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DECOMPOSABILITY OF FINITE RANK OPERATORS IN CERTAIN SUBSPACES AND ALGEBRAS

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Let $\mathcal{S}$ be either a reflexive subspace or a bimodule of a reflexive algebra in $B(H)$, the set of bounded operators on a Hilbert space H. We find some conditions such that a finite rank $T \in \mathcal{S}$ has a rank one summand in $\mathcal{S}$ and $\mathcal{S}$ has strong decomposability. Let $\mathcal{S}(\mathcal{L})$ be the set of all operators on H that annihilate all the operators of rank at most one in $\operatorname{alg}\mathcal{L}$. We construct an atomic Boolean subspace lattice $\mathcal{L}$ on H such that there is a finite rank operator T in $\mathcal{S}(\mathcal{L})$ such that T does not have a rank one summand in $\mathcal{S}(\mathcal{L})$. We obtain some lattice-theoretic conditions on a subspace lattice $\mathcal{L}$ which imply $\operatorname{alg}\mathcal{L}$ is strongly decomposable.

1. INTRODUCTION

Let H be a complex Hilbert space, $B(H)$ the set of bounded linear operators on H, and $F(H)$ the set of operators with finite rank. For convenience we disregard the distinction between a subspace of H and the orthogonal projection onto it. Throughout, all subspaces will be assumed to be closed. By a *subspace lattice* on H, we mean a collection $\mathcal{L}$ of subspaces of H with (0), H in $\mathcal{L}$ and such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\vee M_r$ denotes the closed linear span of $\{M_r\}$. A totally ordered subspace lattice is called a *nest*. A complemented and distributive subspace lattice is called a *Boolean lattice*. An element L of a subspace lattice $\mathcal{L}$ is called an *atom* if the condition $(0) \subsetneq K \subseteq L$ $(K \in \mathcal{L})$ implies either $K = (0)$ or $K = L$. A subspace lattice is *atomic* if each element of the lattice is the closed linear span of the atoms it contains.

For every subspace lattice $\mathcal{L}$ on H, we define $\operatorname{alg}\mathcal{L}$ by

$$\operatorname{alg}\mathcal{L} = \{T \in B(H) : TM \subseteq M, \text{ for every } M \in \mathcal{L}\}.$$

Let $\mathcal{L}^{\perp} = \{I - P : P \in \mathcal{L}\}$. We have $\operatorname{alg}\mathcal{L}^{\perp} = (\operatorname{alg}\mathcal{L})^*$. If e, f are in H, then the rank one operator $x \mapsto f(x)e = (x, f)e$ is denoted by $e \otimes f$. If K, $L \in \mathcal{L}$, we denote by L_- the subspace $L_- = \vee\{M \in \mathcal{L} : L \nsubseteq M\}$, by $K_\# = \vee\{L \in \mathcal{L} : K \nsubseteq L_-\}$ and by $K_+ = \cap\{L \in \mathcal{L} : L \nsubseteq K\}$. By convention $H_+ = \cap\emptyset = H$, $(0)_- = \vee\emptyset = (0)$. The

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complete distributivity of $\mathcal{L}$ is equivalent to $K = K_{\#}$ for all $K \in \mathcal{L}$. An element L in $\mathcal{L}$ is *completely meet prime* if $L \not\supseteq L_+$. An element M in $\mathcal{L}$ is *completely join prime* if $M \not\subseteq M_-$.

If M is a subset of H, we denote by $[M]$ the norm closure of span$\{x : x \in M\}$. In this paper, "$\subseteq$" is used for set inclusion while "$\subset$" is reserved for proper inclusion. Let R and T be finite rank operators on H. We say that R is a *summand* of T if $\operatorname{rank} T = \operatorname{rank} R + \operatorname{rank}(T - R)$. Let $\mathcal{S}$ be a subset of $B(H)$. Then $\mathcal{S}$ is said to be *decomposable* if each finite rank operator in $\mathcal{S}$ is a sum of rank one operators in $\mathcal{S}$. We say that $\mathcal{S}$ is *strongly decomposable* if, for each $r > 1$, each operator in $\mathcal{S}$ of rank r can be expressed as the sum of r rank one operators in $\mathcal{S}$. A subspace $\mathcal{S}$ of $B(H)$ is said to be *reflexive* if whenever T in $B(H)$ satisfies the condition $Tx \in [\mathcal{S}x]$ for all $x \in H$, then T is in $\mathcal{S}$.

Finite rank operators and rank one operators have been used extensively in the study of nest algebras and related non-self-adjoint reflexive algebras. By [**5**], we know that if $\mathcal{L}$ is a nest or an atomic Boolean subspace lattice on H, then $\operatorname{alg}\mathcal{L}$ is strongly decomposable. These results were first proved in [**16**] and [**12**], respectively. These results were improved in [**10**]. Erdos and Power [**3**] proved that if $\mathcal{N}$ is a nest and $\mathcal{S}$ is a σ-weakly closed bimodule of $\operatorname{alg}\mathcal{N}$, then $\mathcal{S}$ is strongly decomposable. In [**6**], Hopenwasser and Moore construct a totally atomic commutative subspace lattice $\mathcal{L}$ and a rank two operator in $\operatorname{alg}\mathcal{L}$ which cannot be written as a sum of rank one operators in $\operatorname{alg}\mathcal{L}$.

Let $\mathcal{S}$ be either a reflexive subspace or a bimodule of a reflexive algebra. For $T \in \mathcal{S} \cap F(H)$, we find some conditions such that T has a rank one summand in $\mathcal{S}$. We also obtain some necessary and sufficient conditions which imply that $\mathcal{S}$ is strongly decomposable. We construct an atomic Boolean subspace lattice $\mathcal{L}$ on H with three atoms for which there is a finite rank operator T in $\mathcal{S}(\mathcal{L})$ such that T does not have a rank one summand in $\mathcal{S}(\mathcal{L})$, where $\mathcal{S}(\mathcal{L})$ is the set of all operators on H that annihilate all the operators of rank at most one in $\operatorname{alg}\mathcal{L}$. This answers a question in [**8**] negatively. We obtain some lattice-theoretic conditions on a subspace lattice $\mathcal{L}$ which imply $\operatorname{alg}\mathcal{L}$ is strongly decomposable. Theorems 2.12 and 2.13 generalise the main results of [**10**].

2. Main results

In [**2**], Erdos gives some necessary and sufficient conditions such that a reflexive subspace of $B(H)$ contains a rank one operator. In the following we obtain another equivalent condition.

Lemma 2.1. *Let $\mathcal{S}$ be a reflexive subspace of $B(H)$. Then $e \otimes f$ belongs to $\mathcal{S}$ if and only if $f \in \left(\operatorname{span}\{y : e \notin [\mathcal{S}y],\ y \in H\}\right)^{\perp}$.*

PROOF: Suppose that $e \otimes f \in \mathcal{S}$. For any y in H, $e \otimes f(y) = (y, f)e \in [\mathcal{S}y]$. Hence if $e \notin [\mathcal{S}y]$, $(y, f) = 0$. So $f \in \left(\operatorname{span}\{y : e \notin [\mathcal{S}y],\ y \in H\}\right)^{\perp}$.

Conversely, suppose $f \in \left(\operatorname{span}\{y : e \notin [\mathcal{S}y],\ y \in H\}\right)^{\perp}$. Let $y \in H$. It follows that $e \otimes f(y) = (y, f)e \in [\mathcal{S}y]$. Since $\mathcal{S}$ is reflexive, it follows that $e \otimes f \in \mathcal{S}$. ∎

The following Lemma will be used repeatedly.

LEMMA 2.2. ([**5**]) *Let T be a finite rank operator and let A be a rank one operator in $B(H)$. Then A is a summand of T if and only if A is of the form $(Ty) \otimes (T^{*}f)$ (or equivalently, $T(y \otimes f)T$), where y and f are vectors in H and $(Ty, f) = 1$.*

THEOREM 2.3. *Suppose that $\mathcal{S}$ is a reflexive subspace of $B(H)$ and T is a finite rank operator in $\mathcal{S}$. Then T has a rank one summand in $\mathcal{S}$ if and only if there is a non-zero e in H such that $e \in T(H)$ and $e \notin \operatorname{span}\{Ty : e \notin [\mathcal{S}y],\ y \in H\}$, where $T(H)$ is the range of T.*

PROOF: Suppose that $0 \neq e \in T(H)$ and $e \notin \operatorname{span}\{Ty : e \notin [\mathcal{S}y],\ y \in H\}$. Choose $g \in H$ such that $g \in \left(\operatorname{span}\{Ty : e \notin [\mathcal{S}y],\ y \in H\}\right)^{\perp}$, $(e, g) = 1$, and take $y \in H$ such that $Ty = e$. For any $\widetilde{y}$ satisfying $e \notin [\mathcal{S}\widetilde{y}]$, we have $(\widetilde{y}, T^{*}g) = 0$. It follows that $T^{*}g \in \left(\operatorname{span}\{\widetilde{y} : e \notin [\mathcal{S}\widetilde{y}],\ \widetilde{y} \in H\}\right)^{\perp}$. Using Lemma 2.1, $e \otimes T^{*}g = (Ty) \otimes (T^{*}g) \in \mathcal{S}$. Using Lemma 2.2, $e \otimes (T^{*}g) = (Ty) \otimes (T^{*}g)$ is a rank one summand of T in $\mathcal{S}$.

Conversely, suppose that T has a rank one summand in $\mathcal{S}$. By Lemma 2.2, there exist m and f in H such that

$$T(m \otimes f)T = (Tm) \otimes (T^{*}f) \in \mathcal{S},$$

and

$$(Tm, f) = 1 = (m, T^{*}f).$$

Let $Tm = e$. Using Lemma 2.1, we have $T^{*}f \in \left(\operatorname{span}\{y : e \notin [\mathcal{S}y],\ y \in H\}\right)^{\perp}$. Hence for any $v \in \operatorname{span}\{y : e \notin [\mathcal{S}y],\ y \in H\}$, $(v, T^{*}f) = (Tv, f) = 0$. Since $(e, f) = (Tm, f) = 1$, it follows that $e \notin T\left(\operatorname{span}\{y : e \notin [\mathcal{S}y],\ y \in H\}\right) = \operatorname{span}\{Ty : e \notin [\mathcal{S}y],\ y \in H\}$. ∎

COROLLARY 2.4. *Let M and N be nonzero subspaces of H satisfying $M \cap N = (0)$ and $M \vee N = H$ and let $\mathcal{L} = \{(0), M, N, H\}$. Then every σ-weakly closed $\operatorname{alg}\mathcal{L}$-bimodule $\mathcal{S}$ is strongly decomposable.*

PROOF: By [**9**, Theorem 2.2] and [**1**, Theorem 3.1], it follows that $\mathcal{S}$ is reflexive. By [**4**, Theorem 2], we know that $\mathcal{S}$ is determined by an order homomorphism ϕ of $\mathcal{S}$. Let ϕ be any order homomorphism of $\mathcal{L}$ and let

$$\mathcal{M} = \{T \in B(H) : (I - \phi(E))TE = 0, E \in \mathcal{L}\}.$$

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By the symmetry of M and N, we only need to consider that in the following cases $\mathcal{M}$ has strong decomposability.

(1) $\phi : M \mapsto M,\ N \mapsto (0)$,
(2) $\phi : M \mapsto N,\ N \mapsto (0)$,
(3) $\phi : M \mapsto N,\ N \mapsto M$,
(4) $\phi : M \mapsto M,\ N \mapsto N$,
(5) $\phi : M \mapsto H,\ N \mapsto (0)$.

For cases (1) to (4), we can easily prove the result using Theorem 2.3.

In case (5), $\mathcal{M} = \{T \in B(H) : TN = 0\}$. Let P denote the projection on N. Then T is in $\mathcal{M}$ if and only if $TP = 0$. Hence $\mathcal{M}$ has strong decomposability. ∎

If $\mathcal{L}$ is a subspace lattice on the Hilbert space H, let $\mathcal{S}(\mathcal{L})$ denote the set of all operators on H that annihilate all the operators of rank at most one in $\operatorname{alg}\mathcal{L}$, that is

$$\mathcal{S}(\mathcal{L}) = \{T \in B(H) : \operatorname{tr}(TR) = 0, \text{ for every } R \in \operatorname{alg}\mathcal{L} \text{ of rank at most one}\}.$$

Lemma 2.5. ([8]) *For any subspace lattice $\mathcal{L}$ on H,*

$$\mathcal{S}(\mathcal{L}) = \{T \in B(H) : T(K) \subseteq K_- \text{ for every } K \in \mathcal{L}\}.$$

Lemma 2.6. ([8]) *Let $\mathcal{L}$ be a subspace lattice on H and $e, f \in H$. The following are equivalent.*

(1) $e \otimes f \in \mathcal{S}(\mathcal{L})$,
(2) $e \in L$ *and* $f \in (L_\#)^\perp$ *for some* $L \in \mathcal{L}$.

Theorem 2.7. *Let $\mathcal{L}$ be a subspace lattice and let $T \in \mathcal{S}(\mathcal{L}) \cap F(H)$. Then T has a rank one summand in $\mathcal{S}(\mathcal{L})$ if and only if there exists an $L \in \mathcal{L}$ such that $T(H) \cap L \not\subseteq T(L_\#)$.*

Proof: Suppose that there exists $L \in \mathcal{L}$ such that $T(H) \cap L \not\subseteq T(L_\#)$. Let $e \in T(H) \cap L$ with $e \notin T(L_\#)$. Let $e = Ty$. Choose $g \in \big(T(L_\#)\big)^\perp$ such that $(e, g) = (Ty, g) = 1$. We have, using Lemma 2.6

$$(Ty, g) = (y, T^* g) = 1 \text{ and } T(y \otimes g)T = (Ty) \otimes (T^* g) \in \mathcal{S}(\mathcal{L}).$$

By Lemma 2.2, it follows that T has a rank one summand in $\mathcal{S}(\mathcal{L})$.

Conversely, suppose T has a rank one summand in $\mathcal{S}(\mathcal{L})$. By Lemma 2.2, there exist e, f in H such that

$$T(e \otimes f)T = (Te) \otimes (T^* f) \in \mathcal{S}(\mathcal{L}) \text{ and } (Te, f) = 1.$$

By Lemma 2.6, there exists L in $\mathcal{L}$ such that $Te \in L$ and $T^* f \in (L_\#)^\perp$. Since $Te \in L$, and $(Te, f) = 1$ and for any $v \in L_\#$, $(Tv, f) = 0$, we have that $T(H) \cap L \not\subseteq T(L_\#)$. ∎

EXAMPLE 2.8. There is an atomic Boolean subspace lattice $\mathcal{L}$ with three atoms such that $\mathcal{S}(\mathcal{L})$ is not strongly decomposable.

PROOF: Let H be a finite-dimensional Hilbert space and let A be an invertible operator in $B(H)$. Define $L_1 = \{(x,0,0):\ x \in H\}$, $L_2 = \{(x,Ax,0):\ x \in H\}$ and $L_3 = \{(x,Ax,Ax):\ x \in H\}$. By [**1**, Lemma 6.3], it follows that $\{L_1, L_2, L_3\}$ is the set of atoms of an atomic Boolean subspace lattice.

Define $T : L_1 \to L_2 \vee L_3$, by $(x,0,0) \mapsto (0,0,Px)$, $T : L_2 \to L_1 \vee L_3$, by $(x,Ax,0) \mapsto (0,Px,Px)$, and $T : L_3 \to L_2 \vee L_1$, by $(x,Ax,Ax) \mapsto (0,Px,0)$, where P is a nonzero finite rank projection in $B(H)$. We can extend T to a bounded finite rank operator in $B(H \oplus H \oplus H)$. In fact $T(x,y,z) = \big(0, PA^{-1}y,\ P(x - A^{-1}z)\big)$, for every $x,y,z \in H$. By the definition of T, it follows that $T \in \mathcal{S}(\mathcal{L})$. We have that $T(H) \cap L_1 = (0)$, $T(H) \cap L_2 = (0)$ and $T(H) \cap L_3 = (0)$. We can check that $T(H) \cap (L_2 \vee L_3) \subsetneq T(L_2 \vee L_3)$, $T(H) \cap (L_2 \vee L_1) \subsetneq T(L_2 \vee L_1)$ and $T(H) \cap (L_1 \vee L_3) \subsetneq T(L_1 \vee L_3)$. Hence by Theorem 2.7, T does not have a rank one summand in $\mathcal{S}(\mathcal{L})$, where $\mathcal{L}$ is the subspace lattice generated by L_1, L_2 and L_3. ∎

REMARK. The above Example answers a question in [**8**, p.31] negatively.

THEOREM 2.9. *Suppose that $\mathcal{L}$ is a subspace lattice and* Rad (alg $\mathcal{L}$) *is the radical of* alg $\mathcal{L}$. *Let $T \in$* Rad (alg $\mathcal{L}$) $\cap F(H)$. *Then T has a rank one summand in* Rad (alg $\mathcal{L}$) *if and only if there exists an M in $\mathcal{L}$ such that $T(H) \cap M \not\subseteq T(M_- \vee M)$.*

PROOF: Suppose that $T(H) \cap M \not\subseteq T(M_- \vee M)$. Choose g in $\big(T(M_- \vee M)\big)^{\perp}$, e in H such that $(Te, g) = 1$ and $Te \in M$. Then $(e, T^*g) = 1$, $(Tx, g) = (x, T^*g) = 0$ for any $x \in M_- \vee M$. By $T^*g \in (M_- \vee M)^{\perp}$, $Te \in M$ and [**7**, Lemma 3], it follows that $(Te) \otimes (T^*g) \in$ Rad (alg $\mathcal{L}$). By Lemma 2.2, T has a rank one summand in Rad (alg $\mathcal{L}$).

Conversely, suppose T has rank one summand in Rad (alg $\mathcal{L}$). It follows that there exist $e, f \in H$ such that $T(e \otimes f)T = (Te) \otimes (T^*f) \in$ Rad (alg $\mathcal{L}$). By [**6**, Lemma 3], there exists M in $\mathcal{L}$ such that $T^*f \in (M_- \vee M)^{\perp}$, $(Te, f) = 1$. Hence $T(H) \cap M \not\subseteq T(M_- \vee M)$. ∎

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspace lattices on Hilbert spaces H_1 and H_2, respectively. Then the ordinal sum $\mathcal{L}_1 + \mathcal{L}_2$ is defined as the subspace lattice on $H_1 \oplus H_2$ given by

$$\mathcal{L}_1 + \mathcal{L}_2 = \{L \oplus (0) :\ L \in \mathcal{L}_1\} \cup \{H_1 \oplus M :\ M \in \mathcal{L}_2\}.$$

THEOREM 2.10. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspace lattices on Hilbert spaces H_1 and H_2. If $\mathcal{S}(\mathcal{L}_1)$ and $\mathcal{S}(\mathcal{L}_2)$ are strongly decomposable, then $\mathcal{S}(\mathcal{L}_1 + \mathcal{L}_2)$ is strongly decomposable.*

PROOF: Since

$$\operatorname{alg}(\mathcal{L}_1 + \mathcal{L}_2) = \left\{ \begin{pmatrix} A_1 & T \\ 0 & A_2 \end{pmatrix} : A_i \in \operatorname{alg}\mathcal{L}_i,\ \text{for } i = 1,2,\ T \in B(H_2, H_1) \right\},$$

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we have

$$(2.1)\quad \mathcal{S}(\mathcal{L}_1+\mathcal{L}_2)=\left\{\begin{pmatrix} B_1 & S \\ 0 & B_2\end{pmatrix} : B_i\in\mathcal{A}(\mathcal{L}_i),\ \text{for } i=1,\ 2,\ \text{and } S\in B(H_2,H_1)\right\}.$$

Let T be a finite rank operator in $\mathcal{S}(\mathcal{L}_1+\mathcal{L}_2)$. Then

$$T=\begin{pmatrix} T_1 & S \\ 0 & T_2\end{pmatrix},\ \text{where } T_i\in\mathcal{S}(\mathcal{L}_i) \text{ for } i=1,\ 2 \text{ and } S\in B(H_2,H_1).$$

Suppose $T_1\neq 0$. Since $\mathcal{S}(\mathcal{L}_1)$ is strongly decomposable, we may choose $e_1\in H_1$, $f_1\in H_1$ such that $T_1(e_1\otimes f_1)T_1$ is a rank one summand of T_1 in $\mathcal{S}(\mathcal{L}_1)$. Let $e=e_1\oplus 0$ and let $f=f_1\oplus 0$. For any $x=x_1\oplus x_2\in H_1\oplus H_2$, $(x,f)=(x_1,f_1)$. It follows that

$$T(e\otimes f)T=\begin{pmatrix} T_1(e_1\otimes f_1)T_1 & T_1(e_1\otimes f_1)S \\ 0 & 0\end{pmatrix}.$$

By $(Te,f)=(T_1e_1,f_1)=1$, (2.1) and $T(e\otimes f)T\in\mathcal{S}(\mathcal{L}_1+\mathcal{L}_2)$, it follows from Lemma 2.2 that $T(e\otimes f)T$ is a rank one summand of T in $\mathcal{S}(\mathcal{L}_1+\mathcal{L}_2)$.

If $T_1=0$ and $T_2\neq 0$, we can similarly prove that T has a rank one summand in $\mathcal{S}(\mathcal{L}_1+\mathcal{L}_2)$.

Suppose that $T_1=T_2=0$. Then $T=\begin{pmatrix} 0 & A \\ 0 & 0\end{pmatrix}$. Since $B(H_2,H_1)$ is strongly decomposable, it follows that T has a rank one summand in $\mathcal{S}(\mathcal{L}_1+\mathcal{L}_2)$.

Since T is any finite rank operator in $\mathcal{S}(\mathcal{L}_1+\mathcal{L}_2)$, it follows that $\mathcal{S}(\mathcal{L}_1+\mathcal{L}_2)$ is strongly decomposable. ∎

Let

$$\mathcal{J}_\mathcal{L}=\{L\in\mathcal{L}:\ L\neq(0) \text{ and } L_-\neq H\},\ \mathcal{P}_\mathcal{L}=\{L\in\mathcal{L}:L\not\subseteq L_-\}.$$

By [**13**], we know that $L\in\mathcal{L}$ is completely meet prime if and only if $L=M_-$ for some $M\in\mathcal{P}_\mathcal{L}$.

Lemma 2.11. ([**15**, Lemma 2.3.1]) *Let K and L be subspaces of H and let $F=\sum_{i=1}^{n}e_i\otimes f_i$ be a rank n operator in $B(H)$. If $F(L)\subseteq K$ and $f_1\notin L^\perp$, then F can be written as $F=\widetilde{e}_1\otimes f_1+\sum_{i=2}^{n}e_i\otimes\widetilde{f}_i$ with $\widetilde{e}_1\in K$.*

Theorem 2.12. *Let $\mathcal{L}$ be a subspace lattice on H such that $\mathcal{J}_\mathcal{L}=\mathcal{P}_\mathcal{L}$ and $\vee\{L:L\in\mathcal{J}_\mathcal{L}\}=H$. Then $\operatorname{alg}\mathcal{L}$ is strongly decomposable.*

Proof: Suppose that $\operatorname{alg}\mathcal{L}$ is not strongly decomposable. Then there is an operator of rank $n>1$ $T=\sum_{i=1}^{n}e_i\otimes f_i$ in $\operatorname{alg}\mathcal{L}$ such that T does not have a rank one

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summand in $\operatorname{alg}\mathcal{L}$. By $H = \vee\{M : M \in \mathcal{J}_{\mathcal{L}}\}$, it follows that there exists an M in $\mathcal{J}_{\mathcal{L}}$ such that $f_1 \notin M^{\perp}$. By Lemma 2.11, T can be written as

$$T = \widetilde{e}_1 \otimes f_1 + \sum_{i=2}^{n} e_i \otimes \widetilde{f}_i,$$

with $\widetilde{e}_1 \in M$. Let

$$N = \cap\{L \in \mathcal{J}_{\mathcal{L}} : \widetilde{e}_1 \in L\}. \tag{2.2}$$

Then $N \in \mathcal{J}_{\mathcal{L}}$ and $\widetilde{e}_1 \in N$.

Now we show that $\widetilde{e}_1 \in N_-$. Suppose $\widetilde{e}_1 \notin N_-$. Since $T^* = f_1 \otimes \widetilde{e}_1 + \sum_{i=2}^{n} \widetilde{f}_i \otimes e_i$, by Lemma 2.11, we have that $T^* = g_1 \otimes \widetilde{e}_1 + \sum_{i=2}^{n} \widetilde{f}_i \otimes h_i$ with $g_1 \in (N_-)^{\perp}$.

By $\widetilde{e}_1 \in N$ and $g_1 \in (N_-)^{\perp}$, we have that $g_1 \otimes \widetilde{e}_1$ is a rank one summand of T^* in $\operatorname{alg}\mathcal{L}^{\perp}$. Hence T has a rank one summand in $\operatorname{alg}\mathcal{L}$. This is a contradiction.

Let $W = N_- \cap N$. We have $\widetilde{e}_1 \in W$ and $W \in \mathcal{J}_{\mathcal{L}}$. By the assumption, $W \subset N$ and $\widetilde{e}_1 \in W$. This contradicts (2.2). ∎

THEOREM 2.13. *Let $\mathcal{L}$ be a subspace lattice on H such that $\mathcal{J}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}}$ and $\cap\{L_- : L \in \mathcal{J}_{\mathcal{L}}\} = 0$. Then $\operatorname{alg}\mathcal{L}$ is strongly decomposable.*

PROOF: By [**14**, Proposition 2.1], it follows that

$$\mathcal{J}_{\mathcal{L}^{\perp}} = \{(M_-)^{\perp} : M \in \mathcal{J}_{\mathcal{L}}\}.$$

Since $\mathcal{J}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}}$, for any $M \in \mathcal{J}_{\mathcal{L}}$, we have that $(M_-)^{\perp}$ is completely join prime. Hence for subspace lattice $\mathcal{L}^{\perp}$, we have $\mathcal{J}_{\mathcal{L}^{\perp}} = \mathcal{P}_{\mathcal{L}^{\perp}}$. By $\cap\{M_- : M \in \mathcal{J}_{\mathcal{L}}\} = (0)$, it follows that $\vee\{N : N \in \mathcal{J}_{\mathcal{L}^{\perp}}\} = H$. By Theorem 2.12, $\operatorname{alg}\mathcal{L}^{\perp}$ is strongly indecomposable. It follows that $\operatorname{alg}\mathcal{L}$ is too. ∎

COROLLARY 2.14. ([**10**]) *Let $\mathcal{L}$ be a subspace lattice on H. If $\mathcal{L}$ satisfies one of the following conditions*

(1) $\vee\{K : K \in \mathcal{J}_{\mathcal{L}}\} = H$ *and for each K in $\mathcal{J}_{\mathcal{L}}$, $K_- \vee K = H$,*

(2) $\cap\{K_- : K \in \mathcal{J}_{\mathcal{L}}\} = (0)$ *and for each K in $\mathcal{J}_{\mathcal{L}}$, $K_- \vee K = H$,*

then $\operatorname{alg}\mathcal{L}$ is strongly decomposable.

If $\mathcal{L}$ is a completely distributive subspace lattice, by [**11**] we have $\vee\{L : L \in \mathcal{J}_{\mathcal{L}}\} = H$ and $\cap\{L_- : L \in \mathcal{J}_{\mathcal{L}}\} = (0)$. By Theorem 2.12 or Theorem 2.13, we have the following result.

COROLLARY 2.15. ([**15**, Theorem 2.3.2]) *Let $\mathcal{L}$ be a finite distributive subspace lattice on H which satisfies $\mathcal{J}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}}$. Then $\operatorname{alg}\mathcal{L}$ is strongly decomposable.*

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