LIFTINGS OF A MONOMIAL CURVE

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Abstract

We study an operation, that we call lifting, creating nonisomorphic monomial curves from a single monomial curve. Our main result says that all but finitely many liftings of a monomial curve have Cohen–Macaulay tangent cones even if the tangent cone of the original curve is not Cohen–Macaulay. This implies that the Betti sequence of the tangent cone is eventually constant under this operation. Moreover, all liftings have Cohen–Macaulay tangent cones when the original monomial curve has a Cohen–Macaulay tangent cone. In this case, all the Betti sequences are just the Betti sequence of the original curve.

1. Introduction

The main theme of this note is an operation that we call lifting and use for spreading a special property of a monomial curve within an infinite family of examples. By a monomial curve C_S in the affine space K^n over a field K, we mean a curve whose points $(t^{m_1}, \ldots, t^{m_n})$ vary with the parameter $t \in K$. The curve has a strong relation with the semigroup S generated minimally by the positive integers $m_1, \ldots, m_n$. The smallest minimal generator of S is called the *multiplicity* of C_S. The monomials $t^{m_1}, \ldots, t^{m_n}$ generate the integral domain $K[S]$ of $K[t]$, which is known as the *numerical semigroup ring* associated to S. The polynomial ring $R = K[x_1, \ldots, x_n]$ is graded by the semigroup S, by setting $\deg_S(x_i) = m_i$, giving rise to a graded map $R \to K[S]$, which sends x_i to t^{m_i}. Its kernel denoted by I_S is called the *toric ideal* of S. When K is algebraically closed, the vanishing ideal $I(C_S)$ of C_S is I_S, and thus $K[S]$ is isomorphic to the coordinate ring R/I_S of the curve C_S. The curve C_S is singular at the origin if $S \neq \mathbb{N}$ in which case one wants to measure how nice the singularity is. An algebraic way to understand the quality of the singularity is to look at the coordinate ring $gr_{\mathfrak{m}}(K[S])$ of the *tangent cone* which is isomorphic to the ring R/I_S^*, where $\mathfrak{m} = \langle t^{m_1}, \ldots, t^{m_n} \rangle$ is the maximal ideal of $K[S]$. Here I_S^* is the ideal generated by the least degree homogeneous summands f^* of f in I_S. The Cohen–Macaulayness of $gr_{\mathfrak{m}}(K[S])$ is investigated for this purpose in [1, 7, 11, 17, 18].

It is well-known that the ideal I_S is generated by finitely many binomials $x_1^{u_1} \cdots x_n^{u_n} - x_1^{v_1} \cdots x_n^{v_n}$ such that $u_i v_i = 0$ and $\deg_S(x_1^{u_1} \cdots x_n^{u_n}) = \deg_S(x_1^{v_1} \cdots x_n^{v_n})$. In general, the ideal I_S has alternative minimal generating sets of the same cardinality. If a binomial B or $-B$ appears in every minimal generating set it is called *indispensable*. Hence, the ideal I_S has a unique minimal generating set if and only if it is generated by indispensable binomials. Uniquely generated ideals are of interest in the emerging field of algebraic statistics (see [8, 19]). Recently, indispensable binomials were also used to characterise monomial curves in K^4 with a Cohen–Macaulay tangent cone (see [2, 16]).

Our main aim in this note is to see how Cohen–Macaulayness spreads among liftings of a monomial curve. It is inspired by a recent paper of Herzog and Stamate [10] on a similar additive operation on S called shifting, which produces a semigroup generated by $m_1 + k, \ldots, m_n + k$ for every positive integer k. Herzog and Stamate show that tangent cones of shiftings are Cohen–Macaulay for all sufficiently large k, and that the Betti numbers of the tangent cones are eventually periodic in k. The second concern of this paper is to examine how lifting effects indispensability of binomials in I_S and strong indispensability of a minimal free resolution of $K[S]$.

Before stating our main results let us introduce more notations. Fix a numerical semigroup S generated minimally by $m_1, m_2, \ldots, m_n$. By a k-lifting S_k of S we mean the numerical semigroup generated by

$$m_1, km_2, \ldots, km_n,$$

where k is a positive integer with $\gcd(k, m_1) = 1$. The last condition is to avoid having isomorphic liftings. In the same vein, the monomial curve C_k corresponding to S_k is called a k-lifting of $C := C_S$. When C has multiplicity m_1, all of its liftings will have multiplicity m_1. There is a closely related operation called simple gluing that has been used to produce more examples with an interesting structure from a single monomial curve. Let T be a semigroup generated minimally by $m_2, \ldots, m_n$. If $m_1 \in T$ and $\gcd(k, m_1) = 1$, then S_k is called a simple gluing of T and $\mathbb{N}$. Notice that S_k is a k-lifting of $S_1 = S$ for any k. This technique has been used for the first time in [21] to prove that S_k is symmetric if and only if T is symmetric. It is used in [13] to produce monomial curves with Noetherian symbolic blow-ups and in [20] to create monomial curves which are set theoretic complete intersections. In [3], the authors proved that the tangent cone $gr_m(K[S_k])$ is Cohen–Macaulay when the same is true for the semigroup T and $k \leq m_2 + \cdots + m_n$. This is extended later in [12]. Finally, it has very recently been used to study the catenary degree (see [15, Theorem 3.3]) which is an invariant of the semigroup measuring complexity of factorisations of elements. As T is independent of k, all these results reveal the common behaviour of the liftings S_k.

We finish the introduction by describing the structure of the paper. In the next section, we establish a one-to-one correspondence between the binomials in both ideals I_S and I_{S_k} with monomials having no common divisor, associating minimal generators of I_S to those of I_{S_k}, and preserving indispensability (see Proposition 2.2). In Section 3, we show that all but finitely many liftings of a monomial curve

have Cohen–Macaulay tangent cones even if the tangent cone of the original curve is not Cohen–Macaulay. Moreover, all liftings have Cohen–Macaulay tangent cones when the original monomial curve has a Cohen–Macaulay tangent cone (see Theorem 3.3). In the last section, we show that the minimal free resolution of $K[S]$ is strongly indispensable if and only if the minimal free resolution of $K[S_k]$ is strongly indispensable (see Proposition 4.1). Finally, we prove that the Betti sequence of the tangent cone is eventually constant under this operation and that all the Betti sequences are just the Betti sequence of the original tangent cone if the latter is Cohen–Macaulay (see Theorem 4.5).

2. Indispensability

In this section we establish the correspondence between the indispensable binomials of a monomial curve and those of its liftings. Let us recall a graph encoding minimal generators of I_S. Let $V(d)$ be the set of monomials of S-degree d. Denote by $G(d)$ the graph with vertices the elements of $V(d)$ and edges $\{M, N\} \subset V(d)$ such that the binomial $M - N$ is generated by binomials in I_S of S-degree strictly smaller than d with respect to $<_S$, where $s_1 <_S s_2$ if $s_2 - s_1 \in S$. In particular, when $\gcd(M, N) \neq 1$, $\{M, N\}$ is an edge of $G(d)$ as $M - N = \gcd(M, N) \cdot [(M - N)/\gcd(M, N)]$ and the binomial $(M - N)/\gcd(M, N)$ in I_S has strictly smaller S-degree. A binomial of S-degree d is indispensable if and only if $G(d)$ has two connected components which are singletons, by [5, Corollary 2.10]. This means that $M - N$ is indispensable exactly when $V(d) = \{M, N\}$ and $M - N$ is not generated by binomials in I_S of S-degree strictly smaller than d.

Let $b \in S$. Every tuple $(v_1, \dots, v_n) \in \mathbb{N}^n$ satisfying $b = v_1 m_1 + \cdots + v_n m_n$ is called a *factorisation* of b. Note that there is a one-to-one correspondence between factorisations $(v_1, \dots, v_n)$ of $b \in S$ and monomials $N = x_1^{v_1} \cdots x_n^{v_n}$ of S-degree b. The following key fact will be used many times in what follows.

LEMMA 2.1. *There is a one-to-one correspondence between S and $kS \subset S_k$ under which $m_1 + S$ is mapped onto $k(m_1 + S)$. Moreover, the factorisation $(v_1, v_2, \dots, v_n)$ of $b \in S$ corresponds to the factorisation $(kv_1, v_2, \dots, v_n)$ of $kb \in kS$.*

PROOF. If $b \in S$, then $kb \in kS \subset S_k$. Conversely, if $kb \in S_k$, then we have $kb = v_1'(m_1) + v_2(km_2) + \cdots + v_n(km_n)$. So, k divides $v_1' m_1$, which forces the existence of $v_1 \in \mathbb{N}$ such that $v_1' = kv_1$, since $\gcd(k, m_1) = 1$. Thus, $b = v_1 m_1 + \cdots + v_n m_n \in S$. So, the map sending $b \in S$ to $kb = kv_1(m_1) + v_2(km_2) + \cdots + v_n(km_n) \in S_k$ is one-to-one and onto. As $v_1 \neq 0$ if and only if $kv_1 \neq 0$, it follows that this map restricts to a one-to-one correspondence between $m_1 + S$ and $k(m_1 + S)$. Clearly, the factorisation $(v_1, v_2, \dots, v_n)$ of b corresponds to the factorisation $(kv_1, v_2, \dots, v_n)$ of kb. □

It is time to describe the correspondence between the binomials in both ideals I_S and I_{S_k} with monomials having no common divisor. This has been noted for the first time by Morales (see [13, Lemma 3.2]). We show that indispensable binomials are associated with indispensable ones under this correspondence. Let B be a binomial in

I_S with monomials having no common divisor. Then, either x_1 divides no monomials in B or it divides only one of them. In the former case, B lies in I_{S_k}, so let $B_k := B$. In the latter, $B = M - N$ with $M = x_2^{u_2} \cdots x_n^{u_n}$ and $N = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$, so we define $B_k := M - N_k \in I_{S_k}$, where $N_k = x_1^{kv_1} x_2^{v_2} \cdots x_n^{v_n}$. In both cases, $\deg_{S_k}(B_k) = k \deg_S(B) \in kS$.

PROPOSITION 2.2. *The map* $\phi_k : I_S \to I_{S_k}$, *given by* $B \to B_k$, *is a one-to-one correspondence between the binomials in both ideals with monomials having no common divisor, associating minimal generators of* I_S *to those of* I_{S_k}, *and preserving indispensability.*

PROOF. It is clear that the assignment is one-to-one. We prove that it is onto by using Lemma 2.1. Now, assume that $M - N'$ is a binomial in I_{S_k}, where $M = x_2^{u_2} \cdots x_n^{u_n}$ and N' is a monomial corresponding to the factorisation $(v_1', v_2, \ldots, v_n)$ with $v_1' > 0$. So, $\deg_{S_k}(N') = \deg_{S_k}(M) = kb$ lies in $k(m_1 + S)$, where $b := \deg_S(M)$. Lemma 2.1 implies that the factorisation $(v_1', v_2, \ldots, v_n)$ of kb corresponds to the factorisation $(v_1, v_2, \ldots, v_n)$ of b, for some $v_1 > 0$. Therefore, there is a monomial $N = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$ with S-degree b such that $N' = N_k$. So, $\phi_k(M - N) = M - N'$.

Let $G(b)$ be the graph of an S-degree b and $G_k(d)$ be the graph of an S_k-degree d. By Lemma 2.1, there is a one-to-one correspondence between the monomials of S-degree b and monomials of S_k-degree kb. So, there is a correspondence between the vertices of the graphs $G(b)$ and $G_k(kb)$. By the first part, $M - N$ is generated by binomials in I_S of S-degree smaller than b if and only if $M - N_k$ is generated by binomials in I_{S_k} of S_k-degree smaller than kb. This correspondence associates edges between $\{M\}$ and $\{N\}$ to edges between $\{M\}$ and $\{N_k\}$. As these graphs determine the minimal generators, this correspondence associates a minimal generating set of I_S to a minimal generating set of I_{S_k}. Assuming that B is an indispensable binomial of S-degree b, we see that $G(b)$ has two connected components $\{M\}$ and $\{N\}$ which are singletons by [5, Corollary 2.10]. Thus, $\{M\}$ and $\{N_k\}$ are the only connected components of $G_k(kb)$ and they are singletons. Hence, B is indispensable if and only if B_k is indispensable. □

3. Cohen–Macaulayness of the tangent cone

In this section, we study local properties of liftings of a monomial curve C with multiplicity m_1. Recall that the S-degree and the usual degree of a monomial are

$$\deg_S(x_1^{u_1} \cdots x_n^{u_n}) = u_1 m_1 + \cdots + u_n m_n \quad \text{and} \quad \deg(x_1^{u_1} \cdots x_n^{u_n}) = u_1 + \cdots + u_n.$$

LEMMA 3.1 [9]. *Let* C *be the monomial curve corresponding to* S. *Then the tangent cone of* C *is Cohen–Macaulay if and only if for every monomial* $M = x_2^{u_2} \cdots x_n^{u_n}$ *with* $\deg_S(M) \in m_1 + S$ *there exists a monomial* $N = x_1^{v_1} \cdots x_n^{v_n}$ *with* $v_1 > 0$ *such that* $\deg_S M = \deg_S N$ *and* $\deg M \leq \deg N$. □

REMARK 3.2. It is sufficient to check the conditions in Lemma 3.1 for monomials M with $u_i < m_1$, where $i = 2, \ldots, n$. This is because when $u_i \geq m_1$ we have $u_i = q_i m_1 + r_i$

and so

$$\deg_S(M) = \sum_{j=2}^{n} u_j m_j = (q_i m_i) m_1 + r_i m_i + \sum_{j \in \{2,\dots,n\} \setminus \{i\}} u_j m_j = \deg_S(N)$$

for $N = x_1^{q_i m_i} \cdots x_i^{r_i} \cdots x_n^{u_n}$. Since $u_i = q_i m_1 + r_i < q_i m_i + r_i$, it follows that

$$\deg(M) = \sum_{j=2}^{n} u_j < (q_i m_i) + r_i + \sum_{j \in \{2,\dots,n\} \setminus \{i\}} u_j = \deg(N).$$

THEOREM 3.3. *Let C be a monomial curve and C_k be its kth lift. If C has a Cohen–Macaulay tangent cone, then C_k has a Cohen–Macaulay tangent cone for all $k > 1$. If not, there is a positive integer k_0 such that C_k has a Cohen–Macaulay tangent cone for all $k \geq k_0$.*

PROOF. Take a monomial $M = x_2^{u_2} \cdots x_n^{u_n}$ whose S_k-degree $\deg_{S_k}(M) = k \deg_S(M)$ lies in $m_1 + S_k$. It follows from Lemma 2.1 that $\deg_S(M) \in m_1 + S$. This means that there is a monomial $N = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$ such that $v_1 > 0$ and $\deg_S(M) = \deg_S(N)$. Clearly, we also have $\deg_{S_k}(M) = \deg_{S_k}(N_k)$, for $N_k = x_1^{k v_1} x_2^{v_2} \cdots x_n^{v_n}$. So, under the correspondence in Proposition 2.2, the binomial $M - N$ in I_S maps to $M - N_k$ in I_{S_k}.

On the other hand,

$$\deg(N_k) - \deg(M) = \deg(N_k) - \deg(N) + \deg(N) - \deg(M)$$
$$= (k-1)v_1 + \deg(N) - \deg(M). \tag{3.1}$$

Now, if C has a Cohen–Macaulay tangent cone, it follows from Lemma 3.1 that there is at least one N with $\deg(N) - \deg(M) \geq 0$. Thus, we have $\deg(N_k) - \deg(M) \geq 0$ for the corresponding N_k by 3.1. So, C_k has a Cohen–Macaulay tangent cone by Lemma 3.1.

If C does not have a Cohen–Macaulay tangent cone, then again by Lemma 3.1 there is some monomial M with $\deg_S(M) \in m_1 + S$ such that $\deg(N) - \deg(M) < 0$ for all N with $\deg_S(M) = \deg_S(N)$. Since S is a numerical semigroup, there are only finitely many monomials N with the S-degree $\deg_S(M)$ for a fixed monomial M. Let N_0 be the one with the biggest degree so that $\deg(N) - \deg(M)$ is the biggest possible negative number. Then $(k_0(M) - 1)v_1 + \deg(N) - \deg(M) \geq 0$ for large enough $k_0(M)$. Hence, it follows that $\deg(N_k) - \deg(M) \geq 0$ by (3.1), for all $k \geq k_0(M)$. By Remark 3.2 it is sufficient to check the condition of Lemma 3.1 for finitely many monomials M. So, if we choose k_0 to be the maximum of all $k_0(M)$ corresponding to these monomials, then Lemma 3.1 completes the proof.   □

4. Minimal free resolutions

Let S be a numerical semigroup and S_k be its kth lift as before. In this section we discuss the relation between their homological invariants. We start with the relation

between minimal free resolutions of the semigroup rings $K[S]$ and $K[S_k]$. Recall that $R = K[x_1, \ldots, x_n]$ is S- and S_k-graded, respectively, via

$$\deg_S(x_1) = \deg_{S_k}(x_1) = m_1 \quad \text{and} \quad \deg_{S_k}(x_i) = k \deg_S(x_i) = km_i, \quad \text{for all } i > 1.$$

As indicated in [14], a minimal S_k-graded free resolution of $K[S_k]$ is obtained from a minimal S-graded free resolution of $K[S]$ via the faithfully flat extension $f : R \to R$, defined by sending $x_1 \to x_1^k$ and $x_i \to x_i$ for all $i > 1$. This proves the following proposition.

PROPOSITION 4.1. *If $K[S]$ has a minimal S-graded free resolution given by*

$$0 \longrightarrow \bigoplus_{j=1}^{\beta_{n-1}} R[-b_{n-1,j}] \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\beta_1} R[-b_{1,j}] \longrightarrow R \longrightarrow K[S] \longrightarrow 0,$$

then $K[S_k]$ has a minimal S_k-graded free resolution given by

$$0 \longrightarrow \bigoplus_{j=1}^{\beta_{n-1}} R[-kb_{n-1,j}] \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\beta_1} R[-kb_{1,j}] \longrightarrow R \longrightarrow K[S_k] \longrightarrow 0.$$

It follows from Proposition 2.2 that I_{S_k} has a unique minimal generating set or equivalently is generated minimally by indispensable binomials if and only if I_S has the same property. This means that one of the first matrices in the resolutions of Proposition 4.1 is unique if and only if the other is so. The corresponding notion introduced by Charalambous and Thoma [6] for the full minimal free resolution is strong indispensability of the resolution. Recall that a resolution $(\mathbf{F}, \phi)$ is called strongly indispensable if for any graded minimal resolution $(\mathbf{G}, \theta)$, we have an injective complex map $i : (\mathbf{F}, \phi) \longrightarrow (\mathbf{G}, \theta)$. As a consequence, we get the following result about indispensability of higher syzygies using Lemma 2.1.

PROPOSITION 4.2. *$K[S_k]$ has a strongly indispensable minimal free resolution $\iff$ $K[S]$ has a strongly indispensable minimal free resolution.*

PROOF. By [4, Lemma 19], the algebra $K[S_k]$ has a strongly indispensable minimal free resolution $\iff$ $b - b' \notin S_k$, for all $b, b' \in \mathcal{B}_i(S_k)$. Proposition 4.1 implies that $\mathcal{B}_i(S_k) = k\mathcal{B}_i(S)$ and thus $b \in \mathcal{B}_i(S_k) \iff b = kd$, for some $d \in \mathcal{B}_i(S)$. Thus, $b - b' \notin S_k$ for all $b, b' \in \mathcal{B}_i(S_k) \iff d - d' \notin S$ for all $d, d' \in \mathcal{B}_i(S)$, by Lemma 2.1, which completes the proof by [4, Lemma 19] again. □

DEFINITION 4.3. For an ideal I, a finite subset $G \subset I$ is called a standard basis of I if the least homogeneous summands of the elements of G generate the ideal I^*. In other words, $G \subset I$ is a standard basis of I, if I^* is generated by g^* for $g \in G$.

We use the following crucial fact to relate Betti numbers of tangent cones to liftings.

Lemma 4.4 [10, Lemma 1.2]. *Let I be an ideal of $R = K[x_1, \ldots, x_n]$ with $I \subset \mathfrak{m} = (x_1, \ldots, x_n)$. Suppose that x_1 is a non zero-divisor on $K[[x_1, \ldots, x_n]]/IK[[x_1, \ldots, x_n]]$. Let $\pi : R \to \bar{R} = K[x_2, \ldots, x_n]$ be the K-algebra homomorphism with $\pi(x_1) = 0$ and $\pi(x_i) = x_i$ for $i > 1$, and set $\bar{I} = \pi(I)$. Let $g_1, \ldots, g_r$ be a standard basis of $\bar{I}$ such that there exist polynomials $f_1, \ldots, f_r \in I$ with $\pi(f_i) = g_i$ and $\deg(f_i^*) = \deg(g_i^*)$, for $i = 1, \ldots, r$. Then,*

(a) *$f_1, \ldots, f_r$ is a standard basis of I;*
(b) *x_1 is regular on $gr_\mathfrak{m}(R/I)$;*
(c) *there is an isomorphism*

$$gr_\mathfrak{m}(R/I)/x_1 gr_\mathfrak{m}(R/I) = gr_{\bar{\mathfrak{m}}}(\bar{R}/\bar{I}),$$

of graded K-algebras, where $\bar{\mathfrak{m}} = \pi(\mathfrak{m})$.

Before we state our final result, recall that the curve C is of homogeneous type if $\beta_i(R/I_S) = \beta_i(gr_\mathfrak{m}(R/I_S))$, for all i.

Theorem 4.5. *Let C_k be the kth lifting of C. Then, there exists a positive integer k_0 such that for all $i = 1, \ldots, n-1$ and $k \geq k_0$,*

$$\beta_i(gr_\mathfrak{m}(R/I_{S_k})) = \beta_i(gr_\mathfrak{m}(R/I_{S_{k_0}})).$$

Furthermore, when the tangent cone of C is Cohen–Macaulay, C is of homogeneous type if and only if C_k is of homogeneous type for all $k > 1$.

Proof. First, $\bar{I} = \pi(I_{S_k})$ is independent of the value of k, since $\pi(B_k) = \pi(B) = B$ if B does not involve x_1 and $\pi(B_k) = \pi(M - N_k) = M$ if x_1 divides N_k. Being the image of ideals with binomial generators, $\bar{I}$ has a standard basis consisting of binomials and monomials. These binomials are images of themselves under π as they do not involve the variable x_1. So, we need to prove that for any monomial $M = x_2^{u_2} \cdots x_n^{u_n}$ in this standard basis, there is a binomial $B_k = M - N_k$ in I_{S_k} with $\pi(B_k) = M$ and $\deg(B_k^*) = \deg(M)$. Since $M \in \bar{I}$, there is always a binomial $B_k = M - N_k$ in I_{S_k} with $\pi(B_k) = M$ but the last condition is satisfied exactly when $\deg(M) \leq \deg(N_k)$. In the proof of Theorem 3.3, we demonstrate that there is some positive integer k_0 such that $\deg(M) \leq \deg(N_k)$ for all $k \geq k_0$ and thus the tangent cone $gr_\mathfrak{m}(R/I_{S_k})$ is Cohen–Macaulay for all $k \geq k_0$. Therefore, the hypothesis of Lemma 4.4 holds as x_1 is always regular on $K[[S_k]]$. Thus, we have the following isomorphism

$$gr_\mathfrak{m}(R/I_{S_k})/x_1 gr_\mathfrak{m}(R/I_{S_k}) = gr_{\bar{\mathfrak{m}}}(\bar{R}/\bar{I}),$$

which implies that

$$\beta_i(gr_\mathfrak{m}(R/I_{S_k})/x_1 gr_\mathfrak{m}(R/I_{S_k})) = \beta_i(gr_{\bar{\mathfrak{m}}}(\bar{R}/\bar{I})), \quad \text{for all } i = 1, \ldots, n-1.$$

Since x_1 is not a zero-divisor on $gr_\mathfrak{m}(R/I_{S_k})$, it follows that

$$\beta_i(gr_\mathfrak{m}(R/I_{S_k})/x_1 gr_\mathfrak{m}(R/I_{S_k})) = \beta_i(gr_\mathfrak{m}(R/I_{S_k})), \quad \text{for all } i = 1, \ldots, n-1.$$

Therefore, for all $k \geq k_0$,

$$\beta_i(gr_\mathfrak{m}(R/I_{S_k})) = \beta_i(gr_{\bar{\mathfrak{m}}}(\bar{R}/\bar{I})) = \beta_i(gr_\mathfrak{m}(R/I_{S_{k_0}})), \quad \text{for all } i = 1, \ldots, n-1.$$

When the tangent cone $gr_\mathfrak{m}(R/I_S)$ of C is Cohen–Macaulay, we have $k_0 = 1$ by the proof of Theorem 3.3. Thus, by the first part,

$$\beta_i(gr_\mathfrak{m}(R/I_{S_k})) = \beta_i(gr_\mathfrak{m}(R/I_S)), \quad \text{for all } i = 1, \ldots, n-1$$

and by Proposition 4.1, we have $\beta_i(R/I_{S_k}) = \beta_i(R/I_S)$, for all $i = 1, \ldots, n-1$. Therefore, C is of homogeneous type if and only if C_k is of homogeneous type. $\square$

Acknowledgement

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