CONVOLUTION OF ORBITAL MEASURES IN SYMMETRIC SPACES

BOUDJEMÂA ANCHOUCHE and SANJIV KUMAR GUPTA ✉

Abstract

Let G/K be a noncompact symmetric space, G_c/K its compact dual, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, H be an element of $\mathfrak{a}$, $a = \exp(H)$, and $a_c = \exp(iH)$. In this paper, we prove that if for some positive integer r, $\nu_{a_c}^r$ is absolutely continuous with respect to the Haar measure on G_c, then ν_a^r is absolutely continuous with respect to the left Haar measure on G, where ν_{a_c} (respectively ν_a) is the K-bi-invariant orbital measure supported on the double coset $K a_c K$ (respectively $K a K$). We also generalize a result of Gupta and Hare ['Singular dichotomy for orbital measures on complex groups', *Boll. Unione Mat. Ital.* (9) **III** (2010), 409–419] to general noncompact symmetric spaces and transfer many of their results from compact symmetric spaces to their dual noncompact symmetric spaces.

1. Introduction

In [1], Dunkl proved that the convolution of the surface measure of the sphere in $\mathbb{R}^n$ with itself is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^n$. Ragozin [11, 12] generalized Dunkl's result to the setup of symmetric spaces; more precisely, he proved [12, Theorem 2.5] that if G/K is a Riemannian symmetric space and $\{\nu_i\}_{i=1}^{\dim G/K}$ are K-bi-invariant continuous zonal measures on G, then the convolution $\nu_1 * \cdots * \nu_{\dim G/K}$ is absolutely continuous with respect to the left Haar measure on G, and he conjectured [12, p. 375] that $\dim G/K$ may be improved to $[(\dim G/K - 1)/j] + 1$, where j is the minimum dimension of the nonfinite K-orbits in G/K. In a series of papers, Hare and the second author of this paper have partially improved Ragozin's result mentioned above. In [6, Theorem 3.1] they proved that if $a \in \mathrm{SU}(n) \backslash N_{\mathrm{SU}(n)}(\mathrm{SO}(n))$, where $N_{\mathrm{SU}(n)}(\mathrm{SO}(n))$ is the normalizer of $\mathrm{SO}(n)$ in $\mathrm{SU}(n)$, and $\nu_a = m_{\mathrm{SO}(n)} * \delta_a * m_{\mathrm{SO}(n)}$, where $m_{\mathrm{SO}(n)}$ denotes the Haar measure on $\mathrm{SO}(n)$ and δ_a denotes the point mass at a, then ν_a^n is absolutely continuous

with respect to the Haar measure on $\mathrm{SU}(n)$. Moreover, they proved that there exists $a \in \mathrm{SU}(n) \backslash N_{\mathrm{SU}(n)}(\mathrm{SO}(n))$ such that v_a^{n-1} is singular with respect to the Haar measure on $\mathrm{SU}(n)$. Note that $\dim \mathrm{SU}(n)/\mathrm{SO}(n) = (n^2 + n - 2)/2$. In [4, Theorem 3.1] they proved that for a compact symmetric space G_c/K, there is a dense subset $D \subseteq G_c$ such that if $a_1, a_2 \in D$, then $v_{a_1} * v_{a_2}$ is absolutely continuous with respect to Haar measure on G_c. Finally, in [2, Theorem 4.1], they proved if G_c is a compact, connected, simple, classical Lie group, $G_c^{\mathbb{C}}$ the complexification of G_c, $\mathfrak{g}_c$ the Lie algebra of G_c, and $H \in \mathfrak{g}_c$ is such that $a = \exp(iH) \in G_c^{\mathbb{C}} \backslash N_{G_c^{\mathbb{C}}}(G_c)$, then v_a is absolutely continuous with respect to the left Haar measure on $G_c^{\mathbb{C}}$ if and only if μ_H^r is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{g}_c$, where μ_H is the G_c-invariant orbital measure supported on the adjoint orbit of H in $\mathfrak{g}_c$. The aim of this paper is to extend some of the results mentioned above.

Our main theorem, Theorem 3.1 below, is an extension of Gupta and Hare's result in [2, Theorem 4.1] to general symmetric spaces (the notation is as in Section 2).

Combining the main theorem with some results of Hare and the second author, which were mentioned above, we obtain several interesting results, which will be listed in Section 4. In order not to duplicate the results, we will skip those results and invite the interested reader to go directly to that section.

This paper is organized as follows. In Section 2 we collect some basic facts about symmetric spaces, restricted roots, and orbital measures. In Section 3 we prove the main theorem. In Section 4 we extend some of the results obtained in [4, 6] to the noncompact symmetric spaces using the main theorem. In the final section we show that the inequality of the general transference result Theorem 3.1(1) can be strict and suggest some open problems.

2. Notation and basic facts

Let M be a symmetric space of noncompact type and let G be the connected component of the isometry group of M which contains the identity. Then G is a semisimple Lie group with trivial center which acts transitively on M. Fix a point $p \in M$ and let K be the subgroup of G that fixes p. Then K is a compact subgroup of G and M is diffeomorphic to G/K via the map

$$\Psi : G/K \longrightarrow M$$
$$gK \longmapsto g \cdot p.$$

Throughout what follows we will identify M with G/K, and put $M = G/K$. Let $\mathfrak{g}$ be the Lie algebra of G, $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ be a Cartan involution, and let

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\} \quad \text{and} \quad \mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

The decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is called the Cartan decomposition of the Lie algebra $\mathfrak{g}$. It is easy to check that

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

Inside the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ we consider the subset

$$\mathfrak{g}_c = \mathfrak{k} \oplus i\mathfrak{p}.$$

The Lie bracket of $\mathfrak{g}$ induces a Lie bracket on the complexification on $\mathfrak{g}^{\mathbb{C}}$, and therefore $\mathfrak{g}_c$ inherits a Lie bracket from $\mathfrak{g}^{\mathbb{C}}$, which makes $\mathfrak{g}_c$ a real subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let G_c be a Lie group with Lie algebra $\mathfrak{g}_c$. Then G_c is a compact Lie group and $M_c = G_c/K$ is a compact symmetric space, called the compact dual of $M = G/K$.

Fix a maximal abelian (as a subalgebra of $\mathfrak{g}$) subspace $\mathfrak{a}$ of $\mathfrak{p}$, and denote by $\mathfrak{a}^*$ its dual. For $\alpha \in \mathfrak{a}^*$, $\alpha \neq 0$, consider the set

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

The set Σ of restricted roots is defined by

$$\Sigma = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq 0\}.$$

Then it is very well known that Σ is finite and nonempty, and we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid [H, X] = 0 \text{ for all } H \in \mathfrak{a}\} = \mathfrak{a} \oplus \mathfrak{m},$$

with

$$\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a}) = \{X \in \mathfrak{k} \mid [X, Y] = 0, \text{ for all } Y \in \mathfrak{a}\}.$$

It is easy to check that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}, \quad \theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}, \quad [X_\alpha, \theta X_\alpha] = B(X_\alpha, \theta X_\alpha)H_\alpha,$$

where H_α is defined by $B(H, H_\alpha) = \alpha(H)$, in which B is the Killing form.

Clearly, we have

$$\mathfrak{k} = \text{span}\{X + \theta X \mid X \in \mathfrak{g}_\alpha, \alpha \in \Sigma\} \oplus \mathfrak{m}.$$

and

$$\mathfrak{p} = \text{span}\{X - \theta X \mid X \in \mathfrak{g}_\alpha, \alpha \in \Sigma\} \oplus \mathfrak{a}.$$

Since on a compact Lie group left Haar measures coincide with right Haar measures, and since all left Haar measures are, up to a constant, the same on a fixed Lie group, throughout the paper we will fix Haar measures on G_c and K and a left Haar measure on G and we will talk of 'the' Haar measure on G_c and K and 'the' left Haar measure on G. A basic reference on symmetric spaces is [8].

2.1. Compact Lie groups seen as symmetric spaces. Let G_c be a compact Lie group and consider the map

$$\theta : G_c \times G_c \longrightarrow G_c \times G_c$$
$$(g_1, g_2) \longmapsto (g_2, g_1).$$

Then θ is an involution, that is, $\theta^2 = \mathrm{id}$, with fixed point set

$$\Delta_{G_c} = \{(g, g) \mid g \in G_c\},$$

which can be clearly identified with G_c. Then the compact group G_c can be identified with the symmetric space $(G_c \times G_c)/\Delta_{G_c}$ via the map

$$\rho : G_c \longrightarrow (G_c \times G_c)/\Delta_{G_c}$$
$$g \longmapsto (g, e)\Delta_{G_c}, \tag{2.1}$$

which is easily seen to be a diffeomorphism.

Now let us describe the dual of G_c, seen as a symmetric space. Put $\widetilde{G} = G_c \times G_c$ and let $\mathfrak{g}_c$ (respectively $\widetilde{\mathfrak{g}}$) be the Lie algebra of G_c (respectively $\widetilde{G}$). Then the decomposition of $\widetilde{\mathfrak{g}}$ under the action of $d\theta$ is given by

$$\widetilde{\mathfrak{g}} = \mathfrak{k} \oplus \mathfrak{p}$$

where

$$\mathfrak{k} = \Delta\mathfrak{g}_c = \{(X, X) \mid X \in \mathfrak{g}_c\} \quad \text{and} \quad \mathfrak{p} = \{(X, -X) \mid X \in \mathfrak{g}_c\}.$$

Let

$$\widetilde{\mathfrak{g}}_{\mathrm{dual}} = \mathfrak{k} \oplus i\mathfrak{p}.$$

Then $\widetilde{\mathfrak{g}}_{\mathrm{dual}}$ is a real Lie subalgebra of the complexification

$$\widetilde{\mathfrak{g}}^{\mathbb{C}} = \widetilde{\mathfrak{g}} \oplus i\widetilde{\mathfrak{g}}$$

of the Lie algebra $\widetilde{\mathfrak{g}}$. It is easy to prove that $\widetilde{\mathfrak{g}}_{\mathrm{dual}}$ is isomorphic to $\mathfrak{g}_c^{\mathbb{C}}$ via the map

$$\sigma : \widetilde{\mathfrak{g}}_{\mathrm{dual}} = \mathfrak{k} \oplus i\mathfrak{p} \longrightarrow \mathfrak{g}_c^{\mathbb{C}}$$
$$(X, X) + i(Y, -Y) \longmapsto X + iY.$$

Then the Lie group with Lie algebra $\widetilde{\mathfrak{g}}_{\mathrm{dual}}$ can be identified with the complexification $G_c^{\mathbb{C}}$ of G_c. Consider the involution

$$\tau : \mathfrak{g}_c^{\mathbb{C}} = \mathfrak{g}_c \oplus i\mathfrak{g}_c \longrightarrow \mathfrak{g}_c^{\mathbb{C}}$$
$$X + iY \longmapsto X - iY,$$

and denote also by τ the involution on $G_c^{\mathbb{C}}$ induced by the involution τ of $\mathfrak{g}_c^{\mathbb{C}}$. The fixed point of τ on $G_c^{\mathbb{C}}$ is G_c. Therefore $G_c^{\mathbb{C}}/G_c$ is a symmetric space, dual of the compact Lie group G_c, seen as a symmetric space. For example, the dual of $\mathrm{SU}(n)$, seen as a symmetric space, is $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$. Let

$$C_a = \{gag^{-1} \mid g \in G_c\}$$

be the orbit of an element $a \in G_c$ under conjugation. Under the diffeomorphism (2.1), C_a corresponds to the double coset $\Delta_G(a, e)\Delta_G$, which we will identify (after identifying Δ_{G_c} with G_c), with $G_c(a, e)G_c$.

2.2. Orbital measures. Before introducing the orbital measures which will be of interest in this paper, let us first recall the definition of the convolution of positive measures on a Lie group. Let $v_1, \ldots, v_r$ be r positive measures on a Lie group G (actually G need only be a locally compact group). Then the convolution $v_1 * \cdots * v_r$ of the measures $v_1, \ldots, v_r$ is defined by

$$\langle v_1 * \cdots * v_r, f \rangle = \int_G f(g) \, d(v_1 * \cdots * v_r)(g)$$

$$= \int_G \cdots \int_G f(g_1 \cdots g_r) \, dv_1(g_1) \cdots dv_r(g_r); \quad f \in C_c(G).$$

If v_i is supported on S_i, $i = 1, \ldots r$, then the measure $v_1 * \cdots * v_r$ is supported on $S_1 \cdots S_r$, and in this case

$$\langle v_1 * \cdots * v_r, f \rangle = \int_{S_1} dv_1(g_1) \int_{S_2} dv_2(g_2)$$

$$\cdots \int_{S_r} f(g_1 \cdots g_r) \, dv_r(g_r); \quad f \in C_c(G).$$

We now define four different types of orbits and continuous singular measures supported on them.

(1) Fix a maximal torus T in G_c. Let $h \in G_c$, and let C_h be the *conjugacy class* containing h, that is,

$$C_h = \{ghg^{-1} : g \in G_c\}.$$

Every conjugacy class contains an element in the maximal torus T. Conjugacy classes are homogeneous submanifolds of proper dimension in G_c (as their co-dimension in G_c is at least $\text{rank}(G_c)$) and thus have Haar measure zero.
For an element $h \in G_c$, the *orbital measure*, v_h, supported on C_h, is the G_c-invariant measure defined by

$$\langle v_h, f \rangle = \int_{G_c} f \, dv_h$$

$$= \int_{G_c} f(ghg^{-1}) \, dm_{G_c}(g); \quad f \in C(G_c)$$

where m_{G_c} is the Haar measure on G_c. Orbital measures v_h are singular (with respect to the Haar measure m_{G_c} on G_c [10]), probability measures, and are continuous measures if and only if h does not belong to the center $Z(G_c)$ of G_c. They are also *central* measures, meaning they commute with all other measures under convolution.

(2) The group G_c acts on its Lie algebra $\mathfrak{g}_c$ by the adjoint action. Let $H \in \mathfrak{g}_c$ and O_H denote its adjoint orbit,

$$O_H = \{\text{Ad}(g)H : g \in G_c\}.$$

Let $\mathfrak{t}$ be the Lie algebra of the maximal torus T mentioned above. Every adjoint orbit O_H contains an element in the Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}_c$. Adjoint orbits are submanifolds of proper dimension in $\mathfrak{g}_c$ [13] (as their co-dimension in $\mathfrak{g}_c$ is at least rank($\mathfrak{g}_c$) and so have the Lebesgue measure zero.

For any $H \in \mathfrak{g}_c$, the G_c invariant orbital measure μ_H, supported on O_H, is given by

$$\langle \mu_H, f \rangle = \int_{\mathfrak{g}_c} f \, d\mu_H$$
$$= \int_{G_c} f(\mathrm{Ad}(g)H) \, dm_{G_c}(g); \quad f \in C(\mathfrak{g}_c).$$

Orbital measures μ_H are singular (with respect to Lebesgue measure on $\mathfrak{g}_c$) and are continuous measures if and only if $H \neq 0$.

(3)   Let $H \in \mathfrak{p}$. The orbit of H under the adjoint action of K on $\mathfrak{p}$, denoted by $O_{K,H}$, is given by

$$O_{K,H} = \{\mathrm{Ad}(k)H \mid k \in K\}.$$

Since $\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}$, every orbit $O_{K,H}$ contains an element of $\mathfrak{a}$. Orbits $O_{K,H}$ are of proper dimension in $\mathfrak{p}$ (as their co-dimension in $\mathfrak{p}$ is greater than or equal to the rank of the noncompact symmetric space M).

For $H \in \mathfrak{p}$, define a K-invariant measure $\mu_{K,H}$ supported on $O_{K,H}$ as follows: for $f \in C(\mathfrak{p})$, we put

$$\langle \mu_{K,H}, f \rangle := \int_K f(\mathrm{Ad}(g)H) \, dm_K(g),$$

where m_K is the Haar measure of K. It is clear that $\mu_{K,H}$ is a singular measure (with respect to Lebesgue measure on $\mathfrak{p}$) and is continuous if and only if $H \neq 0$.

(4)   Let G/K be a symmetric space (not necessarily noncompact). Consider the natural action of $K \times K$ on G defined by

$$\chi : (K \times K) \times G \longrightarrow G$$
$$((k_1, k_2), a) \longmapsto k_1 a k_2.$$

The orbit of a point $a \in G$ is the double coset space KaK, which we can assume without loss of generality to be of the form $K\exp(H)K$, for some $H \in \mathfrak{a}$ [9, p. 485]. Double cosets KaK are of proper dimension in G (as their co-dimension in G is greater than or equal to the rank of the symmetric space G/K). Each orbit is equipped with a unique $K \times K$ invariant (bi-K-invariant) measure ν_a, defined as follows: for $f \in C(G)$, we put

$$\langle \nu_a, f \rangle = \int_G f(g) \, d\nu_a(g)$$
$$= \int_K \int_K f(k_1 a k_2) \, dm_K(k_1) \, dm_K(k_2).$$

The set of the measures ν_a is a Banach algebra under convolution and is denoted by $M(K \backslash G/K)$. It can be seen that $\nu_a = m_K * \delta_a * m_K$, where δ_a denotes the point mass at a. It is clear that ν_a is singular with respect to the left Haar measure on the group G. Clearly, ν_a is continuous if and if only a is not in the normalizer of K in G.

3. Convolution of orbital measures in symmetric spaces

Let $M = G/K$ be a noncompact Riemannian symmetric space and G_c/K its compact dual. In this section we give a proof of our main theorem which consists of two parts: the first describes a transference result from a compact symmetric space to its dual, and the second generalizes Gupta and Hare's result in [2, Theorem 4.1] to general symmetric spaces.

THEOREM 3.1. *Let H be an element of $\mathfrak{a}$, $H \neq 0$, $a = \exp(H)$, $a_c = \exp(iH)$, and let*

$$n_a \overset{\text{def}}{:=} \min\{r \in \mathbb{N} \mid \nu_a^r \ll m_G\},$$

$$n_{a_c} \overset{\text{def}}{:=} \min\{r \in \mathbb{N} \mid \nu_{a_c}^r \ll m_{G_c}\},$$

$$n(K, a) \overset{\text{def}}{:=} \min\{r \in \mathbb{N} \mid \mu_{K,H}^r \ll \mu_{\mathfrak{p}}\},$$

where the symbol '$\ll$' means 'is absolutely continuous with respect to', and m_G (respectively m_{G_c}; $\mu_{\mathfrak{p}}$) denotes the left Haar measure on G (respectively Haar measure on G_c; Lebesgue measure on $\mathfrak{p}$). Then:

(1) $n_a \leq n_{a_c}$;
(2) $n_a = n(K, a)$.

REMARK 3.2. In Section 5 it will be shown that the inequality in part (1) of Theorem 3.1 can be strict.

Before proving the theorem, we need the following preliminary results.

LEMMA 3.3. *Let $H \in \mathfrak{a}$, and let $X_\alpha \in \mathfrak{g}_\alpha$. Then:*

(1) $\operatorname{Ad}(\exp(iH))(X_\alpha \pm \theta X_\alpha) = \cos(\alpha(H))(X_\alpha \pm \theta X_\alpha) + i \sin(\alpha(H))(X_\alpha \mp \theta X_\alpha)$;
(2) $\operatorname{Ad}(\exp(H))(X_\alpha \pm \theta X_\alpha) = \sinh(\alpha(H))(X_\alpha \mp \theta X_\alpha) + \cosh(\alpha(H))(X_\alpha \pm \theta X_\alpha)$.

PROOF. (1) From the identities

$$\operatorname{Ad}(\exp(iH))(X_\alpha) = \exp(\operatorname{ad}(iH))X_\alpha = e^{i\alpha(H)}X_\alpha,$$

$$\operatorname{Ad}(\exp(iH))(\theta X_\alpha) = \exp(\operatorname{ad}(iH))\theta X_\alpha = e^{-i\alpha(H)}(\theta X_\alpha),$$

we deduce that

$$\operatorname{Ad}(\exp(iH))(X_\alpha \pm \theta X_\alpha)$$
$$= \exp(\operatorname{ad}(iH))(X_\alpha \pm \theta X_\alpha) = e^{i\alpha(H)}X_\alpha \pm e^{-i\alpha(H)}\theta X_\alpha$$
$$= \cos(\alpha(H))(X_\alpha \pm \theta X_\alpha) + i \sin(\alpha(H))(X_\alpha \mp \theta X_\alpha).$$

(2) Similarly, from the identities

$$\begin{aligned}
\mathrm{Ad}(\exp(H))(X_\alpha) &= e^{\alpha(H)}X_\alpha, \\
\mathrm{Ad}(\exp(H))(\theta X_\alpha) &= e^{-\alpha(H)}(\theta X_\alpha),
\end{aligned}$$

we deduce that

$$\begin{aligned}
\mathrm{Ad}(\exp(H))(X_\alpha \pm \theta X_\alpha) &= \exp(\mathrm{ad}(H))(X_\alpha \pm \theta X_\alpha) \\
&= e^{\alpha(H)}X_\alpha \pm e^{-\alpha(H)}\theta X_\alpha \\
&= \sinh(\alpha(H))(X_\alpha \mp \theta X_\alpha) + \cosh(\alpha(H))(X_\alpha \pm \theta X_\alpha).
\end{aligned}$$

This concludes the proof. □

Put

$$\begin{aligned}
\mathcal{V}(a_c, k_2, \ldots, k_n) &= \mathfrak{k} + \mathrm{Ad}(a_c)\mathfrak{k} + \mathrm{Ad}(a_c k_2 a_c)\mathfrak{k} + \cdots + \mathrm{Ad}(a_c k_2 a_c \cdots k_n a_c)\mathfrak{k}, \\
\mathcal{V}(a, k_2, \ldots, k_n) &= \mathfrak{k} + \mathrm{Ad}(a)\mathfrak{k} + \mathrm{Ad}(ak_2 a)\mathfrak{k} + \cdots + \mathrm{Ad}(ak_2 a \cdots k_n a)\mathfrak{k},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{W}(a_c, k_2, \ldots, k_n) &= \mathfrak{k} + i(\mathcal{T}_{a_c} + \mathrm{Ad}(a_c k_2)\mathcal{T}_{a_c} \\
&\qquad + \mathrm{Ad}(a_c k_2 a_c k_3)\mathcal{T}_{a_c} + \cdots + \mathrm{Ad}(a_c k_2 \cdots a_c k_n)\mathcal{T}_{a_c}), \\
\mathcal{W}(a, k_2, \ldots, k_n) &= \mathfrak{k} + \mathcal{T}_a + \mathrm{Ad}(ak_2)\mathcal{T}_a \\
&\qquad + \mathrm{Ad}(ak_2 ak_3)\mathcal{T}_a + \cdots + \mathrm{Ad}(ak_2 \cdots ak_n)\mathcal{T}_a,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_{a_c} &= \mathrm{span}\{(X_\alpha - \theta X_\alpha) \mid \alpha \text{ restricted root such that } \alpha(H) \not\equiv 0 (\mathrm{mod}\ \pi)\}, \\
\mathcal{T}_a &= \mathrm{span}\{(X_\alpha - \theta X_\alpha) \mid \alpha \text{ restricted root such that } \alpha(H) \neq 0\}.
\end{aligned}$$

PROPOSITION 3.4. *Let* $\mathcal{V}(a_c, k_2, \ldots, k_n)$ *and* $\mathcal{W}(a_c, k_2, \ldots, k_n)$ *be as above. Then:*
(1) $\mathcal{V}(a_c, k_2, \ldots, k_n) = \mathcal{W}(a_c, k_2, \ldots, k_n)$;
(2) $\mathcal{V}(a, k_2, \ldots, k_n) = \mathcal{W}(a, k_2, \ldots, k_n)$.

PROOF. Let $\alpha \in \Sigma$ and $M \in \mathfrak{m}$. From Lemma 3.3(1),

$$\begin{aligned}
\mathrm{Ad}(\exp(iH))(X_\alpha + \theta X_\alpha + M) &= \cos(\alpha(H))(X_\alpha + \theta X_\alpha) \\
&\qquad + i\sin(\alpha(H))(X_\alpha - \theta X_\alpha).
\end{aligned}$$

Hence

$$\mathrm{Ad}(a_c)\mathfrak{k} \subseteq \mathfrak{k} + i\mathcal{T}_{a_c}. \tag{3.1}$$

Consequently

$$\begin{aligned}
\mathrm{Ad}(a_c k_2 a_c)\mathfrak{k} &\subseteq \mathrm{Ad}(a_c k_2)(\mathfrak{k} + i\mathcal{T}_{a_c}) \\
&\subseteq \mathfrak{k} + i(\mathcal{T}_{a_c} + \mathrm{Ad}(a_c k_2)(\mathcal{T}_{a_c})).
\end{aligned}$$

By induction we get

$$\mathcal{V}(a_c, k_2, \ldots, k_n) \subseteq \mathcal{W}(a_c, k_2, \ldots, k_n). \tag{3.2}$$

Again, from Lemma 3.3,

$$i \sin(\alpha(H))(X_\alpha - \theta X_\alpha) = -\cos(\alpha(H))(X_\alpha + \theta X_\alpha) \\ + \mathrm{Ad}(\exp(iH))(X_\alpha + \theta X_\alpha). \tag{3.3}$$

Therefore

$$\mathfrak{k} + i\mathcal{T}_{a_c} \subseteq \mathfrak{k} + \mathrm{Ad}(a_c)\mathfrak{k}.$$

From (3.3), we deduce that

$$i \sin(\alpha(H)) \, \mathrm{Ad}(a_c k_2)(X_\alpha - \theta X_\alpha) \\ = \mathrm{Ad}(a_c k_2)[-\cos(\alpha(H))(X_\alpha + \theta X_\alpha) + \mathrm{Ad}(\exp(iH))(X_\alpha + \theta X_\alpha)].$$

Hence

$$i\,\mathrm{Ad}(a_c k_2)\mathcal{T}_{a_c} \subseteq \mathrm{Ad}(a_c)\mathfrak{k} + \mathrm{Ad}(a_c k_2 a_c)\mathfrak{k}.$$

Then

$$\mathcal{W}(a_c, k_2) \subseteq \mathcal{V}(a_c, k_2).$$

By induction,

$$\mathcal{W}(a_c, k_2, \ldots, k_n) \subseteq \mathcal{V}(a_c, k_2, \ldots, k_n). \tag{3.4}$$

Part (1) of the proposition is a consequence of (3.2) and (3.4).

The proof of part (2) of the proposition follows the same lines. □

Let

$$\mathcal{F}_c(a, k_2, \ldots, k_n) = \mathfrak{k} + i(\mathcal{T}_a + \mathrm{Ad}(k_2)\mathcal{T}_a + \mathrm{Ad}(k_2 k_3)\mathcal{T}_a + \cdots + \mathrm{Ad}(k_2 \cdots k_n)\mathcal{T}_a)$$

and

$$\mathcal{F}(a, k_2, \ldots, k_n) = \mathfrak{k} + \mathcal{T}_a + \mathrm{Ad}(k_2)\mathcal{T}_a + \mathrm{Ad}(k_2 k_3)\mathcal{T}_a + \cdots + \mathrm{Ad}(k_2 \cdots k_n)\mathcal{T}_a.$$

COROLLARY 3.5.
(1) $\mathcal{V}(a_c, k_2, \ldots, k_n) \subseteq \mathcal{F}_c(a, k_2, \ldots, k_n).$
(2) $\mathcal{V}(a, k_2, \ldots, k_n) = \mathcal{F}(a, k_2, \ldots, k_n).$

REMARK 3.6. The inclusion in (1) of Corollary 3.5 is strict as will be seen in Section 5.

PROOF. (1) By Proposition 3.4, it is enough to prove that

$$\mathcal{W}(a_c, k_2, \ldots, k_n) \subseteq \mathcal{F}_c(a, k_2, \ldots, k_n).$$

Then

$$\mathrm{Ad}(k_2)(X_\alpha - \theta X_\alpha) = \sum_{\beta \in \Sigma} c_\beta (X_\beta - \theta X_\beta) + X \tag{3.5}$$

for some $X \in \mathfrak{a}$ depending on k_2 and α. Combining Lemma 3.3 and (3.5) gives

$$
\begin{aligned}
i\operatorname{Ad}(a_c k_2)(X_\alpha - \theta X_\alpha) &= i\operatorname{Ad}(a_c)\left(\sum_{\beta \in \Sigma} c_\beta(X_\beta - \theta X_\beta) + X\right) \\
&= i\sum_{\beta \in \Sigma} c_\beta \operatorname{Ad}(a_c)(X_\beta - \theta X_\beta) + i\operatorname{Ad}(a_c)X \\
&= -\sum_{\beta \in \Sigma} c_\beta \sin \beta(H)(X_\beta + \theta X_\beta) \\
&\quad + i\sum_{\beta \in \Sigma} c_\beta \cos \beta(H)(X_\beta - \theta X_\beta) + iX.
\end{aligned}
\tag{3.6}
$$

Combining (3.5) and (3.6), we get

$$
\begin{aligned}
i\operatorname{Ad}&(a_c k_2)(X_\alpha - \theta X_\alpha) - i\operatorname{Ad}(k_2)(X_\alpha - \theta X_\alpha) \\
&= -\sum_{\beta \in \Sigma} c_\beta \sin \beta(H)(X_\beta + \theta X_\beta) \\
&\quad + i\sum_{\beta \in \Sigma} c_\beta(\cos \beta(H) - 1)(X_\beta - \theta X_\beta) \\
&\in \mathfrak{k} + i\mathcal{T}_a.
\end{aligned}
$$

Therefore

$$
i\operatorname{Ad}(a_c k_2)\mathcal{T}_a \subseteq \mathfrak{k} + i(\mathcal{T}_a + \operatorname{Ad}(k_2)\mathcal{T}_a).
\tag{3.7}
$$

Since $\mathcal{T}_{a_c} \subseteq \mathcal{T}_a$, it follows that

$$
i\operatorname{Ad}(a_c k_2)\mathcal{T}_{a_c} \subseteq \mathfrak{k} + i(\mathcal{T}_a + \operatorname{Ad}(k_2)\mathcal{T}_a).
\tag{3.8}
$$

From (3.7) and (3.8) we deduce that

$$
\begin{aligned}
i\operatorname{Ad}(a_c k_2 a_c k_3)\mathcal{T}_{a_c} &= \operatorname{Ad}(a_c k_2)(i\operatorname{Ad}(a_c k_3)\mathcal{T}_{a_c}) \\
&\subseteq \mathfrak{k} + i(\mathcal{T}_a + \operatorname{Ad}(k_2)\mathcal{T}_a + \operatorname{Ad}(k_2 k_3)\mathcal{T}_a).
\end{aligned}
$$

By induction, we get

$$
\mathcal{W}(a_c, k_2, \ldots, k_n) \subseteq \mathcal{F}_c(a, k_2, \ldots, k_n).
$$

Hence part (1) of the corollary.

(2) Using (3.5) and Lemma 3.3,

$$
\begin{aligned}
\operatorname{Ad}&(a k_2)(X_\alpha - \theta X_\alpha) - \operatorname{Ad}(k_2)(X_\alpha - \theta X_\alpha) \\
&= \sum_{\beta \in \Sigma} c_\beta(\cosh \beta(H) - 1)(X_\beta - \theta X_\beta) \\
&\quad + \sum_{\beta \in \Sigma} c_\beta \sinh \beta(H)(X_\beta + \theta X_\beta) \in \mathfrak{k} + \mathcal{T}_a.
\end{aligned}
$$

Therefore

$$
\operatorname{Ad}(a k_2)\mathcal{T}_a \subseteq \mathfrak{k} + \mathcal{T}_a + \operatorname{Ad}(k_2)\mathcal{T}_a \quad \text{and} \quad \operatorname{Ad}(k_2)\mathcal{T}_a \subseteq \mathfrak{k} + \mathcal{T}_a + \operatorname{Ad}(a k_2)\mathcal{T}_a. \tag{3.9}
$$

Thus
$$\mathcal{V}(a, k_2) = \mathcal{F}(a, k_2).$$

From arguments similar to Corollary 3.5(1), we obtain
$$\mathcal{V}(a, k_2, k_3) \subseteq \mathcal{F}(a, k_2, k_3).$$

Also, using (3.9), we get
$$
\begin{aligned}
\mathcal{F}(a, k_2, k_3) &\subseteq \mathfrak{k} + \mathcal{T}_a + \mathrm{Ad}(ak_2)\mathcal{T}_a + \mathrm{Ad}(ak_2)\,\mathrm{Ad}(k_3)\mathcal{T}_a \\
&\subseteq \mathfrak{k} + \mathcal{T}_a + \mathrm{Ad}(ak_2)\mathcal{T}_a + \mathrm{Ad}(ak_2)(\mathfrak{k} + \mathcal{T}_a + \mathrm{Ad}(ak_3)\mathcal{T}_a) \\
&= \mathcal{V}(a, k_2, k_3).
\end{aligned}
$$

Hence
$$\mathcal{V}(a, k_2, k_3) = \mathcal{F}(a, k_2, k_3).$$

Continuing inductively, we get
$$\mathcal{V}(a, k_2, \ldots, k_n) = \mathcal{F}(a, k_2, \ldots, k_n).$$

This concludes the proof. $\qquad\square$

PROPOSITION 3.7. *Let G/K be a symmetric space and $a_1, \ldots, a_r \in G$. Suppose that $v_{a_1}, \ldots, v_{a_r}$ are the orbital measures on G supported on the double cosets $Ka_1K, \ldots, Ka_rK$. Then:*

(1) $v_{a_1} * \cdots * v_{a_r}$ *is absolutely continuous with respect to the left Haar measure on G if and only if the function Ψ given by*

$$
\begin{aligned}
\Psi : K^{r+1} &\longrightarrow G \\
(k_1, \ldots, k_{r+1}) &\longmapsto k_1 a_1 k_2 \cdots k_r a_r k_{r+1}
\end{aligned}
$$

has full rank at some point $(k_1, \ldots, k_{r+1}) \in K^{r+1}$;

(2) *for $X_1, \ldots, X_{r+1} \in \mathfrak{k}$ and $(k_1, \ldots, k_{r+1}) \in K^{r+1}$,*

$$
\begin{aligned}
(d\Psi)_{(k_1,\ldots,k_{r+1})}&(X_1, \ldots, X_{r+1}) \\
&= X_1 + \mathrm{Ad}(k_1 a_1)X_2 + \cdots + \mathrm{Ad}(k_1 a_1 \cdots k_r a_r)X_{r+1}.
\end{aligned}
$$

PROOF. (1) Ragozin [12, Theorem 2.5] proved that if the rank of Ψ is equal to the dimension of G at one point $(k_1, k_2, \ldots, k_{r+1})$ in K^{r+1}, then $v_{a_1} * \cdots * v_{a_r}$ is absolutely continuous. Conversely, if rank Ψ is less than the dimension of G at every $(k_1, k_2, \ldots, k_{r+1})$, then Sard's theorem says the measure of the image of Ψ is zero. But the image is the product of the double cosets, $Ka_1K \cdots Ka_rK$, which supports the measure $v_{a_1} * \cdots * v_{a_r}$, and hence this measure is singular.

(2) Let X_1 be an element of $T_{k_1}K \cong \mathfrak{k}$, and f a smooth function in a neighborhood of $k_1 a_1 k_2 a_2 \cdots k_r a_r k_{r+1}$. Then

$$
\begin{aligned}
d\Psi_{(k_1,\ldots,k_{r+1})}(X_1, 0, \ldots, 0)f &= \frac{d}{dt} f(\exp(tX_1)k_1 a_1 \cdots k_r a_r k_{r+1})\Big|_{t=0} \\
&= X_1 f.
\end{aligned}
$$

Similarly, for X_2 an element of $T_{k_2} K \cong \mathfrak{k}$, and f as above,

$$d\Psi_{(k_1,\ldots,k_{r+1})}(0, X_2, \ldots, 0) f$$
$$= \frac{d}{dt} f(k_1 a_1 \exp(t X_2) k_2 a_2 \cdots k_r a_r k_{r+1}) \Big|_{t=0}$$
$$= \frac{d}{dt} f(\exp(\mathrm{Ad}(k_1 a_1) t X_2) k_1 a_1 k_2 a_2 \cdots k_r a_r k_{r+1}) \Big|_{t=0}$$
$$= \mathrm{Ad}(k_1 a_1) X_2.$$

Hence

$$d\Psi_{(k_1,\ldots,k_{r+1})}(0, X_2, \ldots, 0) = \mathrm{Ad}(k_1 a_1) X_2.$$

By induction, we get

$$d\Psi_{(k_1,\ldots,k_{r+1})}(0, \ldots, 0, X_i, 0, \ldots, 0) = \mathrm{Ad}(k_1 a_1 \cdots k_{i-1} a_{i-1}) X_i.$$

Therefore

$$d\Psi_{(k_1,\ldots,k_{r+1})}(X_1, X_2, \ldots, X_{r+1})$$
$$= X_1 + \mathrm{Ad}(k_1 a_1) X_2 + \cdots + \mathrm{Ad}(k_1 a_1 \cdots k_{i-1} a_{i-1}) X_i + \cdots$$
$$+ \mathrm{Ad}(k_1 a_1 \cdots k_{r-1} a_{r-1}) X_r + \mathrm{Ad}(k_1 a_1 \cdots k_r a_r) X_{r+1}.$$

This completes the proof.   □

REMARK 3.8. Note that $d\Psi_{(k_1,\ldots,k_{r+1})}$ is independent of k_{r+1}.

Our next proposition is similar to Proposition 3.7.

PROPOSITION 3.9. *Let $H_1, \ldots, H_r \in \mathfrak{p}$ and suppose that $\mu_{K,H_1}, \ldots, \mu_{K,H_r}$ are the orbital measures on $\mathfrak{p}$ supported on the orbits O_{K,H_i}. Then:*

(1) $\mu_{K,H_1} * \cdots * \mu_{K,H_r}$ *is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{p}$ if and only if the function*

$$\psi : O_{K,H_1} \times \cdots \times O_{K,H_r} \longrightarrow \mathfrak{p}$$
$$(X_1, \ldots, X_r) \longmapsto X_1 + \cdots + X_r$$

   has full rank at some point $(X_1, \ldots, X_r) \in O_{K,H_1} \times \cdots \times O_{K,H_r}$;

(2)   *letting $Y \in O_{K,H}$, the tangent space $T_Y(O_{K,H})$ of $O_{K,H}$ at Y is $\{[X, Y] \mid X \in \mathfrak{k}\}$;*

(3)   *letting $Y_1, \ldots, Y_r \in \mathfrak{k}$, $(X_1, \ldots, X_r) \in O_{K,H_1} \times \cdots \times O_{K,H_r}$, we have*

$$(d\psi)_{(X_1,\ldots,X_r)} : T_{X_1}(O_{K,H_1}) \times \cdots \times T_{X_r}(O_{K,H_r}) \longrightarrow \mathfrak{p}$$
$$([X_1, Y_1], \ldots, [X_r, Y_r]) \longmapsto [X_1, Y_1] + \cdots + [X_r, Y_r].$$

PROOF. (1) This is similar to the proof of Proposition 3.7.

(2) Let $X \in \mathfrak{k}$. Then the curve $\gamma(t) = \mathrm{Ad}(\exp(t X)) Y$ is inside $O_{K,H}$ and passes through Y at $t = 0$. Since the derivative of γ at $t = 0$ is $[X, Y]$, we deduce that

$$T_Y(O_{K,H}) = \{[X, Y] \mid X \in \mathfrak{k}\}.$$

(3) This follows from (2).   □

PROOF OF THEOREM 3.1. (1) Suppose that there exists a positive integer r such that $\nu_{a_c}^r$ is absolutely continuous with respect to the Haar measure on G_c. Then, by Proposition 3.7, we deduce that there exist $(k_1, k_2, \ldots, k_r)$ such that

$$\mathfrak{k} + \mathrm{Ad}(k_1 a_c)\mathfrak{k} + \mathrm{Ad}(k_1 a_c k_2 a_c)\mathfrak{k} + \cdots + \mathrm{Ad}(k_1 a_c \cdots k_r a_c)\mathfrak{k} = \mathfrak{g}_c. \quad (3.10)$$

The fact that $\mathrm{Ad}(k_1^{-1})\mathfrak{k} = \mathfrak{k}$, along with the definition of $\mathcal{V}(a_c, k_2, \ldots, k_n)$, shows that (3.10) is equivalent to

$$\mathcal{V}(a_c, k_2, \ldots, k_n) = \mathfrak{g}_c. \quad (3.11)$$

From Corollary 3.5(1) and (3.11) we get

$$\mathfrak{g}_c = \mathcal{V}(a_c, k_2, \ldots, k_n) \subseteq \mathcal{F}_c(a, k_2, \ldots, k_n). \quad (3.12)$$

From (3.12) and the definition of $\mathcal{V}(a_c, k_2, \ldots, k_n)$ it follows that

$$\mathfrak{g}_c = \mathfrak{k} + i(\mathcal{T}_a + \mathrm{Ad}(k_2)\mathcal{T}_a + \mathrm{Ad}(k_2 k_3)\mathcal{T}_a + \cdots + \mathrm{Ad}(k_2 \cdots k_n)\mathcal{T}_a).$$

Hence

$$\mathfrak{p} = \mathcal{T}_a + \mathrm{Ad}(k_2)\mathcal{T}_a + \mathrm{Ad}(k_2 k_3)\mathcal{T}_a + \cdots + \mathrm{Ad}(k_2 \cdots k_n)\mathcal{T}_a.$$

Consequently

$$\mathcal{F}(a, k_2, \ldots, k_n) = \mathfrak{k} + \mathcal{T}_a + \mathrm{Ad}(k_2)\mathcal{T}_a + \mathrm{Ad}(k_2 k_3)\mathcal{T}_a + \cdots + \mathrm{Ad}(k_2 \cdots k_n)\mathcal{T}_a$$
$$= \mathfrak{g}.$$

Combining Corollary 3.5(2), and Proposition 3.7(2), we deduce that ν_a^r is absolutely continuous with respect to the left Haar measure on G. Therefore $n_a \leq n_{a_c}$.

(2) $\mu_{K,H}^r$ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{p}$ if and only if there exist points $x_1, \ldots, x_r \in K$, such that for $i = 1, \ldots, r$, $X_i = \mathrm{Ad}(x_i)H$ and

$$T_{X_1}(O_{K,H}) + \cdots + T_{X_r}(O_{K,H}) = \mathfrak{p}. \quad (3.13)$$

Since $T_{X_i}(O_{K,H}) = \mathrm{Ad}(x_i)T_H(O_{K,H})$ for $i = 1, \ldots, r$, Equation (3.13) holds if and only if

$$\mathrm{Ad}(x_1)T_H(O_{K,H}) + \cdots + \mathrm{Ad}(x_r)(O_{K,H}) = \mathfrak{p}. \quad (3.14)$$

Without loss of generality we can assume that $x_1 = e$ in the identity (3.13). Let $k_2 = x_2$, $k_3 = k_2^{-1} x_2$ and $k_r = k_{r-1}^{-1} \cdots k_2^{-1} x_r$. Using (3.14), Corollary 3.5 and the observation $T_H(O_{K,H}) = \mathcal{T}_a$, we deduce that (3.14) holds if and only if

$$\mathfrak{k} + \mathcal{T}_a + \mathrm{Ad}(k_2)\mathcal{T}_a + \cdots + \mathrm{Ad}(k_2 \cdots k_r)\mathcal{T}_a = \mathfrak{g}. \quad (3.15)$$

But (3.15) holds if and only if ν_a^r is absolutely continuous with respect to the left Haar measure on G. □

The following corollary of Theorem 3.1(2) was proved by second author and Hare in [2, Theorem 4.1].

COROLLARY 3.10. *Let G_c be a compact connected simple Lie group and $\mathfrak{g}_c$ be its Lie algebra. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}_c$, $H \in \mathfrak{t}$, $a = \exp(iH)$ and r a positive integer. Then ν_a^r is absolutely continuous with respect to the left Haar measure on $G_c^{\mathbb{C}}$ if and only if μ_H^r is absolutely continuous with respect to Lebesgue measure on $\mathfrak{g}_c$.*

PROOF. Consider the noncompact symmetric space $G_c^{\mathbb{C}}/G_c$. In this case $\mathfrak{p}$ is $i\mathfrak{g}_c$ and $\mathfrak{a}$ is $i\mathfrak{t}$. Therefore the corollary follows from Theorem 3.1(2). □

4. Transference of results

In this section we transfer results of the second author with Hare, mentioned in the introduction, to noncompact symmetric spaces. Before we state our first transference result, we need the following definition.

DEFINITION 4.1. Given a restricted root $\alpha \in \Sigma$ and $a \in \exp \mathfrak{a}$, say $a = \exp(H)$ for $H \in \mathfrak{a}$, we set $\alpha(a) = \alpha(H)$ (this is well defined, as $\exp$ is a diffeomorphism from $\mathfrak{a}$ onto $\exp(\mathfrak{a})$). We call the element $a \in \exp \mathfrak{a}$ *regular* if $\alpha(a) \neq 0$ for all $\alpha \in \Sigma$. We say that $a_c = \exp(iH)$ is regular if $\alpha(a) \neq 0 \pmod{\pi}$ for all $\alpha \in \Sigma$. It is clear from the definitions of regular elements that if a_c is regular then a is also regular.

The following theorem is proved in [4, Theorem 3.1].

THEOREM 4.2. *Suppose that A, $B \in \mathfrak{a}$, and put $a_c = \exp(iA)$ and $b_c = \exp(iB)$. If a_c and b_c are regular elements in G_c and ν_{a_c}, ν_{b_c} are the associated K-orbital measures, then $\nu_{a_c} * \nu_{b_c}$ is absolutely continuous with respect to the Haar measure on G_c.*

Using Theorems 3.1(1) and 4.2, we have the following analogue of Theorem 4.2 on noncompact symmetric spaces.

THEOREM 4.3. *Let A, $B \in \mathfrak{a}$, $a = \exp(A)$, $b = \exp(B)$, and let ν_a, ν_b be the associated K-orbital measures on G. If a and b are regular elements in G then $\nu_a * \nu_b$ is absolutely continuous with respect to the left Haar measure on G.*

PROOF. Let $t_0 \in \mathbb{R}$ be such that $a_c(t_0) = \exp(it_0 A)$ and $b_c(t_0) = \exp(it_0 B)$ are regular elements in G_c. Then by Theorem 4.2, $\nu_{a_c(t_0)} * \nu_{b_c(t_0)}$ is absolutely continuous with respect to the Haar measure on G_c. Therefore by Theorem 3.1(1), $\nu_{a(t_0)} * \nu_{b(t_0)}$ is absolutely continuous on G with respect to the left Haar measure on G. Hence by Theorem 3.1(2), $\mu_{K, t_0 A} * \mu_{K, t_0 B}$ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{p}$. So, $O_{K, t_0 A} + O_{K, t_0 B} = t_0(O_{K, A} + O_{K, B})$ contains an open set in it and therefore $(O_{K, A} + O_{K, B})$ has an open set in it. Hence $\mu_{K, A} * \mu_{K, B}$ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{p}$. Therefore $\nu_a * \nu_b$ is absolutely continuous with respect to the left Haar measure on G by Theorem 3.1(2). □

For the next result, take $G = SL(n, \mathbb{R})$ and $K = SO(n)$. Then $G_c = SU(n)$. For $a \in G \setminus N_G(K)$ (respectively $a_c \in G_c \setminus N_{G_c}(K)$), Ragozin [12, Theorem 2.5] had shown that $v_a^r \in L^1(G)$ (respectively $v_{a_c}^r \in L^1(G_c)$) for $r = \dim G/K = (n^2 + n - 2)/2$. He had speculated that $\dim G/K$ is not sharp, but had conjectured that (approximately) $r = n/2$ would work [12, p. 375]. Improving upon the result of Ragozin, the following result was proved in [6, Theorem 3.1].

THEOREM 4.4. *If $A \in \mathfrak{a}$, $a = \exp(A)$, and a_c is not in the normalizer of $SO(n)$ in $SU(n)$, then $v_{a_c}^n$ is absolutely continuous with respect to the Haar measure on $SU(n)$.*

Using Theorems 3.1(1) and 4.4, we obtain the following analogue of Ragozin's result for the noncompact symmetric space $SL(n, \mathbb{R})/SO(n)$.

THEOREM 4.5. *If $A \in \mathfrak{a}$, $a = \exp(A)$, and a is not in the normalizer of $SO(n)$ in $SL(n, \mathbb{R})$, then v_a^n is absolutely continuous with respect to the left Haar measure on $SL(n, \mathbb{R})$.*

PROOF. The proof is similar to the proof of Theorem 4.3 and is left to the reader. $\square$

5. Final remarks

The aim of this section is to show that the inequality in Theorem 3.1(1) can be strict. For this let $G_c = SO(2n + 1)$, the group of $(2n + 1) \times (2n + 1)$ orthogonal matrices of determinant 1. Then the complexification $G_c^{\mathbb{C}}$ of G_c is equal to

$$SO(n, \mathbb{C}) = \{g \in SL(n, \mathbb{C}) : g^t g = 1\}.$$

As was explained in Section 2.1, the dual of the compact Lie group $G_c = SO(2n + 1)$, seen as a symmetric space, is $G_c^{\mathbb{C}}/G_c$. Let E_{ij} denote the $(2n + 1) \times (2n + 1)$ matrix with (i, j)th entry 1 and all other entries zero. Set $H = \sum_{i=1}^{n} i\pi(E_{2i-1,2i} - E_{2i,2i-1})$, $a = \exp(H)$ and $a_c = \exp(iH)$.

It is shown in [5, Theorem 6.8] (respectively [7, Theorem 8.2] or [3, Theorem 6.6]) that $v_{a_c}^{(2n-1)}$ is singular with respect to the Haar measure on G_c, (respectively μ_{iH}^n is absolutely continuous with respect to Lebesgue measure on the Lie algebra $\mathfrak{so}(2n + 1)$ of $SO(2n + 1)$).

Also, it follows from Corollary 3.10 that v_a^n is absolutely continuous with respect to the left Haar measure on $SO(n, \mathbb{C})$ if and only if μ_{iH}^n is absolutely continuous with respect to Lebesgue measure on $\mathfrak{so}(2n + 1)$. Therefore v_a^n is absolutely continuous with respect to the left Haar measure on $SO(n, \mathbb{C})$ but $v_{a_c}^n$ is singular with respect to the Haar measure on $SO(2n + 1)$. This shows that inequality in Theorem 3.1(1) can be strict.

OPEN PROBLEM. *Let G/K be a noncompact symmetric space. Let $H \in \mathfrak{a}$ and r be a positive integer. It is shown in [2, Theorem 3.1] that for a noncompact symmetric space $G_c^{\mathbb{C}}/G_c$, $\mu_{\mathfrak{g}_c, H}^r \in L^2(\mathfrak{g}_c)$ if and only if $v_{\exp(H)}^r \in L^2(G_c^{\mathbb{C}})$. It is an open problem whether $\mu_{K,H}^r \in L^2(\mathfrak{p})$ if and only if $v_{\exp(H)}^r \in L^2(G)$.*

Acknowledgement

References

[1] C. Dunkl, 'Operators and harmonic analysis on the sphere', *Trans. Amer. Math. Soc.* **125** (1966), 250–263.

[2] S. Gupta and K. Hare, 'Singular dichotomy for orbital measures on complex groups', *Boll. Unione Mat. Ital.* (9) **III** (2010), 409–419.

[3] S. Gupta and K. Hare, 'Singularity of orbits in classical Lie algebras', *Geom. Funct. Anal.* **13** (2003), 815–844.

[4] S. Gupta and K. Hare, 'Convolutions of generic orbital measures in compact symmetric spaces', *Bull. Aust. Math. Soc.* **79**(3) (2009), 513–522.

[5] S. Gupta and K. Hare, 'L^2-singular dichotomy for orbital measures of classical compact Lie groups', *Adv. Math.* **222** (2009), 1521–1573.

[6] S. Gupta and K. Hare, 'Smoothness of convolution powers of orbital measures on the symmetric space $SU(n)/SO(n)$', *Monatsh. Math.* **159** (2010), 27–43.

[7] S. Gupta, K. Hare and S. Seyfaddini, 'L^2-singular dichotomy for orbital measures of classical simple Lie algebras', *Math. Z.* **262** (2009), 91–124.

[8] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, New York, 1978).

[9] A. W. Knapp, *Lie Groups: Beyond an Introduction* (Birkhäuser, Boston, 2002).

[10] M. Mimura and H. Toda, *Topology of Lie Groups*, Translations of Mathematical Monographs, 91 (American Mathematical Society, Providence, RI, 1991).

[11] D. Ragozin, 'Central measures on compact simple Lie groups', *J. Funct. Anal.* **10** (1972), 212–229.

[12] D. Ragozin, 'Zonal measure algebras on isotropy irreducible homogeneous spaces', *J. Funct. Anal.* **17** (1974), 355–376.

[13] V. S. Varadarajan, *Lie Groups and Lie Algebras and their Representations* (Springer, New York, 1984).

BOUDJEMÂA ANCHOUCHE, Department of Mathematics and Statistics,
Sultan Qaboos University, PO Box 36, Al-Khoud 123, Muscat, Sultanate of Oman
e-mail: anchouch@squ.edu.om

SANJIV KUMAR GUPTA, Department of Mathematics and Statistics,
Sultan Qaboos University, PO Box 36, Al-Khoud 123, Muscat, Sultanate of Oman
e-mail: gupta@squ.edu.om