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COMPACTNESS AND ALMOST PERIODICITY OF MULTIPLIERS

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ABSTRACT. The question as to the existence of nontrivial compact or weakly compact multipliers between spaces of functions on groups has been investigated for several years. Until now, however, no general method which is applicable to a large class of function spaces seems to be known.

In this paper we prove that the existence of nontrivial compact multipliers between Banach function spaces on which a group acts is related to the existence of nonzero almost periodic functions.

1. **Introduction.** The question as to the existence of nontrivial compact or weakly compact multipliers between spaces of functions on groups has been investigated during the last several years, and answers pertaining to the particular spaces under consideration have been furnished. Until now, however, no general method applicable to a large class of function spaces seems to be known; only for compact multipliers on commutative Banach algebras (which are not necessarily connected with groups) some generality is obtained ([5], [6]).

In this paper we investigate the case of the existence of nontrivial compact or weakly compact multipliers between two Banach spaces (whether equal or not) on which a (not necessarily commutative) group acts. Using the notion of almost periodicity we arrive in section 3 at a good general criterion to determine the non-existence of such multipliers. In particular, many known special results may be derived from it. In section 4 we show that compactness of a multiplier is connected with the notion of equi-almost periodicity; this leads to other characterizations of compact multipliers.

2. **Definitions and notations.** Let X be a Banach space, and G a group. We say that G acts to the left on X if there exists a mapping from the cartesian product $G \times X$ to X such that, to $a \in G$ and $f \in X$ there corresponds an element ${}_af$ of X; we call ${}_af$ the left translate of f; we assume that ${}_a(f+g) = {}_af + {}_ag$ and that $\|{}_af\| \leq \|f\|$. Examples of such an action are easily furnished by taking for X the space $L_p(G)$ $(1 \leq p \leq \infty)$ with respect to left Haar measure on a (non necessarily commutative) locally compact group G, or a Segal algebra

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on G, while the action is the usual left translation. We call an element f of X left almost periodic (l.a.p.) [respectively left weakly almost periodic (l.w.a.p.)] if the set $\{_af : a \in G\}$ of left translates of f is relatively compact in the norm topology [respectively the weak topology] of X. The set of l.a.p. [l.w.a.p.] elements of X is denoted by $ap(X)[w(X)]$.

Suppose that G acts in the described manner on two Banach spaces X and Y. A bounded linear transformation $T: X \to Y$ is called a multiplier if T "commutes with translation", i.e. if $T(_af) = {}_a(Tf)$, for all $a \in G$ and all $f \in X$. For a multiplier T we are able to speak of the translate $_aT$, which is that bounded linear transformation which to each f in X associates the element $(_aT)(f) = {}_a(Tf) = T(_af)$ in Y. A multiplier T is called uniformly almost periodic [strongly almost periodic] [weakly almost periodic] if the set $\{_aT : a \in G\}$ of translates of T is relatively compact with respect to the uniform operator topology [strong operator topology] [weak operator topology] on the set $B(X, Y)$ of bounded linear transformations from X to Y. The corresponding sets of multipliers are denoted respectively by UAP, SAP, WAP. Obviously, $UAP \subset SAP \subset WAP$.

3. **Compactness and almost periodicity.** Consider a fixed f in X, and let $T: X \to Y$ be a multiplier. The mapping $F: B(X, Y) \to Y$ such that $F(U) = U(f)$ for all U in $B(X, Y)$ is continuous when Y has the norm topology while $B(X, Y)$ has the uniform or strong operator topology, and when Y has the weak topology while $B(X, Y)$ has the weak operator topology; also $F(\{_aT : a \in G\}) = \{_a(Tf) : a \in G\}$. This leads us to the following conclusions, which we state in the form of a proposition.

PROPOSITION 3.1. *If $T \in UAP[SAP][WAP]$, then $Tf \in ap(Y)[ap(Y)][w(Y)]$ for all $f \in X$. Hence, if $T \in UAP$ or SAP and if $T(X)$ contains no l.a.p. elements except zero, then $T = 0$; if $T \in WAP$ and if $T(X)$ contains no l.w.a.p. elements except zero, then $T = 0$.*

PROPOSITION 3.2. *Let $T: X \to Y$ be a multiplier. If each element in $T(X)$ is l.a.p., then T belongs to SAP. If each element in $T(X)$ is l.w.a.p., then T belongs to WAP.*

Proof. This result may be derived from exercise VI.9.2 in [3]. We present here a short proof for the first part of the proposition.

Suppose that Tf is l.a.p. for each f in X. Consider the cartesian product $\prod_{f \in X} Y_f$, where each Y_f is Y with its norm topology. Then $B(X, Y) \subset \prod_{f \in X} Y_f$, and the relative topology of $B(X, Y)$ induced by the product topology is precisely the strong operator topology. For each f in X let Z_f denote the closure of $\{_a(T_f) : a \in G=$ in the norm topology of Y. Since each Z_f is compact by hypothesis, we have that $\prod_{f \in X} Z_f$ is compact in the product topology.

Now $\{_aT : a \in G\}$ is contained in $\prod_{f\in X} Z_f$ be definition and so its closure is compact in the product topology and hence compact in the strong operator topology. Thus, T belongs to SAP.

The second part of the proposition is proved analogously by giving each Y_f the weak topology. ■

From propositions 3.1 and 3.2 we derive

COROLLARY 3.3.

$$T \in SAP \Leftrightarrow Tf \in ap(Y), \qquad \forall f \in X.$$
$$T \in WAP \Leftrightarrow Tf \in w(Y), \qquad \forall f \in X.$$

PROPOSITION 3.4. *Let $T: X \to Y$ be a multiplier. If T is compact, then T belongs to SAP. If T is weakly compact, then T belongs to WAP.*

Proof. Suppose that T is compact. If $b(X)$ denotes the closed unit ball in X we have, for $f \in b(X)$:

$$\{_a(Tf) : a \in G\} = \{T(_af) : a \in G\} \subset \{T(g) : g \in b(X)\},$$

and the last set is relatively compact with respect to the norm topology of Y. Hence $Tf \in ap(Y)$, for each $f \in b(X)$. The same is of course true for $f \in X$ with $\|f\| > 1$ since $\|f\|^{-1} f \in b(X)$. So, in view of corollary 3.3, T belongs to SAP.

The statement about a weakly compact multiplier is proved analogously. ■

From corollary 3.3 and proposition 3.4 we immediately see that, *if Y has no nonzero l.a.p. elements, then there do not exist nontrivial compact multipliers from X to Y; if Y has no nonzero l.w.a.p. elements, then there do not exist nontrivial weakly compact multipliers.* In particular, if we take for Y the space $L_p(G)$ $(1 \le p < \infty)$ with its usual norm and G is not compact, then there are no compact multipliers from any X to $L_p(G)$; if Y is $L_1(G)$ for non-compact G, or any Segal algebra $S(G)$ [11] on a non-compact group G, then there are no weakly compact multipliers from any X to $L_1(G)$ or $S(G)$. These facts generalize many of the results appearing in the literature (e.g. [2], [4], [7], [8]), and they all are a consequence of the absence of l.a.p. or l.w.a.p. elements in the mentioned spaces (see e.g. [1], [2], [10]).

4. **Compactness and equi-almost periodicity**

DEFINITION. Let A be a subset of X invariant under left translation. We call A left equi-almost periodic (l.e.a.p.) if the following condition is true: given $\varepsilon > 0$, there exists a finite subset $F \subset G$ with the property that for every $a \in G$ there exists $b \in F$ such that $\|_af - {}_bf\| < \varepsilon$ for all $f \in A$. This definition is motivated by Lemma 3 in Loomis [9], which states that a finite set of l.a.p. functions is l.e.a.p. In this paper the property is named "uniformly almost periodic".

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PROPOSITION 4.1. *Let $T:X \to Y$ be a multiplier. If T is compact then $T(b(X))$ is l.e.a.p.*

Proof. Let $\varepsilon > 0$ be given. Since T is compact, the set $T(b(X)) = \{Tf : f \in b(X)\}$ is totally bounded in Y. This means that there exist a finite number of elements $\{f_j\}_{j=1}^n$ in $b(X)$ such that, for given f in $b(X)$, an f_k ($k \in \{1, \ldots, n\}$) may be found such that $\|Tf - Tf_k\| < \varepsilon/3$. From proposition 3.4 we deduce that each Tf_j is l.a.p. in Y; hence the finite set $\{Tf_1, \ldots, Tf_n\}$ is l.e.a.p. by the previously mentioned result of Loomis [9], i.e., there exist $a_1, \ldots, a_m$ in G such that for each a in G an a_i ($i \in \{1, \ldots, m\}$) may be found such that $\|_a(Tf_j) - {}_{a_i}(Tf_j)\| < \varepsilon/3$ for $j = 1, \ldots, n$. Now, let $a \in G$ be given and a_i chosen as above. For f in $b(X)$ we have

$$\|_a(Tf) - {}_{a_i}(Tf)\| \leq \|_a(Tf) - {}_a(Tf_k)\| + \|_a(Tf_k) - {}_{a_i}(Tf_k)\| + \|_{a_i}(Tf_k) - {}_{a_i}(Tf)\|.$$

Since, by definition, $\|_a(Tf) - {}_a(Tf_k)\| = \|_a(Tf - Tf_k)\| \leq \|Tf - Tf_k\|$, each term in the right hand side of the triangle inequality is smaller than $\varepsilon/3$. Hence the result. ■

PROPOSITION 4.2. *Let $T:X \to Y$ be a multiplier. If T belongs to UAP, then $T(b(X))$ is l.e.a.p.*

We omit the easy proof. ■

It is not at all clear whether or not the converse of proposition 4.1 is always true. We show that the converse certainly holds when Y is the space $B(G)$ of bounded functions on G with the supremum norm $\| \ \|_\infty$.

PROPOSITION 4.3. *Let $T:X \to B(G)$ be a multiplier. If $T(b(X))$ is l.e.a.p., then T is compact.*

Proof. Let $\varepsilon > 0$ be given. There exist $a_1, \ldots, a_n$ in G such that, for each a in G, a point a_i ($1 \leq i \leq n$) may be found such that $\|_{a^{-1}}(Tf) - {}_{a_i^{-1}}(Tf)\|_\infty < \varepsilon/3$, for all f in $b(X)$. Since for each a in G and each f in $b(X)$, $|(Tf)(a)| \leq \|T\|$, the set $\{((Tf)(a_1), \ldots, (Tf)(a_n)) : f \in b(X)\}$ is bounded (= totally bounded) in $\mathscr{C}^n$, the n-dimensional complex space. So there exist $f_1, \ldots, f_m$ in $b(X)$ such that, for each f in $b(X)$, an f_j ($1 \leq j \leq m$) may be found such that $|(Tf)(a_i) - (Tf_j)(a_i)| < \varepsilon/3$ for all $i \in \{1, \ldots, n\}$. Using the triangle inequality we obtain $|(Tf)(a) - (Tf_j)(a)| < \varepsilon$ for all a in G, and so $\|Tf - Tf_j\|_\infty = \sup_{a \in G} |(Tf)(a) - (Tf_j)(a)| \leq \varepsilon$. Hence the set $\{Tf : f \in b(X)\}$ is totally bounded, which means that T is compact. ■

5. Concluding remarks

5.1. The combination of propositions 4.2 and 4.3 gives: if $T:X \to B(G)$ belongs to UAP, then T is compact.

5.2. By specialization of the spaces X and Y, other relations between the

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different kinds of multipliers introduced in this paper may be obtained. For instance, if $X \equiv L_1(G)$ and $Y \equiv L_\infty(G)$, each multiplier $T: L_1(G) \to L_\infty(G)$ is a convolution-type operator H_θ induced by an element θ of $L_\infty(G)$; it is known ([1], [12]) that H_θ is compact iff θ is l.a.p. Since $\|\theta\| = \|H_\theta\|$, we obtain in this case: $T: L_1(G) \to L_\infty(G)$ is compact iff T belongs to UAP iff $T(b(L_1(G)))$ is l.e.a.p.

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