EMBEDDINGS INTO GROUPS WITH ONLY A FEW DEFINING RELATIONS

Dedicated to the memory of Hanna Neumann

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It is a trivial consequence of Magnus' solution to the word problem for one-relator groups [9] and the existence of finitely presented groups with unsolvable word problem [4] that not every finitely presented group can be embedded in a one-relator group. We modify a construction of Aanderaa [1] to show that any finitely presented group can be embedded in a group with twenty-six defining relations. It then follows from the well-known theorem of Higman [7] that there is a fixed group with twenty-six defining relations in which every recursively presented group is embedded.

The results of the present paper are analogous for groups of the results of [2] about semigroups; however, no knowledge of [2] is required to read this paper.

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Let A be any finitely presented group. In view of the nature of our proposed theorem we may, without loss of generality, assume that A has two-generators—say $A = \langle a_1, a_2 \rangle$. (See for example [8]).

We begin by regarding A as a semigroup. In this role A has four generators, namely a_1, a_2, a_1^{-1} and a_2^{-1}. When convenient, we shall sometimes write a_3 and a_4 for a_1^{-1} and a_2^{-1} respectively.

$$\text{Let } A = (a_1, a_2, a_3, a_4;\ R_i = 1,\ i = 1, 2, \cdots, n).$$

We begin by applying to A a construction of the type given on p. 307 of [4]. It will turn out that there is one delicate point in the argument. In order that this point will be clear, and to make the paper more easily readable, we specify the various constructions in detail.

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Our first step is to define the semigroup

$$A_* = (a_1, a_2, a_3, a_4, p;\ R_i p = p,\ a_j p = p a_j;\ i = 1,2,\cdots,\ j = 1,2,3,4).$$

Next we give an alternative presentation of A_*. For this we write a_5 for p. Then by introducing additional generators and relations we can present A_* as

$$A_* = (a_1, a_2, \cdots, a_r;\ A_j = B_j,\ j = 1,2,\cdots,s)$$

where each relation is of the form $a_\lambda = a_\mu a_v$, $1 \leqq \lambda, \mu, \nu \leqq r$. By suitable repetition, if necessary, we may assume $s = 2^t$, for some t.

We now embed A_* in a semigroup $\mathfrak{L}$ on generators β and γ. The embedding is defined by the mapping ψ given by $a_i^\psi \equiv \beta\beta\gamma^i\beta\gamma^{r+1-i}$ and $(UV)^\psi \equiv U^\psi V^\psi$. Then $\mathfrak{L} = (\beta,\gamma;\ A_j^\psi = B_j^\psi,\ j = 1,2,\cdots,s)$.

Let $u = r+4$; then A_j^ψ contains u symbols and B_j^ψ contains $2u$ symbols. Writing $x_{i,j}$ for the i-th symbol of A_j^ψ and $y_{i,j}$ for the i-th symbol of B_j^ψ we define

$$M \equiv x_{1,1}x_{1,2}\cdots x_{1,s}x_{2,1}\cdots x_{2,s}\cdots x_{u,1}\cdots x_{u,s}$$

and

$$N \equiv y_{1,1}y_{1,2}\cdots y_{1,s}y_{2,1}\cdots y_{2,s}\cdots y_{2u,1}\cdots y_{2u,s}$$

(recall that $s = 2^t$).

Let $\mathfrak{M}$ be the semigroup with presentation

$$\beta, \gamma, \varepsilon;$$

$$\varepsilon\beta\beta = \beta\varepsilon \quad \varepsilon\gamma\beta = \beta\varepsilon$$

$$\varepsilon\beta\gamma = \gamma\varepsilon \quad \varepsilon\gamma\gamma = \gamma\varepsilon$$

$$M = N$$

We now define the semigroup $\mathfrak{N} = (\alpha, \sigma;\ \sigma\alpha\alpha = \alpha\alpha\sigma\alpha\sigma,\ \sigma\alpha\alpha = \alpha\alpha\sigma\sigma, M^\tau N^\tau)$, where

$$\beta^\tau \equiv \sigma\alpha,\ \gamma^\tau \equiv \sigma,\ \varepsilon^\tau \equiv \alpha\alpha \text{ and } (YZ)^\tau \equiv Y^\tau Z^\tau.$$

We write χ for the composition of ψ and τ.

THEOREM 1. *Let U and V be any two words of A. Then*

(i) $(UV)^\chi \equiv U^\chi V^\chi$ (i.e. *are identical as words*);
(ii) *If $U =_A V$ then $U^\chi p^\chi \sigma\alpha^{2t} =_{\mathfrak{N}} V^\chi p^\chi \sigma\alpha^{2t}$;*
(iii) *If Z is a word of $\mathfrak{N}$ and $Zp^\chi\sigma\alpha^{2t} =_{\mathfrak{N}} V^\chi p^\chi \sigma\alpha^{2t}$,*
then there exists a word W of A such that $Z \equiv W^\chi$ and $W =_A V$.

PROOF. Part (i) is trivial. Parts (ii) and (iii) are more or less in the literature. Initially one must note that where U and V are words of $Up =_{A_*} Vp$ if and only if $U =_A V$. Thereafter the most detailed source is probably [2] ([10] contains only a sketch).

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[Unfortunately it has not been possible for us simply to quote results exactly as they occur in the literature. Also we have not been able to make our notation mesh precisely with that in the literature.]

We now employ the $G(\mathfrak{T}, \Phi_0)$ construction on page 307 of [5] with $\mathfrak{R}$ in the role of $\mathfrak{T}$. We rewrite $\mathfrak{R}$ as

$$(s_1, s_2;\ P_i = Q_i,\ i = 1,2,3)$$

and define

$$\mathfrak{R}_* = (s_1, s_2, q;\ P_i q = qQ_i,\ s_j q = qs_j,\ i = 1,2,3,\ j = 1,2).$$

For convenience we write the relations of $\mathfrak{R}_*$ as $F_i q = qK_i$, $i = 1,2,\cdots,5$ and write $\Phi_0 \equiv p^{\times}\sigma\alpha^{2t} \equiv p^{\times}s_2 s_1^{2t}$. Then the group $G = G(\mathfrak{R}, \Phi_0)$ has presentation:

$$s_1, s_2, q, t, k, a, d;$$

$$as_j = s_j a \qquad ds_j = s_j d^6 a d^6$$

$$d^i a d_i \bar{F}_i q = qK_i d^i a d^i$$

$$ta = at \qquad td = dt$$

$$ka = ak \qquad kd = dk$$

$$k(\Phi_0^{-1} q^{-1} tq\Phi_0) = (\Phi_0^{-1} q^{-1} tq\Phi_0)k$$

where $i = 1,2,\cdots 5$, $j = 1,2$ and $\bar{F}_i$ is obtained from F_i by replacing s_j by s_j^{-1}.

For any word W of A, let W^θ denote the word obtained from $W^\times$ by replacing s_j by s_j^{-1}.

THEOREM 2. *For any word W of A, $W =_A 1$ if and only if*

$$k^{-1}(\Phi_0^{-1} q^{-1} (W^\theta)^{-1} tW^\theta q\Phi_0)k =_G \Phi_0^{-1} q^{-1} (W^\theta)^{-1} tW^\theta q\Phi_0 .$$

PROOF. After sorting out the notation one sees that this is an immediate consequence of Theorem 1 (ii) above and Lemma C and Technical Result (i) of [5].

At this point we begin to follow Aanderaa [1]. Let $k_0 \equiv q\Phi_0 k\Phi_0^{-1} q^{-1}$; then of course Theorem 2 says that $W =_A 1$ if and only if $k_0^{-1}(W^\theta)^{-1} tW^\theta k_0 =_G (W^\theta)^{-1} tW^\theta$. We shall sometimes regard k_0 as an abbreviation and at other times as a new generating symbol.

Let $C = \langle c_1, c_2 \rangle$ be an isomorphic copy of A. It is worth stressing that we shall in fact embed C rather than A.

Let $K_1 = G * C$. We shall define a sequence of groups using the well-known *HNN*-construction (see [8] or [4]) or, to use alternative standard terminology in a 'word problem' context, Britton extensions. We shall call such a construction an *HNNB*-extension. It will not always be trivial to verify that we have a legitimate instance of the construction, i.e. that Britton's isomorphism condition holds. For the moment we simply give the presentations, reserving the verifications till later. They are:

$K_2 = (K_1, b_1, b_2;\ b_i^{-1}s_jb_i = s_j,\ b_i^{-1}c_jb_i = c_j,\ b_i^{-1}k_0b_i = k_0c_i^{-1};\ i,j = 1,2);$

$K_3 = (K_2, f; f^{-1}(a_i^{\varepsilon})^{\theta}b_i^{\varepsilon}f = (a_i^{\varepsilon})^{\theta},\ f^{-1}k_0f = k_0,\ i = 1,2\ \varepsilon = \pm 1);$

$K_4 = (K_3, h;\ h^{-1}th = tf,\ h^{-1}k_0h = k_0,\ h^{-1}s_jh = s_j,\ j = 1,2).$

THEOREM 3. (i) *K_1 is naturally embedded in K_4.*

(ii) *The defining relations of C are consequences of the remaining thirty-three defining relations of K_4.*

Before proving Theorem 3, we take the final step in our argument. It will be observed that K_4 has a number of commuting relations. Using a technique due essentially to Borisov [3] we can eliminate some of these.

THEOREM 4. *K_4 can be embedded in a group K_4' with twenty-six defining relations*

PROOF. Define $a_0 \equiv q\Phi_0 a\Phi_0^{-1}q^{-1}$ and $d_0 \equiv q\Phi_0 d\Phi_0^{-1}q^{-1}$. Then we can present K_4 in such a way that the generator k_0 commutes with the generators a_0, d_0, f, h and t. Also b_1 and b_2 commute with s_1, s_2, c_1 and c_2. If we twice apply Borisov's theorem in the form in which it appears in [6], it is clear that with our first application we can reduce the number of commuting relations by three, and with our second application by another four.

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We now prove Theorem 3, beginning with part (ii).

Let $W = {}_A1$; we shall write W_b and W_c for the copies of W in b_1, b_2 and c_1, c_2 respectively. We shall deduce $W_c = 1$ from the relations of K_4 excluding the relations of C. We have, by Theorem 2

$$k_0^{-1}(W^{\theta})^{-1}t(W^{\theta})k_0 = (W^{\theta})^{-1}tW^{\theta}. \qquad (*)$$

If we conjugate both sides by h and use the relations involving h we obtain

$$k_0^{-1}(W^{\theta})^{-1}tfW^{\theta}k_0 = (W^{\theta})^{-1}tfW^{\theta}.$$

If we introduce suitable inverse pairs and use (*) we obtain

$$k_0^{-1}(W^{\theta})^{-1}fW^{\theta}k_0 = (W^{\theta})^{-1}fW^{\theta}.$$

Now it follows from the definition of χ and θ that if

$$W \equiv a_{i_1}a_{i_2}\cdots a_{i_n} \text{ then } W^{\theta} \equiv a_{i_1}^{\theta}a_{i_2}^{\theta}\cdots a_{i_n}^{\theta}.$$

Hence we may pass f from left to right across W^{θ} to obtain $k_0^{-1}W_bk_0 = W_b$. Since $k_0^{-1}b_ik_0 = b_ic_i$, we have $W_bW_c = W_b$ and so $W_c = 1$.

To prove part (i) we must establish three isomorphisms.

(1) K_2 is an $HNNB$-extension of K_1. Here we must show that for each i, $\langle s_j, k_0, c_l; j, l = 1, 2\rangle$ and $\langle s_j, k_0 c_i^{-1}, c^i; j, l = 1, 2\rangle$ are isomorphic under $s_j \to s_j$, $k_0 \to k_0 c^{-1}$, $c_l \to c_l$. This follows from the fact that $\langle s_j, k_0, c_l\rangle$ is the free product $\langle s_j, k_0\rangle * C$ and $\langle s_j, k_0\rangle$ is free on these generators.
(2) K_3 is an $HNNB$-extension of K_2. This time we must look at $(a_i^\varepsilon)^\theta b_i^\varepsilon \to (a_i^\varepsilon)^\theta$, $k_0 \to k_0$. We shall show that $\langle (a_i^\varepsilon)^\theta, k_0\rangle$ is free on these generators. Then when we map $K_2 \to G$ by putting $b_i = 1$, $i = 1, 2$ we map $\langle (a_i^\varepsilon)^\theta b_i^\varepsilon, k_0\rangle$ onto $\langle (a_i^\varepsilon)^\theta, k_0\rangle$. It suffices to prove that $\langle (a_i^\varepsilon)^\theta$ is free in $\langle s_1, s_2\rangle$.

We recall the definition of θ. Writing a_3 and a_4 for a_1^{-1} and a_2^{-1}, we see that

$$a_i^\theta \equiv (s_2^{-1} s_1^{-1})^2 s_2^{-(i+1)} s_1^{-1} s_2^{-(r+1-i)}.$$

Let us call the right most s_1^{-1} the central symbol of a_i^θ. Then it is not hard to see that in a product $(a_i^\theta)^\eta (a_j^\theta)^\nu$, $\eta, \nu = \pm 1$ the central symbols of a_i^θ and a_j^θ are not cancelled unless $i = j$ and $\eta + \nu = 0$. This extends to arbitrary products and the desired assertion follows.

(3) K_4 is an $HNNB$-extension of K_3. In this case we must prove that $D = \langle s_j, t, k_0\rangle$ and $E = \langle s_j, tf, k_0\rangle$ are naturally isomorphic. To do so we shall show that D has presentation

$$(s_1, s_2, k_0; k_0^{-1}(W^\theta)^{-1} t W^\theta k_0 = (W^\theta)^{-1} t W^\theta)$$

where W ranges over all relators of A.

Let X be any word in s_1, s_2, t and k_0, such that $X =_G 1$. We want to reduce X to the empty word using only the displayed relations. If X contains a subword $t^\eta W^\theta k$, where $W =_A 1$, we replace this subword by $W^\theta k^\nu (W^\theta)^{-1} t^\eta W^\theta$ and freely reduce the resulting word. This operation is clearly equivalent to an application of the given relations. Let X_1 be the result of iterating this procedure for as long as possible.

Suppose X_1 is non-empty. Since both $\langle s_j, t\rangle$ and $\langle s_j, k_0\rangle$ are free we may assume both t and k_0 appear in X_1. We can apply Britton's Lemma [4] to G with k_0 as stable letter. Hence X_1 contains a subword $Z_1 t^{m_1} Z_2 \cdots Z_r t^{m_r} Z_{r+1}$ which belongs to $\langle a_0, d_0, t\rangle$. Next we may use t as stable letter to deduce that there exist words R_1 and R_2 on a and b such that $Z_{r+1} q \Phi_0 = R_1 q \Phi_0 R_2$ in the group G_2 of [5]. (For the moment we use the notation of [5].) This last equality yields the two equalities

$$R_1^{-1} Z_{r+1} A_{i_1}^{\varepsilon_1} \cdots A_{i_n}^{\varepsilon_n} =_{G_4} 1 \text{ and } \Phi_0 R_2 \Phi_0^{-1} B_{i_1}^{\varepsilon_1} \cdots B_{i_n}^{\varepsilon_n} =_{G_4} 1.$$

We require a lemma.

LEMMA. *If $RZA_{i_1}^{\varepsilon_1} \cdots A_{i_n}^{\varepsilon_n} =_{G_4} 1$, then Z is a negative word on the s_j-symbols.*

Proof. The s-reduced word obtainable from $A_{i_1}^{\varepsilon_1} \cdots A_{i_n}^{\varepsilon_n}$ has form $\Pi_{k=1}^n (D_{m_k}^{i_k} a D_{m_k}^{i_k} \overline{\Delta}_k)^{\varepsilon_k}$, by Lemma 14 of [5] (correcting a misprint). Moreover $m_1 = 0$

(for $\varepsilon_1 = +1$ or $\varepsilon_1 = -1$). It then follows from Lemma 12 of [5] that Z cannot contain positive occurrences of s_j.

Picking up the main argument again we see that we now have exactly the hypotheses of Lemma 19 of [5] and thus $Z_{r+1} \equiv \bar{\Delta}$ where $\Delta q \Phi_0 = _{\mathfrak{N}_*} q\Phi_0$. In turn we obtain $\Delta\Phi_0 = _{\mathfrak{N}}\Phi_0$. After sorting out the notation, we see that, by Theorem 1 (iii), $\Delta \equiv W^{\times}$ for some relator W of A. Thus $Z_{r+1} \equiv W^{\theta}$ and this is a contradiction.

We complete the whole argument by showing that E has presentation

$$(s_1, s_2, t_1, k_0; \kappa_0^{-1}(W^{\theta})^{-1}t_1 W^{\theta} k_0 = (W^{\theta})^{-1} t_1 W^{\theta})$$

where we write $t_1 \equiv tf$ and W ranges over all relators of A.Us ng the relations of K_3 we see that

$$k_0^{-1}(W^{\theta})^{-1}t_1 W^{\theta} k_0 = _{K_3} k_0^{-1}(W^{\theta})^{-1} t W^{\theta} k_0 \, W_b \, W_c$$

and $(W^{\theta})^{-1}t_1 W^{\theta} = _{K_3}(W^{\theta})^{-1} t W^{\theta} W_b$. So the displayed relations certainly hold in E. Using the fact that k_0 is a stable letter for D over $\langle s_j, t\rangle$ we can then easily check that the natural map from D to E is an isomorphism.

One point in our construction is perhaps worth comment. Instead of Theorem 2 as stated, we could have asserted that $W = _A 1$ if and only if

$$k^{-1}(\Phi_0^{-1}(W^{\times})^{-1}q^{-1}tqW^{\times}\Phi_0)k = _G \Phi_0^{-1}(W^{\times})^{-1}q^{-1}tqW^{\times}\Phi_0.$$

Defining $t_0 \equiv q^{-1}tq$ and $k_0 \equiv \Phi_0 k \Phi_0^{-1}$ we then have $W = _A 1$ if and only if $k_0^{-1}(W^{\times})^{-1}t_0 W^{\times} k_0 = _G (W^{\times})^{-1} t_0 W^{\times}$. What is interesting is that if we attempt to repeat the construction with t_0 in the role of t (and χ instead of θ) the argument breaks down.

To see this we observe that there certainly exists a word Φ of $\mathfrak{N}$ such that $\Phi = _{\mathfrak{N}}\Phi_0$ and $Y \equiv \Phi\Phi_0^{-1}$ is freely reduced. Then it follows that

$$k_0^{-1}Y^{-1}t_0 Y k_0 =_G Y^{-1}t_0 Y \text{ while } k_0^{-1}Y^{-1}t_0 f Y k_0 \neq _{K_3} Y^{-1} t_0 f Y.$$

Thus D and E cannot be isomorphic in the natural way.

We conclude with a few speculative remarks concerning the best possible result along the lines of our theorem. It is of course tempting to conjecture that every finitely presented group is embeddable in a two-relator group. If this be so, then there exist two-relator groups with unsolvable word problem, – but the best result presently known for this question is that there exist groups with twelve relators and unsolvable word problem.

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