how they may be extended to any number of coaxial cylindrical layers of different substances, solid and fluid. We would only remark that a diminution of density in the gas surrounding the cylinder has exactly the same effect in diminishing the variable part of the pressure, and so lowering the sound, as it had in the case of the spherical shell.

Eighth Meeting, June 11th, 1886.

DR FERGUSON, F.R.S.E., President, in the Chair.

A Problem in Combinations.

By ALEXANDER ROBERTSON, M.A.

I.

Given sets of balls of different colours, in how many ways may they be arranged in line so that no two balls of the same colour shall come together.

If we have two colours only, and the same number m of each colour, there are evidently two arrangements possible; if we have $m, m-1$ respectively, only one arrangement is possible; if we have $m, m-2$; $m, m-3$, &c., no arrangement is possible. We may write these results

$(m, m)=2$, $(m, m-1)=1$, $(m, m-2)=0$, $(m, m-3)=0$, &c.

In the sequel we shall consider three colours only, say m white balls, n black balls, and p red balls. The number of arrangements is denoted by (mnp), and in general we shall take m greater than n and n greater than p, but not always.

Keeping m and n constant, the smallest value of p is $m-n-1$. There is also a superior limit to p; taking n as then the smallest we should have, smallest value of $n=p-m-1$, therefore $p=m+n+1$, and therefore the total number of values of p is $2n+3$. For example, let $m=5$ and $n=2$, the smallest value of p is 2 and the greatest 8; or p may have seven values, so that arrangements are possible with 522, 523, 524, 525, 526, 527, 528. If, however, m and n are equal, the case is slightly different; then the smallest value of p is 0, and the greatest is $2m+1$, or p has $2m+2$ values.

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We proceed now to discuss the number of arrangements of 4 white (W), 3 black (B), and 2 red (R) balls, or (432). The natural way would be to write the 9 numerals on 9 cards, and first find the possible arrangements of the 4 white balls, by considering in how many ways 123......9 may be arranged 4 at a time so that no two consecutive numbers shall come together. We find that we have 15 such arrangements, which are given in the table, with the remainders of 5 places for the other balls.

		Remainders.	Values.
(1)	1357	24689	6
(2)	1358	24679	6
(3)	1359	24678	3
(4)	1368	24579	6
(5)	1469	24578	4
(6)	1379	24568	3
(7)	1468	23579	6
(8)	1469	23578	4
(9)	1479	23568	4
(10)	1579	23468	3
(11)	2468	13579	10
(12)	2469	13578	6
(13)	2479	13568	6
(14)	2579	13468	6
(15)	3579	12468	6
			79 in all.

We have now to take these remainders and divide each into groups of 3's and 2's, rejecting all those in which two consecutive numbers come together. Thus, taking the first remainder 24689, it gives

246	~~89~~
248	69
249	68
268	49
269	48
~~289~~	
468	29
469	28
~~489~~	
~~689~~	

So that rejecting the unsuitable ones, we get six possible arrangements. We may call this the value due to this first remainder. Treating them all in the same way we get the third column of the table, or values due to the different remainders, and the sum of these being 79 gives us the result

$$(432)=79.$$

Generalizing this method, the number of arrangements of the m W balls is C_m^{n+p+1}. The following table gives specimens of the value due to a good many remainders. The value is found to depend entirely on the number and range of the sequences in the different remainders. This will abridge the work very much. The value is a maximum when there is no sequence, and is then C_n^{n+p} as in the 11th remainder in the above table.

Species of sequences.	Values.	Species of sequences.	Values.
(0)	C_n^{n+p}	(3, 3)	C_{n-2}^{n+p-4}
(2)	$2C_{n-1}^{n+p-2}$	(2, 4)	$4C_{n-3}^{n+p-6}$
(3)	C_{n-1}^{n+p-2}	(2, 5)	$2C_{n-3}^{n+p-6}$
(4)	$2C_{n-2}^{n+p-4}$	(3, 4)	$2C_{n-3}^{n+p-6}$
(5)	C_{n-2}^{n+p-4}	(2, 2, 2)	$8C_{n-3}^{n+p-6}$
(6)	$2C_{n-3}^{n+p-6}$	(2, 2, 3)	$4C_{n-3}^{n+p-6}$
(2, 2)	$4C_{n-2}^{n+p-4}$	(2, 3, 3)	$2C_{n-3}^{n+p-6}$
(2, 3)	$2C_{n-2}^{n+p-4}$	(2, 2, 2, 2)	$16C_{n-3}^{n+p-6}$

The work for (432) may therefore be put down as follows—

Species.	Number.	Value.	Arrangements.
(0)	1	10	10
(2)	8	6	48
(3)	3	3	9
(22)	3	4	12
	—		—
	15		79

Application to (543).

Number of arrangements of the $5=C_5^8=56$.

On analysing the remainders as to their sequences we get the following results—

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Species.	Number.	Value.	Arrangements.
(2)	6	20	120
(3)	10	10	100
(4)	4	6	24
(2, 2)	20	12	240
(2, 3)	12	6	72
(2, 2, 2)	4	8	32
			588

II.

We have now to discuss a general theorem which will enable us to make the number of arrangements in any case depend on similar arrangements with a smaller number of elements.

General Theorem.

$$(mnp) = (m-1,\ n-1,\ p) + (m-1,\ n,\ p-1) + (m,\ n-1,\ p-1)$$
$$+ 2(m-1,\ n-1,\ p-1)$$

Let W_{mnp}, B_{mnp}, R_{mnp} denote those of (mnp) which commence with a W, a B, or a R respectively, so that

$$(mnp) = (W + B + R)_{mnp}.$$

If we have one W ball more, and put it on at the beginning of the B and R portions of (mnp) we evidently get the W part of $(m+1, n, p)$ that is

$$W_{m+1,\,n,\,p} = (B + R)_{mnp}$$

Similarly $\qquad W_{mnp} = (B + R)_{\overline{m-1},\,n,\,p}$

so $\qquad B_{mnp} = (W + R)_{m,\,\overline{n-1},\,p}$

and $\qquad R_{mnp} = (W + B)_{m,\,n,\,\overline{p-1}}$

therefore $\qquad (mnp) = (W + B + R)_{mnp}$

$$= (B + R)_{m-1,\,n,\,p} + (W + R)_{m,\,n-1,\,p} + (W + B)_{m,\,n,\,p-1}$$

But $\qquad B_{\overline{m-1},\,n,\,p} = (W + R)_{\overline{m-1},\,\overline{n-1},\,p}$

$$R_{\overline{m-1},\,n,\,p} = (W + B)_{m-1,\,n,\,p-1}$$
$$W_{m,\,\overline{n-1},\,p} = (B + R)_{\overline{m-1},\,\overline{n-1},\,p}$$
$$R_{m,\,\overline{n-1},\,p} = (W + B)_{m,\,\overline{n-1},\,\overline{p-1}}$$
$$W_{m,\,n,\,\overline{p-1}} = (B + R)_{\overline{m-1},\,n,\,\overline{p-1}}$$
$$B_{m,\,n,\,\overline{p-1}} = (W + R)_{m,\,\overline{n-1},\,\overline{p-1}}$$

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therefore $(mnp) = (2W + B + R)_{m,\overline{n-1},\overline{p-1}} + (W + 2B + R)_{\overline{m-1},n,\overline{p-1}}$
$+ (W + B + 2R)_{\overline{m-1},\overline{n-1},p}$
$= (m, n-1, p-1) + (m-1, n, p-1) + (m-1, n-1, p)$
$+ W_{m,n-1,p-1} + B_{m-1,n,p-1} + R_{m-1,n-1,p}$

But $W_{m,n-1,p-1} = (B+R)_{m-1,n-1,p-1}$
$B_{m-1,n,p-1} = (W+R)_{m-1,n-1,p-1}$
$R_{m-1,n-1,p} = (W+B)_{m-1,n-1,p-1}$

therefore $W_{m,n-1,p-1} + B_{m-1,n,p-1} + R_{m-1,n-1,p} = 2(m-1, n-1, p-1)$

therefore $(mnp) = (m-1, n-1, p) + (m-1, n, p-1) + (m, n-1, p-1)$
$+ 2(m-1, n-1, p-1)$.

This gives us a quick method of calculating the number of arrangements in any case.

Application to find (432).

(211)	(311)	(221)	(321)
(101) = 2	(201) = 1	(111) = 6	(211) = 6
(110) = 2	(210) = 1	(120) = 1	(220) = 2
(200) = 0	(300) = 0	(210) = 1	(310) = 0
2(100) = 2	2(200) = 0	2(110) = 4	2(210) = 2
6	2	12	10

(421)	(331)	(322)	(432)
(311) = 2	(221) = 12	(212) = 12	(322) = 38
(320) = 1	(230) = 1	(221) = 12	(331) = 18
(410) = 0	(320) = 1	(311) = 2	(421) = 3
2(310) = 0	2(220) = 4	2(211) = 12	2(321) = 20
3	18	38	79

By this method the following table of the number of arrangements has been computed—

(111)	6
(211)	6
(221)	12
(222)	30
(311)	2
(321)	10
(322)	38
(331)	18
(332)	74
(333)	174
(421)	3
(422)	24
(431)	14
(432)	79
(433)	248
(441)	24
(442)	138
(443)	480
(444)	1092
(522)	6
(531)	4
(532)	44
(533)	212
(541)	18
(542)	135
(543)	588
(544)	1668
(551)	30
(552)	222
(553)	1026
(554)	3228
(555)	7188
(632)	10
(633)	100
(641)	5
(642)	70
(643)	445
(644)	1700
(651)	22
(652)	206
(653)	1150
(654)	4315
(655)	11492
(661)	36
(662)	326
(663)	1882
(664)	7580
(665)	22274
(666)	48852
(777)	339720
(888)	2403588
(999)	17236524

Thus, to find (543)

$$\begin{array}{r} (543) \\ \hline (433) = 248 \\ (442) = 138 \\ (532) = 44 \\ 2(432) = 158 \\ \hline 588 \end{array}$$

To extend the theorem. Let $\overline{rst}$ denote the arrangement $(m-r, n-s, p-t)$, then the theorem may be written

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$$
\begin{aligned}
&\overline{000} = \overline{110} + \overline{101} + \overline{011} + 2.\overline{111}\\
\text{therefore}\quad &\overline{110} = \overline{220} + \overline{211} + \overline{121} + 2.\overline{221}\\
&\overline{101} = \overline{211} + \overline{202} + \overline{112} + 2.\overline{212}\\
&\overline{011} = \overline{121} + \overline{112} + \overline{022} + 2.\overline{122}\\
&2\overline{111} = 2(\overline{221}) + 2(\overline{212}) + 2(\overline{122}) + 4(\overline{222})
\end{aligned}
$$

therefore

$$
\begin{aligned}
\overline{000} = {} & \overline{220} + \overline{202} + \overline{022}\\
& + 2(\overline{211} + \overline{121} + \overline{112})\\
& + 4(\overline{221} + \overline{212} + \overline{122})\\
& + 4.\overline{222}
\end{aligned}
$$

that is

$$
\begin{aligned}
(mnp) = {} & (m-2,\ n-2,\ p) + (m-2,\ n,\ p-2) + (m,\ n-2,\ p-2)\\
& + 2\{(m-2,\ n-1,\ p-1) + (m-1,\ n-2,\ p-1)\\
& \qquad + (m-1,\ n-1,\ p-2)\}\\
& + 4\{(m-2,\ n-2,\ p-1) + (m-2,\ n-1,\ p-2)\\
& \qquad + (m-1,\ n-2,\ p-2)\}\\
& + 4(m-2,\ n-2,\ p-2)
\end{aligned}
$$

and so on to any number of steps.

Another mode of extending it is as follows, and is much simpler.

$$
\begin{aligned}
\text{As before}\quad & \overline{000} = \overline{110} + \overline{101} + \overline{011} + 2.\overline{111}\\
\text{therefore}\quad & \overline{111} = \overline{221} + \overline{212} + \overline{122} + 2.\overline{222}\\
& \overline{222} = \overline{332} + \overline{323} + \overline{233} + 2.\overline{333}\\
& \overline{333} = \overline{443} + \overline{434} + \overline{344} + 2.\overline{444}
\end{aligned}
$$

Add and reject terms which occur on both sides, we get

$$
\begin{aligned}
\overline{000} = {} & \overline{110} + \overline{101} + \overline{011}\\
& + \overline{221} + \overline{212} + \overline{122}\\
& + \overline{332} + \overline{323} + \overline{233}\\
& + \overline{443} + \overline{434} + \overline{344}\\
& + \overline{111} + \overline{222} + \overline{333} + 2.\overline{444}
\end{aligned}
$$

whose interpretation is obvious.

III.

Expression of the number of arrangements by an algebraic formula. Keep n and p constant, say $n = 5$, $p = 2$, and m variable, we have

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$$(052) = 0$$
$$(152) = 0$$
$$(252) = 6$$
$$(352) = 44$$
$$(452) = 135$$
$$(552) = 222$$
$$(652) = 206$$
$$(752) = 102$$
$$(852) = 21$$

If we divide by the co-efficients of the 8th power of a binomial we get a regular series.

$$0 \quad 0 \quad \tfrac{3}{14} \quad \tfrac{11}{14} \quad \tfrac{27}{14} \quad \tfrac{111}{28} \quad \tfrac{103}{14} \quad \tfrac{51}{4} \quad 21$$

Multiply by 28, the L.C.M. of the denominators, and difference

0	0	6	22	54	111	206	357	588
	0	6	16	32	57	95	151	231
		6	10	16	25	38	56	80
			4	6	9	13	18	24
				2	3	4	5	6
					1	1	1	1

therefore by finite differences

$$(m52) = \frac{6C_2^m + 4C_3^m + 2C_4^m + C_5^m}{28} \times C_m^8$$

It would, perhaps, be simpler to reject the 0, 0 and divide by co-efficients of the 6th power instead of the 8th.

Applying this process to $m11$, $m21$, $m31$, we would get $mn1$

then to $m12$, $m22$, $m32$, „ „ $mn2$

to $m13$, $m23$, $m33$, „ „ $mn3$

and so on.

(We do not actually require to find them all, for $(m21) = (m12)$, &c.)

There results

$$(mn1) = \frac{C_{n-1}^m + 2C_{n-2}^m}{\frac{1}{2}C_3^{n+2}} C_m^{n+2}$$

$$(mn2) = \frac{C_1^n(C_n^m + 2C_{n-1}^m) + 10(2C_{n-2}^m + 3C_{n-3}^m)}{2C_4^{n+3}} \times C_m^{n+3}$$

$$(mn3) = C_2^{n+1}(C_{n+1}^m + 2C_n^m) + 10C_1^{n-1}(2C_{n-1}^m + 3C_{n-2}^m) + 70(3C_{n-3}^m + 4C_{n-4}^m)$$
$$\times \frac{C_m^{n+4}}{5C^{n+4}}$$

$$(mn4) = C_3^{n+2}(C^m_{n+2} + 2C^m_{n+1}) + 10C_2^n(2C^m_n + 3C^m_{n-1})$$
$$+ 70C_1^{n-2}(3C^m_{n-2} + 4C^m_{n-3}) + 420(4C^m_{n-4} + 5C^m_{n-5})$$
$$\times \frac{C_m^{n+5}}{10C_6^{n+5}}$$

$$(mn5) = C_4^{n+3}(C^m_{n+3} + 2C^m_{n+2}) + 10C_3^{n+1}(2C^m_{n+1} + 3C^m_n)$$
$$+ 70C_2^{n-1}(3C^m_{n-1} + 4C^m_{n-2}) + 420C_1^{n-3}(4C^m_{n-3} + 5C^m_{n-4})$$
$$+ 2310(5C^m_{n-5} + 6C^m_{n-6})$$
$$\times \frac{C_m^{n+6}}{\frac{35}{2}C_7^{n+6}}$$

The numerical factors $\frac{1}{2}$, 2, 5, 10 $\frac{35}{2}$, &c., in the denominators are evidently

$$\tfrac{1}{2}C_3^3,\ \tfrac{1}{2}C_3^4,\ \tfrac{1}{2}C_3^5,\ \tfrac{1}{2}C_3^6,\ \tfrac{1}{2}C_3^7,\ \&c.$$

In the numerators occur the factors 1, 10, 70, 420, 2310, &c., whose law is not so obvious. Denoting them at present by A_1, A_2, A_3, A_4, A_5, &c., and by comparing the above results we may write

$$(mnp) = A_1C_{p-1}^{n+p-2}(C^m_{n+p-2} + 2C^m_{n+p-3})$$
$$+ A_2C_{p-2}^{n+p-4}(2C^m_{n+p-4} + 3C^m_{n+p-5})$$
$$+ A_3C_{p-3}^{n+p-6}(3C^m_{n+p-6} + 4C^m_{n+p-7})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ A_p(pC^m_{n-p} + \overline{p+1}C^m_{n-p-1}) \times \frac{C_m^{n+p+1}}{\frac{1}{2}C_3^{p+2}C_{p+2}^{n+p+2}}$$

If in this we put $m = n - p$, all the terms vanish except the last,

therefore $$(n-p,\ n,\ p) = A_p(pC_{n-1}^{n-p} + \overline{p+1}C_{n-p-1}^{n-p} \times \frac{C_{n-p}^{n+p+1}}{\frac{1}{2}C_3^{p+2}C_{p+2}^{n+p+1}}$$

In this put $n = p + 1$, we get after some reductions

$$(p+1,\ p,\ 1) = A_p\frac{(2p+1)C_{2p+1}^{2p+2}}{\frac{1}{2}C_3^{p+2}C_p^{2p+2}}$$

but we have already seen that

$$(p+1,\ p,\ 1) = \frac{C_{p-1}^{p+1} + 2C_{p-2}^{p+1}}{\frac{1}{2}C_3^{p+2}} \times C_{p+1}^{p+2}$$
$$= \frac{C_2^{p+1} + 2C_3^{p+1}(p+2)}{\frac{1}{2}C_3^{p+2}}$$

Equate these values and reduce, we get

$$A_p = \frac{p(p+2)}{12}C_p^{2p+2} = \frac{(2p+2)!}{12(p-1)!(p+1)!}$$

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Hence

$$\begin{aligned}(mnp) = \frac{1}{12}\Bigg[& \frac{4\,!}{2\,!}C_{p-1}^{n+p-2}(C_{n+p-2}^{m} + 2C_{n+p-3}^{m}) \\ & + \frac{6\,!}{3\,!}C_{p-2}^{n+p-4}(2C_{n+p-4}^{m} + 3C_{n+p-5}^{m}) \\ & + \frac{8\,!}{2!4!}C_{p-3}^{n+p-6}(3C_{n+p-6}^{m} + 4C_{n+p-7}^{m}) \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + \frac{(2p+.2)\,!}{(p-1)!\,(p+1)!}(pC_{n-p}^{m} + \overline{p+1}C_{n-p-1}^{m})\Bigg] \\ & \qquad \times \frac{C_{m}^{n+p+1}}{\tfrac{1}{2}C_{3}^{p+2}C_{n-1}^{n+p+1}}\end{aligned}$$

By finding the r^{th} term of this expression, and making some obvious reductions, we get

$$(mnp) = \Sigma\frac{\overline{m-1}r+n+p}{r(r+1)}C_{r-1}^{n-1}C_{r-1}^{p-1}C_{n+p+1-m}^{2r+2}$$

the number of terms is not greater than p, but may be less, from the vanishing of the last factor.

This result may also be written more symmetrically

$$(mnp) = \Sigma\left\{\begin{array}{l}(C_r^{n}C_{r-1}^{p-1} + C_r^{p}C_{r-1}^{n-1})C_{n+p+1-m}^{2r+1} \\ +\,(C_r^{n-1}C_{r-1}^{p-1} + C_r^{p-1}C_{r-1}^{n-1})C_{n+p-m}^{2r+1}\end{array}\right\}$$

The following method will be found convenient when we wish to calculate all the arrangements possible when two elements m, n are kept constant.

Calculate the coefficients K by the formulæ

$$K_{2r} = C_r^{n-1}C_{n-r}^{m-1} + C_{r-1}^{n-1}C_{n-r-1}^{m-1}$$
$$K_{2r+1} = 2C_r^{n-1}C_{n-r-1}^{m-1}$$

Then

$$\begin{aligned}(m,\ n,\ m-n-1) &= K_0 \\ (m,\ n,\ m-n) &= C_1^{2n+2}K_0 + K_1 \\ (m,\ n,\ m-n+1) &= C_2^{2n+2}K_0 + C_1^{2n+1}K_1 + K_2 \\ (m,\ n,\ m-n+2) &= C_3^{2n+2}K_0 + C_2^{2n+1}K_1 + C_1^{2n}K_2 + K_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (m,\ n,\ m+n-2) &= C_3^{2n+2}K_0 + C_3^{2n+1}K_1 + C_3^{2n}K_2\ldots\ldots + K_{2n-1} \\ (m,\ n,\ m+n-1) &= C_2^{2n+2}K_0 + C_2^{2n+1}K_1 + C_2^{2n}K_2\ldots\ldots + C_2^{3}K_{2n-1} \\ (m,\ n,\ m+n) &= C_1^{2n+2}K_0 + C_1^{2n+1}K_1 + C_1^{2n}K_2\ldots\ldots + C_1^{3}K_{2n-1} \\ (m,\ n,\ m+n+1) &= K_0 \quad + \quad K_1 + \quad K_2\ldots\ldots + \quad K_{2n-1}\end{aligned}$$

It may be noticed that the sums of alternate terms of these are equal, and that the total sum is always divisible by 16.