ISOMORPHISMS ON COUNTABLE VECTOR SPACES WITH RECURSIVE OPERATIONS

ROBERT I. SOARE

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Terminology and notation may be found in Dekker [1] and [2]. Briefly, we fix a *recursively enumerable* (r.e.) field F with recursive structure, and let $\bar{U}$ be the vector space over F consisting of ultimately vanishing countable sequences of elements of F with the usual definitions of vector addition and multiplication by a scalar. A subspace V of $\bar{U}$ is called an *α-space* if V has a basis B which is contained in some r.e. linearly independent set S.

Definition. For subspaces $V, W \subseteq \bar{U}$, we write

(i) $V \simeq W$ if there is a 1:1 partial recursive function ψ such that domain ψ (denoted by dom ψ) and range ψ (ran ψ) are subspaces of $\bar{U}$, and ψ is a (vector space) isomorphism from dom ψ to ran ψ mapping V onto W.

(ii) $V \cong W$ if $V \simeq W$ via some ψ such that dom ψ = ran $\psi = \bar{U}$.

J. N. Crossley and A. G. Hamilton have asked whether the Karp-Myhill theorem [3, p. 200] can be extended to vector spaces, namely, whether:

(1) $$V_1 \oplus V_2 = \bar{U} = W_1 \oplus W_2, \text{ and}$$

(2) $$V_1 \simeq W_1 \,\&\, V_2 \simeq W_2 \text{ imply}$$

(3) $$V_1 \cong W_1.$$

We settle the question by proving:

Theorem 1. (1) *and* (2) *do not imply* (3) *even if both* V_1 *and* V_2 *are α-spaces, and even via the same r.e. linearly independent set* S (*that is* S *contains bases for both* V_1 *and* V_2).

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THEOREM 2. (1) *and* (2) *do imply* (3) *if* V_1 *and* V_2 *are* α*-spaces via the same r.e. linearly independent set* S, *and if there exist functions* ψ_i, *for* $i \in \{1,2\}$, *witnessing* $V_i \simeq W_i$ *which satisfy* $\psi_i(S \cap V_j) \subseteq W_j$ *for all* $i,j \in \{1,2\}$.

Theorem 2 follows by an extension of the standard Karp-Myhill technique. Theorem 1 is proved by a priority argument like that which the author used in [6, Theorem 1] to prove the failure of the Karp-Myhill analogue for partial recursive order preserving maps on Dedekind cuts. Theorem 2 above and Theorem 1 of [6] together suggest that while the original Karp-Myhill theorem holds for unstructured *sets*, it rarely holds when the maps are required to preserve even weak structure.

Let $\{\phi_e\}_{e \in N}$ be an acceptable numbering of all partial recursive functions as in Rogers [5, p. 41], and let $\phi_e^s(x)$ denote the result (if any) after performing s steps in the computation of $\phi_e(x)$. Let u, v, w, x, y, z (possibly with subscripts) denote vectors in $\bar{U}$; a, b, c denote scalars in F; and $e, i, j, k, m, n, p, q, s$, and t denote members of N, the set of all natural numbers. Given vectors $x_1, x_2, \cdots \in \bar{U}$, let $L(x_1, x_2, \cdots)$ denote the subspace spanned by them, and let $V_1 \oplus V_2 = \bar{U}$ denote the usual vector space decomposition.

1. Concerning Theorem 1

The diagonalization device to be used in the proof of Theorem 1 suggests the following very short proof of a result of Osofsky [1, p. 385], which has been generalized [4, p. 93].

THEOREM (Osofsky). *There is a subspace* $V \subseteq \bar{U}$ *which is not an* α*-space.*

PROOF. Let $\{A_n\}_{n \in N}$ be a (noneffective) enumeration of all infinite r.e. linearly independent sets $\subseteq \bar{U}$. $S^0 = \varnothing$. Given S^n, let x_n and y_n be the first two elements of A_n such that

$$L(x_n, y_n) \cap L(S^n) = (0),$$

the zero vector. Let $S^{n+1} = S^n \cup \{x_n, y_n\}$. Let $V = L(\{x_n + y_n\}_{n \in N})$. Clearly, V is not an α-space since if $B \subseteq A_n$ is a basis for V, then $x_n + y_n \in V$ implies $x_n, y_n \in B$, but $x_n, y_n \notin V$.

THEOREM 1. *There exists an r.e. linearly independent set* S *and* α*-spaces* V_1, V_2, W_1, *and* $W_2 \subseteq \bar{U}$ *which satisfy* (1) *and* (2), *but not* (3), *and such that* S *contains bases for both* V_1 *and* V_2.

PROOF OF THEOREM 1. We must construct partial recursive functions ψ and θ, and α-spaces V_1, V_2, W_1, W_2 such that $V_1 \simeq W_1$ via ψ and $V_2 \simeq W_2$ via θ, but $V_1 \not\simeq W_1$ via any ϕ_e. Let $\{w_n\}_{n \in N}$ be a recursive basis for $\bar{U}$. Define $S = \{w_n\}_{n \in N}$. Let $x_n = w_{2n}$, $y_n = w_{2n+1}$, for all $n \in N$.

For V_1 we shall define below a basis A which contains exactly one of x_n, y_n for each $n \in N$. We then let $B = S - A$ be a basis for V_2. Clearly $V_1 = L(A)$ and $V_2 = L(B)$ are α-spaces via S, and $V_1 \oplus V_2 = \bar{U}$.

We let θ be the identity map, and $W_2 = L(B)$. We shall define $\psi(x_n) = x_n$, for all $n \in N$, and in addition if $y_n \in A$, then $\psi(y_n)$ is defined and in $L(x_n, y_n)$ but not in $L(x_n)$. Note that $\psi(y_n) \notin L(x_n)$ insures that ψ (canonically extended to $L(\mathrm{dom}\ \psi)$) is an isomorphism, and that $W_1 \oplus W_2 = \bar{U}$.

We shall define A and ψ by a sequence of stages during which we may remove from A some x_n, replace it by y_n, and define $\psi(y_n)$. Let A^s and ψ^s denote the approximations to A and ψ at the end of stage s. Once added to A, y_n is never removed, so $A = \lim_s A^s$ is well-defined. Define $W_1^s = L(\psi^s(A^s))$.

To insure that $V_1 \not\cong W_1$ via ϕ_e we shall select at each stage s a certain index $\gamma(s,e)$ and attempt to arrange that if $\phi_e^s(y_{\gamma(s,e)})$ is defined then either:

$$y_{\gamma(s,e)} \notin A^s \ \&\ \phi_e^s(y_{\gamma(s,e)}) \in W_1^s; \text{ or} \tag{4}$$

$$y_{\gamma(s,e)} \in A^s \ \&\ \phi_e^s(y_{\gamma(s,e)}) \notin W_1^s, \tag{5}$$

in which case we say that the eth requirement (denoted by R_e) *is satisfied at stage s.*

Once requirement R_e is satisfied at some stage $s+1$ we must attempt to preserve the second clause of (4) or (5) by preventing W_1^{s+1} from later changing with respect to members relevant to ϕ_e^{s+1} $(y_{\gamma(s,e)})$. We cannot accomplish this absolutely, but it will suffice to prevent any requirements R_i of *lower priority* namely, $i > e$, from causing such a change. This is easily accomplished by defining $\gamma(s+1, i)$, for all $i > e$, to be sufficiently large. It is helpful to visualize a sequence of movable markers $\{\Gamma_j\}_{j \in N}$ resting on distinct integers, such that $\gamma(s, e)$ denotes the integer occupied by Γ_e at the end of stage s.

Stage $s = 0$. Define $A^0 = \{x_n\}_{n \in N}$, $\psi^0(x_n) = x_n$ and $\gamma(0, n) = n$, for all $n \in N$.

Stage $s+1$. Let e be the least $i \leqq s$ such that $\phi_i^{s+1}(y_{\gamma(s,i)})$ is defined, but requirement R_i is not satisfied at stage s. If no such i exists, let $A^{s+1} = A^s$, $\psi^{s+1} = \psi^s$, and $\gamma(s+1, n) = \gamma(s, n)$, for all $n \in N$. Otherwise, we say that requirement R_e *receives attention at stage* $s+1$. Relative to our basis S for $\bar{U}$, choose scalars $\{a_i, b_i\}_{i \in N}$ such that

$$\phi_e^{s+1}(y_{\gamma(s,e)} = \sum_{i=0}^{m} a_i x_i + b_i y_i. \tag{6}$$

(Since each vector $v \in \bar{U}$ is ultimately vanishing, such m exists.) For notational convenience, abbreviate $\gamma(s, e)$ by γ. Define $\psi^{s+1}(y_\gamma)$ to be any vector v in $L(x_\gamma, y_\gamma)$ but not in

$$L(x_\gamma) \cup L(a_\gamma x_\gamma + b_\gamma y_\gamma).$$

Let n be the greatest i such that $y_i \in A^s$, and let $p = 1 + \max\{n, m\}$. For all $i \leqq e$ leave marker Γ_i fixed. For each $i > e$, move marker Γ_i in order of i to an integer q, such that $q \geqq p$, $q \geqq \gamma(s, i)$, the integer previously occupied by Γ_i, and for all $j < i$, $q \geqq \gamma(s+1, j)$, the integer now occupied by Γ_j.

Case 1. $\phi_e^{s+1}(y_\gamma) \in W_1^s$. Define $A^{s+1} = A^s$. (Note that $y_\gamma \notin A^s$.)

Case 2. $\phi_e^{s+1}(y_\gamma) \notin W_1^s$. Define $A^{s+1} = \{y_\gamma\} \cup (A^s - \{x_\gamma\})$. To complete the construction, define $A = \lim_s A^s$, and $\psi = \bigcup_s \psi^s$.

In most constructions it is obvious that the action taken at stage $s+1$ succeeds (at least temporarily) in satisfying the requirement being considered. Here, it is not obvious because in Case 2, $A^{s+1} \neq A^s$ implies $W_1^{s+1} \neq W_1$. The following lemma is the crux of the whole argument.

LEMMA 1. *If requirement R_e receives attention at stage $s+1$, then R_e is satisfied at stage $s+1$.*

PROOF. If Case 1 applies in the above definition of A^{s+1}, clearly R_e is satisfied at stage $s+1$, since $W_1^{s+1} = W_1^s$. If Case 2 applies, note that our choice of $\psi^{s+1}(y_\gamma)$ has insured that adding y_γ to A^{s+1} and removing x_γ will not cause $\phi_e(y_\gamma) \in W_1^{s+1}$. For suppose to the contrary that

$$\phi_e(y_\gamma) \in W_1^{s+1} = L(\psi^{s+1}(A^{s+1}))$$

where $A^{s+1} = \{u_i\}_{i \in N}$, and $u_i = x_i$ or y_i, all $i \in N$, and where $u_\gamma = y_\gamma$. Then there exist scalars $\{c_i\}_{i \in N}$, such that

$$\phi_e^{s+1}(y_\gamma) = c_\gamma \psi^{s+1}(u_\gamma) + \sum_{i \neq \gamma} c_i \psi^{s+1}(u_i), \text{ where } c_\gamma \neq 0. \tag{7}$$

But since $\psi^{s+1}(u_i) \in L(x_i, y_i)$, all $i \in N$, there exist scalars $\{a_i', b_i'\}_{i \in N}$ such that

$$\psi^{s+1}(u_i) = a_i' x_i + b_i' y_i, \text{ all } i \in N. \tag{8}$$

Now by combining (7) with (8),

$$\phi_e^{s+1}(y_\gamma) = c_\gamma(a_\gamma' x_\gamma + b_\gamma' y_\gamma) + \sum_{i \neq \gamma} c_i(a_i' x_i + b_i' y_i). \tag{9}$$

However, comparing (9) with (6) we conclude that

$$a_i = c_i a_i' \text{ and } b_i = c_i b_i', \text{ all } i \in N. \tag{10}$$

By assumption $u_\gamma = y_\gamma$, but then (8) and (10) contradict our definition of $\psi^{s+1}(y_\gamma)$ which by construction is not in $L(a_\gamma x_\gamma + b_\gamma y_\gamma)$.

LEMMA 2. *Each requirement R_e receives attention at most finitely often.*

PROOF OF LEMMA 2. Fix e and assume by induction that no R_i, $i < e$, receives attention after some stage say s'. Then Γ_e never moves after s' because

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Γ_e is moved only when some R_i, $i < e$, receives attention. But if R_e receives attention at some stage $s+1 > s'$, then all Γ_i, $i > e$, are moved to integers $q > m$, where m is defined in (6). Now no y_j, $j \leqq m$, enters A after stage $s+1$, because $\gamma(i, s') > m$ for all $t \geqq s+1$ and $i > e$. Therefore, no $v \in L(x_j, y_j)$ for $j \leqq m$, enters or leaves W_1 at any stage $t > s+1$ because $\psi(x_j)$, $\psi(y_j) \in L(x_j, y_j)$ when defined. Hence,

$$\phi_e^{s+1}(y_\gamma) \in W_1^{s+1} \Leftrightarrow (\forall t \geqq s)[\phi_e^t(y_\gamma) \in W_1^t],$$

so that requirement R_e is satisfied at all $t \geqq s+1$.

LEMMA 3. $V_1 \ncong W_1$.

PROOF. Assume $V_1 \cong W_1$ via ϕ_e. Choose s' sufficiently large so that $\gamma(s, i) = \gamma(s', i)$, for all $i \leqq e$, and for all $s \geqq s'$. Now $\phi_e^s(y_{\gamma(s,e)})$ will be defined for some $s \geqq s'$. Hence, requirement R_e will become satisfied at some stage $s \geqq s'$, and will remain satisfied thereafter.

2. Concerning Theorem 2

THEOREM 2. *For α-spaces V_1, V_2, W_1, $W_2 \subseteq \bar{U}$, (1) and (2) do imply (3) if V_1 and V_2 are α-spaces via the same r.e. linearly independent set, say S, and if there exists functions ψ_i witnessing $V_i \simeq W_i$ which satisfy* $\psi_i(S \cap V_j) \subseteq W_j$ for all i, $j \in \{1, 2\}$.

PROOF OF THEOREM 2. We only sketch the proof which is a variation of the standard Karp-Myhill method of [3]. Let $V_1 \simeq W_1$ via ψ_1, and $V_2 \simeq W_2$ via ψ_2. Then $V_1 \cong W_1$ via ϕ which is defined as the union of finite functions ϕ^s as follows. At even stages $s+1$, enumerate an element $x \in A$ such that $x \notin L(\text{dom } \phi^s)$, and choose the first of $\psi_1(x)$, $\psi_2(x)$ which is defined, say $\psi_1(x)$. If $\psi_1(x) \notin L(\text{ran } \phi^s)$, let $\phi^{s+1}(x) = \psi_1(x)$. Otherwise, choose a set

$$\{y_1, y_2, \cdots, y_n\} \subseteq \text{ran}(\phi^s(A))$$

of minimal cardinality n, such that $\psi_1(x) \in L(y_1, y_2, \cdots, y_n)$, and choose $z_i \in A$ such that $\phi^s(z_i) = y_i$, $1 \leqq i \leqq n$.

The key observation is that $x, z_1, z_2, \cdots, z_n$ are *linked*, that is all lie in V_1 or all in V_2. This follows by the minimality of n; the fact that all $v \in A$ are in V_1 or V_2 (and therefore, all $v \in \text{ran } \psi_1(A) \cup \text{ran } \psi_2(A)$ lie in W_1 or W_2); and because we may assume by induction on s that $\phi^s(V_1) \subseteq W_1$, and $\phi^s(V_2) \subseteq W_2$.

Hence, just as in the standard Karp-Myhill method, either ψ_1 or ψ_2 must eventually be defined on *all* $n+1$ linearly independent vectors $\{x, z_1, z_2, \cdots, z_n\}$, thereby producing a vector $v \notin L(y_1, y_2, \cdots, y_n)$. If $v \notin L(\text{ran } \phi^s)$, let $\phi^{s+1}(x) = v$. Otherwise, choose a set

$$\{y_1', y_2', \cdots, y_m'\} \subseteq \text{ran}(\phi^s(A))$$

of minimal cardinality n, such that $v \in L(y'_1, y'_2, \cdots, y'_m)$, and repeat the above process with

$$\{y_1, y_2, \cdots, y_m\} \cup \{y'_1, y'_2, \cdots, y'_m\}$$

in place of $\{y_1, y_2, \cdots, y_n\}$. Since $L(\text{ran } \phi^s)$ has finite dimension, the process terminates yielding some $v \notin L(\text{ran } \phi^s)$ which is an appropriate image for $\phi^{s+1}(x)$.

On odd stages $s+1$, enumerate an element $x \in \text{ran } \psi_1(A) \cup \text{ran } \psi_2(A)$ such that $x \notin L(\text{ran } \phi^s)$ and proceed similarly.

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University of Illinois
at Chicago Circle
Chicago, Illinois, 60680
U. S. A.