x + L_y, \quad L_x = -u_{xx} + (d/2)u \quad \text{and} \quad L_y = -u_{yy} + (d/2)u. \tag{6.9}$$

THEOREM 5. ([19, page 136], [17, Theorem 5.2])

(1) *Let $\tilde{\beta}_N^x$ and $\tilde{\beta}_M^y$ be the stiffness matrices associated with finite element discretization of L_x and L_y respectively in the finite element space $S_{\pi,1}$. Let M_N and M_M be the corresponding mass matrices. Then*

$$\tilde{\beta}_{N,M} = \tilde{\beta}_N^x \otimes M_M + M_N \otimes \tilde{\beta}_M^y. \tag{6.10}$$

(2) *Let $\hat{B}_N^x(i, m) = (L_x\phi_m)(\xi_i)$, $\hat{B}_M^y(j, n) = (L_y\phi_n)(\eta_j)$ and $\tilde{B}_N^x = W_N\hat{B}_N^x$, $\tilde{B}_M^y = W_M\hat{B}_M^y$, $\tilde{B}_{N,M} = W_{N,M}\hat{B}_{N,M}$.*

Then the matrix representation of the collocation discretization $\hat{B}_{N,M}$ of the operator L_d in $S_{\pi,2}$ is given by

$$\hat{B}_{N,M} = \hat{B}_N^x \otimes I_M + I_N \otimes \hat{B}_M^y, \tag{6.11}$$

and the symmetrized collocation matrix is given by

$$\tilde{B}_{N,M} = \tilde{B}_N^x \otimes W_M + W_N \otimes \tilde{B}_M^y. \tag{6.12}$$

THEOREM 6. *For every* $\alpha = (\alpha_1, \cdots, \alpha_N)^{\mathsf{T}}$, *we have* $(M_N\alpha, \alpha) \sim (W_N\alpha, \alpha)$.

PROOF. Let $u \in S_{\Delta,1}$. Then $\|u\|_0^2 = (M_N\alpha, \alpha)$ and $\langle I_N u, I_N u\rangle_N = (W_N\alpha, \alpha)$.. By Lemma 11, the conclusion holds.

LEMMA 12. *For every vector* $\alpha = (\alpha_1, \cdots, \alpha_{NM})^{\mathsf{T}}$ *we have*

(1) $((\tilde{\beta}_N^x \otimes M_N)\alpha, \alpha) \sim ((\tilde{B}_N^x \otimes W_M)\alpha, \alpha)$,

(2) $((M_N \otimes \tilde{\beta}_M^y)\alpha, \alpha) \sim ((W_N \otimes \tilde{B}_M^y)\alpha, \alpha)$.

PROOF. Because of Theorem 4 and Theorem 6, we have the conclusions following a proof similar to Lemma 5.4 in [14].

Now we will close this section with one of the main results.

THEOREM 7. *For every vector* $\alpha = (\alpha_1, \cdots, \alpha_{NM})^{\mathsf{T}}$ *we have*

$$(\tilde{\beta}_{N,M}\alpha, \alpha) \sim (\tilde{B}_{N,M}\alpha, \alpha).$$

PROOF. By Lemma 12 and Theorem 5, we have the conclusion.

7. General preconditioning

For a uniformly elliptic operator with variable coefficients L_a defined in Section 1, define $a_{N,M}(u, v)$ on $S_{\pi,2} \times S_{\pi,2}$ by

$$a_{N,M}(u, v) = \langle L_a u, v\rangle_{N,M}, \tag{7.1}$$

which defines the operator

$$A_{N,M} : S_{\pi,2} \to S_{\pi,2} \tag{7.2a}$$

by

$$a_{N,M}(u, v) = \langle A_{N,M} u, v\rangle_{N,M} \quad \text{for} \quad u, v \in S_{\pi,2}. \tag{7.2b}$$

Let $\hat{A}_{N,M}$ be the matrix representation of the operator $A_{N,M}$. Recall the uniformly elliptic operators L_d defined in Section 6. Note that those two elliptic operators have the same boundary conditions. In this section we will discuss the matrix

$$L_{N,M} = \tilde{\beta}_{N,M}^{-1} W_{N,M} \hat{A}_{N,M} = \tilde{\beta}_{N,M}^{-1} \tilde{A}_{N,M} \tag{7.3}$$

and its $\tilde{\beta}_{N,M}$ condition number.

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LEMMA 13. *Let $u, v \in S_{\Delta,2}$. With the notation $t = x$ or y we have for any $\epsilon > 0$ and different constants C independent of N and M for each case satisfying*

(1) $|\langle u, v\rangle_{N,M}| \le C\left[\int_\Omega u^2 dxdy\right]^{\frac{1}{2}}\left[\int_\Omega v^2 dxdy\right]^{\frac{1}{2}}$,

(2) $|\langle u_t, u_t\rangle_{N,M}| \le C\left[\int_\Omega u_t^2 dxdy\right]$,

(3) $|\langle u_t, v\rangle_{N,M}| \le C\left[\epsilon \int_\Omega u_t^2 dxdy + \frac{1}{\epsilon}\int_\Omega v^2 dxdy\right]$,

(4) $|\langle -u_{tt}, v\rangle_{N,M}| \le C\left[\frac{1}{\epsilon}\int_\Omega u_t^2 dxdy + \epsilon \int_\Omega v_t^2 dxdy\right]$.

PROOF. Inequality (1) follows from the definition of $\langle\cdot,\cdot\rangle_{N,M}$ and the Schwartz inequality, and follows also from Theorem 3. For (2) note that, using a proof similar to Lemma 3.1 of [9], we have $\langle f', f'\rangle_N \le (f', f')$ for $f \in S_{\Delta,2}$. With this inequality, (2) follows from the definition of $\langle\cdot,\cdot\rangle_{N,M}$ and Theorem 3. For (3), first note that for any positive ϵ,

$$ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2. \tag{7.4}$$

Then (3) follows from the Schwartz inequality, (7.4) and inequalities (1) and (2) of this lemma. For (4) note that from Lemma 3.1 of [9] and [10], for $f, g \in S_{\Delta,2}$

$$\langle -f'', g\rangle_N = (f', g') + p_2 \sum_{i=1}^{N} f'''g'''h^5, \tag{7.5a}$$

where p_2 is a positive constant independent of h. Using the Schwartz inequality and the inverse estimate (see [6]), we have a positive constant c independent of h such that

$$\left|\sum_{i=1}^{N} f'''g'''h^5\right| \le c\|f'\|_0\|g'\|_0. \tag{7.5b}$$

Then applying the Schwartz inequality to (f', g'), we have from (7.5),

$$|\langle -f'', g\rangle_N| \le c\left(\int_I |f'(x)|^2\,dx\right)^{\frac{1}{2}}\left(\int_I |g'(x)|^2\,dx\right)^{\frac{1}{2}}. \tag{7.6}$$

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Finally by the definition of $\langle\cdot,\cdot\rangle_{N,M}$, (7.6) (7.4) and Theorem 3,

$$
\begin{aligned}
\left|\langle -u_{xx}, v\rangle_{N,M}\right| &\le \sum_{j=1}^{M} s\left|\langle -u_{xx}(x,\eta_j), v(x,\eta_j)\rangle_N\right| \\
&\le c\sum_{j=1}^{M} s\left[\int_I u_x^2(x,\eta_j)dx\right]^{\frac{1}{2}}\left[\int_I v_x^2(x,\eta_j)dx\right]^{\frac{1}{2}} \\
&\le C\left[\frac{1}{\epsilon}\int_I \sum_{j=1}^{M} s u_x^2(x,\eta_j)dx + \epsilon\int_I\sum_{j=1}^{M} s v_x^2(x,\eta_j)dx\right] \\
&\le C\left[\frac{1}{\epsilon}\int_\Omega u_x^2 + \epsilon\int_\Omega v_x^2\right].
\end{aligned} \tag{7.7}
$$

In order to discuss the operator

$$S_{N,M} = B_{N,M}^{-1}A_{N,M} : S_{\pi,2} \to S_{\pi,2}, \tag{7.8}$$

let us define a differential operator L_K with the same boundary conditions as L_a,

$$L_K u = -[u_{xx} + u_{yy}] + a_1 u_x + a_2 u_y + K u, \tag{7.9}$$

where K is a sufficiently large positive constant. Define an operator

$$A_{N,M}^{K} : S_{\pi,2} \to S_{\pi,2} \tag{7.10a}$$

by

$$\langle A_{N,M}^{K} u, v\rangle_{N,M} = \langle L_K u, v\rangle_{N,M}. \tag{7.10b}$$

THEOREM 8. *For* $u \in S_{\pi,2}$

(1) $\displaystyle\int_\Omega u_x^2 + u_y^2 dxdy \sim \langle -\Delta u, u\rangle_{N,M}$,

(2) $\displaystyle\int_\Omega u^2 dxdy \sim \langle u, u\rangle_{N,M}$.

PROOF. By definition of $\langle\cdot,\cdot\rangle_{N,M}$, we have

$$\langle -u_{xx}, u\rangle_{N,M} = \sum_{l=1}^{M} s\langle -u_{xx}(x,\eta_l), u(x,\eta_l)\rangle_N. \tag{7.11}$$

Then by Theorem 2 and the fact that (see Lemma 3.3 in [10])

$$\langle -u'', u\rangle_N \ge \int_I |u'|^2\, dx,$$

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we have

$$\langle -u_{xx}, u\rangle_{N,M} \geq C \int_\Omega u_x^2 dxdy. \tag{7.12a}$$

Putting $u = v$ and $\epsilon = 1$ in (7.7), we have

$$\langle -u_{xx}, u\rangle_{N,M} \leq C \int_\Omega u_x^2 dxdy. \tag{7.12b}$$

Hence by (7.12)

$$\langle -u_{xx}, u\rangle_{N,M} \sim \int_\Omega u_x^2 dxdy. \tag{7.13a}$$

In a similar way,

$$\langle -u_{yy}, u\rangle_{N,M} \sim \int_\Omega u_y^2 dxdy. \tag{7.13b}$$

Therefore we have (1) by (7.13).

By the definition of $\langle \cdot, \cdot\rangle_{N,M}$, we have

$$\langle u, u\rangle_{N,M} = \sum_{l=1}^{M} s\langle u(x,\eta_l), u(x,\eta_l)\rangle_N. \tag{7.14}$$

Then by Theorem 2,

$$\langle u, u\rangle_{N,M} \geq c\sum_{l=1}^{M} s\int_I u^2(x,\eta_l)dx \geq c^2 \int_\Omega u^2(x,y)dxdy. \tag{7.15a}$$

By applying Theorem 3 to (7.14)

$$\langle u, u\rangle_{N,M} \leq c^2 \int_\Omega u^2(x,y)dxdy. \tag{7.15b}$$

Hence we have (2).

Using the argument of Theorem 3.2 in [16] or following the argument of Theorem 6.1 in [14], we have the following result.

THEOREM 9. *Let $u \in S_{\pi,2}$. If $\min(N, M) \geq N_0$ for some positive integer N_0 then*

$$\|u\|_1 \sim \|B_{N,M}^{-1} A_{N,M}^K u\|_1.$$

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PROOF. Let $v = B_{N,M}^{-1} A_{N,M}^{K} u$. Then by Theorem 8 there is a constant c such that

$$\begin{aligned}\langle A_{N,M}^{K} u, v\rangle_{N,M} &= \langle B_{N,M} v, v\rangle_{N,M} \\ &= \langle -\Delta v, v\rangle_{N,M} + d\langle v, v\rangle_{N,M} \\ &\geq c(\|\nabla v\|_0^2 + \|v\|_0^2).\end{aligned} \tag{7.16a}$$

On the other hand, by Lemma 13 and (7.4)

$$\begin{aligned}|\langle A_{N,M}^{K} u, v\rangle_{N,M}| &\leq |\langle -\Delta u, v\rangle_{N,M}| + |\langle a_1 u_x, v\rangle_{N,M}| \\ &\quad + |\langle a_2 u_y, v\rangle_{N,M}| + K|\langle u, v\rangle_{N,M}| \\ &\leq c(\|\nabla u\|_0^2 + \|u\|_0^2).\end{aligned} \tag{7.16b}$$

Hence by (7.16)

$$\|v\|_1 = \|B_{N,M}^{-1} A_{N,M}^{K} u\|_1 \leq c\|u\|_1. \tag{7.17}$$

Since $A_{N,M}^{K} u = B_{N,M} v$, we have

$$\langle A_{N,M}^{K} u, u\rangle_{N,M} = \langle B_{N,M} v, u\rangle_{N,M}. \tag{7.18}$$

Now by the fact that K is sufficiently large, Lemma 13, (7.17) and (7.18),

$$\|u\|_1 \leq C\|B_{N,M}^{-1} A_{N,M}^{K} u\|_1 = C\|v\|_1. \tag{7.19}$$

Therefore by (7.17) and (7.19) the conclusion follows.

REMARK. The existence of $B_{N,M}^{-1}$ comes from Lemma 6.1 in [17] if $\min(N, M) \geq N_0$ for some positive constant N_0.

Because of Theorem 9, we have the following lemma using the same argument of Lemma 6.7, Theorem 6.2 in [14] *verbatim*.

LEMMA 14. *For every $u \in S_{\pi,2}$, if $\min(N, M) \geq N_0$ then*

$$\|u\|_1 \sim \|B_{N,M}^{-1} A_{N,M} u\|_1.$$

Translating Lemma 14 into a statement about matrices, we have the following

THEOREM 10. *For any $U = (u_1, \cdots, u_{N,M})^{\mathsf{T}}$ let $V = \hat{B}_{N,M}^{-1} \hat{A}_{N,M} U$. Then $(\tilde{\beta}_{N,M} U, U) \sim (\tilde{\beta}_{N,M} V, V)$.*

PROOF. Since $\|u\|_1^2 \sim (\tilde{\beta}_{N,M} U, U)_{l_2}$, by Lemma 14 we have the conclusion.

Finally we will discuss the matrix

$$L_{N,M} = \beta_{N,M}^{-1} W_{N,M} \hat{A}_{N,M} = \tilde{\beta}_{N,M}^{-1} \tilde{A}_{N,M}$$

and its $\tilde{\beta}_{N,M}$ condition number and $\tilde{\beta}_{N,M}$ singular values. Since $\tilde{\beta}_{N,M}$ is positive definite, we can define an inner product $(U, V)_{\tilde{\beta}_{N,M}} := (\tilde{\beta}_{N,M} U, V)$. Let S be any other real matrix. The $\tilde{\beta}_{N,M}$ adjoint of S is that unique matrix S^* such that

$$(SU, V)_{\tilde{\beta}_{N,M}} = (U, S^* V)_{\tilde{\beta}_{N,M}}.$$

Then $S^* = \tilde{\beta}_{N,M}^{-1} S^{\mathrm{T}} \tilde{\beta}_{N,M}$, where S^{T} is the transpose of S. The $\tilde{\beta}_{N,M}$-singular values of $L_{N,M}$ are the square roots of the eigenvalues of $L_{N,M}^* L_{N,M}$. Now we have the main theorem.

THEOREM 11. *Assume $min(N, M) \geq N_0$. Then for every vector $U = (u_1, \cdots u_{N,M})^{\mathrm{T}}$*

$$(\tilde{\beta}_{N,M} U, U)_{l_2} \sim (\tilde{\beta}_{N,M} L_{N,M} U, L_{N,M} U)_{l_2}.$$

That is, there are two positive constants α and β, independent of N and M, such that the $\tilde{\beta}_{N,M}$-singular values of $L_{N,M}$, denoted by $\sigma_j(N, M)$, satisfy

$$0 < \alpha \leq \sigma_j(N, M) \leq \beta.$$

PROOF. Recall first $Q_{N,M} = \tilde{\beta}_{N,M}^{-1} W_{N,M} \hat{B}_{N,M} = \tilde{\beta}_{N,M}^{-1} \tilde{B}_{N,M}$ which is similar to $\tilde{\beta}_{N,M}^{-\frac{1}{2}} \tilde{B}_{N,M} \tilde{\beta}_{N,M}^{-\frac{1}{2}}$. Therefore by Theorem 4 we have

$$(\tilde{\beta}_{N,M} U, U) \sim (\tilde{\beta}_{N,M} Q_{N,M} U, Q_{N,M} U). \tag{7.20}$$

Write

$$L_{N,M} = Q_{N,M} \hat{S}_{N,M}, \tag{7.21}$$

where $\hat{S}_{N,M}$ is the matrix representation of the operator $S_{N,M}$ defined in (7.8). Let

$$V = \hat{S}_{N,M} U = \hat{B}_{N,M}^{-1} \hat{A}_{N,M} U. \tag{7.22}$$

Then by (7.21) and (7.22),

$$(\tilde{\beta}_{N,M} V, V) \sim (\tilde{\beta}_{N,M} L_{N,M} U, L_{N,M} U). \tag{7.23}$$

Hence by Theorem 10 and (7.23), we have

$$(\tilde{\beta}_{N,M} U, U) \sim (\tilde{\beta}_{N,M} L_{N,M} U, L_{N,M} U).$$

Therefore the proof is complete.

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