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SPECULATIONS CONCERNING THE RANGE OF MAHLER'S MEASURE

BY

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I would like to express my thanks to the Canadian Mathematical Society for inviting me to present this lecture. I would also like to express my appreciation to C.J. Smyth for numerous helpful conversations during his visit this year at the University of British Columbia.

This paper follows reasonably closely the outline of the lecture presented in Ottawa. More details are given here though and a number of proofs which would not be otherwise accessible have been added as Appendices. The attentive reader will soon realize the appropriateness of the title.

1. **Lehmer's question.** Our subject begins with a question of Lehmer concerning a certain function $M(P)$ defined on polynomials $P(z)=a_0z^d+\cdots+a_d$, $(a_0\neq 0)$. If P has zeros $\alpha_1,\ldots,\alpha_d$ then the *measure* of P is defined by

$$M(P)=|a_0|\prod_{i=1}^{d}\max(|\alpha_i|,1). \tag{1}$$

If P has integer coefficients then $M(P)\geq 1$. Furthermore, if in this case $M(P)=1$ and $a_d\neq 0$ then it is immediate that $|\alpha_i|=1$ for all i and that $|a_0|=1$ so a classical theorem of Kronecker [17], [9] tells us that the α_i must be roots of unity; so P is cyclotomic.

Lehmer's question [19], which was motivated by the study of the sequences

$$\Delta_m=\prod_{i=1}^{d}(\alpha_i^m-1),$$

asks how small $M(P)$ can be in case P is not cyclotomic. Specifically, he asks whether,

"Given $\varepsilon>0$, are there P with integer coefficients for which $1<M(P)<1+\varepsilon$?".

Although one might be tempted to guess that this should be possible, no smaller $M(P)>1$ has yet been found than Lehmer's example:

$$M(P)=\sigma_1=1.17628\ldots \tag{2}$$

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where $P(z)=z^{10}+\cdots+1=1\ 1\ 0-1-1-1-1-1\ 0\ 1\ 1$, using an obvious notation.

The functional $M(P)$ comes up naturally in the theory of ergodic automorphisms of the torus $\mathcal{T}^n$, where $\log M(P)$ is the entropy of a mapping associated with P. Lind [20] has shown that an affirmative answer to Lehmer's question is equivalent to the existence of ergodic automorphisms of the infinite dimensional torus $\mathcal{T}^\omega$ which have finite entropy. (See also section 7(1)).

Some of the most impressive progress on the question has been the work of Blanksby and Montgomery [2], Stewart [30] and Dobrowolski [11] aimed at obtaining lower bounds on $M(P)$ which depend on the degree d. Of course the existence of such bounds is not in question since the set of P of a fixed degree with integer coefficients and bounded $M(P)$ is a finite set. One cannot fail to be impressed with Dobrowolski's result: if P is not cyclotomic, then

$$M(P)\geq 1+c\left(\frac{\log\log d}{\log d}\right)^3, \tag{3}$$

where c is an explicit constant. This interesting direction of research has been treated in survey articles of Stewart [31], Mignotte [23] and Waldschmidt [33] so we will not deal with it further in this paper.

Instead, we take the point of view that an appropriate object of study is the set of numbers $L=\{M(P): P \text{ has integer coefficients}\}$. Lehmer's question then simply asks whether 1 is a limit point of L so an ambitious line of attack would be to attempt to characterize all limit points of L.

Observe that L is a countable set of algebraic numbers contained in the interval $[1,\infty)$. Furthermore, L is a semigroup under multiplication since clearly $M(PQ)=M(P)M(Q)$. Thus, if 1 is a limit point of L then a familiar argument shows that L is dense in $[1,\infty)$. In fact, all that this requires is that if $a\in L$ then $a^n\in L$ for all $n>1$ (compare the argument in [24]). In this case the derived set $L^{(1)}$ of L (the set of limit points of L) would be $[1,\infty)$ and hence all successive derived sets $L^{(k)}$ would also be $[1,\infty)$. Thus, if 1 were a limit point of L then L would be a rather uninteresting set.

In order to provide a negative answer to Lehmer's question it would suffice to show that L is nowhere dense or just that $\min L^{(k)}>1$ for some specific k. But what possible reasons could one have for even suspecting that this might be true?

2. **The Pisot-Vijayaraghavan numbers.** Consider for the moment a few of the known facts concerning a certain subset S of L called the set of Pisot-Vijayaraghavan numbers or simply the Pisot numbers. We say that $\theta>1$ is in S if it is the root of a monic polynomial with integer coefficients all of whose remaining roots lie in the unit disk $|z|<1$. Clearly $M(P)=\theta$ in this case, so we do have $S\subset L$.

The following remarkable facts about S are known:

(a) (Salem [24]) S is closed, hence nowhere dense (since countable), hence

$$\min S = \theta_0 > 1 \text{ (since } 1 \notin S).$$

(b) (Siegel [27]) $\theta_0 = 1.32471\ldots$ is the real zero of $z^3 - z - 1$.

(c) (Dufresnoy and Pisot [14])

$$\min S^{(1)} = (\sqrt{5}+1)/2 = 1.61803\ldots$$

(d) (Grandet-Hugot [16]) $\min S^{(2)} = 2$

(e) (Salem [24], Dufresnoy and Pisot [13], Boyd [7]). The set $S^{(k)}$ is non-empty for all finite k, but $S^{(\omega)} = \emptyset$ and in fact $\min S^{(k)} > \sqrt{k}$.

Even more detail is known about S, but the above should be enough to indicate that S is an extremely remarkable set. This perhaps suggests that L may also have a similar interesting structure.

A note of caution should now be sounded. The proofs of (a)–(e) depend in an essential way on the fact that the irreducible polynomials satisfied by members of S are non-reciprocal (with a few easily handled exceptions). Here, the reciprocal of P is $P^*(z) = z^d P(z^{-1}) = a_d z^d + \cdots + a_0$, whose roots are the reciprocals of the roots of P (still assuming $a_d \neq 0$). Thus if P has one root in $|z| > 1$ and $d-1$ in $|z| < 1$ then P^* has $d-1$ roots in $|z| > 1$ and hence $P \neq \pm P^*$ unless possibly $d = 2$. The proofs of all known facts about S use Salem's observation that the function $f(z) = P(z)/P^*(z)$ is a non-constant rational function which has $|f(z)| = 1$ for $|z| = 1$ and has integer coefficients in its expansion about $z = 0$.

The set of Salem numbers T is defined in a similar way to S. A number $\theta > 1$ is in T if it is the root of a monic irreducible P with integer coefficients all of whose other roots lie in $|z| \leq 1$ with at least one on $|z| = 1$. This last condition forces P to be reciprocal [25]. In contrast with S, it is not even known whether T is dense in $[1, \infty)$. It is known that $S \subset T^{(1)}$, (Salem [25]), and the results of [3], [4] and [6] perhaps suggest that $S \cup T$ is closed and $S = T^{(1)}$, but this has not yet been proved. Whether one believes that T is dense in $[1, \infty)$ or nowhere dense depends on whether one believes that proofs of results like (a)–(e) must use $P \neq \pm P^*$ or not.

In view of the success with S as opposed to T, it is natural to single out the following subset of L:

$$L_0 = \{M(P) : P \text{ is non-reciprocal}\}.$$

And, indeed, Smyth [28] has shown that

$$\min L_0 = \theta_0 = \min S, \tag{4}$$

thus answering Lehmer's question for non-reciprocal polynomials. We should observe that $\sigma_1 = 1.17628\ldots$ of (2) is a Salem number, so perhaps (4) suggests that

(5) $$\min L = \sigma_1 = \min T\ (?).$$

3. **Mahler's measure.** By analogy with S, it might seem possible that L_0 or even L is a closed set. We shall show that this is highly unlikely and will propose sets $L_0^{\#}$ and $L^{\#}$ which may be the closures of L_0 and L respectively. We shall need to consider Mahler's [21] definition of the measure of $F(z_1, \ldots, z_n)$, a polynomial in n variables.

It is not obvious how (1) could be extended to polynomials in several variables until we recall Jensen's formula which states that

$$\int_0^1 \log |P(e^{2\pi i\theta})|\, d\theta = \log |a_0| + \sum_{i=1}^{d} \log(\max |\alpha_i|, 1)$$

Thus

(6) $$M(P) = \exp\left\{\int_0^1 \log |P(e^{2\pi i\theta})|\, d\theta\right\},$$

so $M(P)$ is just the geometric mean of $|P(z)|$ on the torus $\mathcal{T}$. Hence a natural candidate for $M(F)$ is

(7) $$M(F) = \exp\left\{\int_0^1 dt_1 \cdots \int_0^1 \log |F(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})|\, dt_n\right\}.$$

Now, even if F is a polynomial with integer coefficients it seems unlikely that $M(F)$ is an algebraic number. For example, Smyth has just shown (see Appendix 1) that

(8) $$M(1 + z_1 + z_2 + z_3) = \exp\left\{\frac{7}{2\pi^2}\zeta(3)\right\} = 1.53154\ldots,$$

which is most probably transcendental although a proof of this is not immediate. Here, of course, $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$, a number which Apéry has recently proved is irrational [1], [22], [32].

Let us define $L^{\#} = \{M(F) : F \text{ has integer coefficients}\}$ and let $L_0^{\#}$ be the corresponding set where F is non-reciprocal. Thus $L^{\#}$ is a countable subset of $[1, \infty)$ which contains L and presumably also some transcendental numbers. It would seem bizarre to propose studying these larger sets except for the following fact:

THEOREM 1. *$L^{\#}$ is contained in the closure of L.*

This is a consequence of the following results:

(9) $$\lim_{n\to\infty} M(F(z, z^n)) = M(F(z_1, z_2))$$

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and

$$\lim_{r_2\to\infty}\cdots\lim_{r_n\to\infty} M(F(z^{r_1},\ldots,z^{r_n}))=M(F(z_1,\ldots,z_n)), \tag{10}$$

where an iterated limit is intended in (10). The proof of (9) or (10) is fairly elementary if F does not vanish on the torus. A proof of (9) when $F(z_1, z_2)$ may vanish on $\mathcal{T}^2$ is given in [5] and that proof is reproduced here in Appendix 3. The proof of (10) is given in Appendix 4.

A more general result is expected to be true but has yet only been proved in case F does not vanish on $\mathcal{T}^n$. Given a vector of integers $\mathbf{r}=(r_1,\ldots,r_n)$, define

$$\mu(\mathbf{r})=\min\{|\mathbf{m}| : \mathbf{m}\in Z^n,\ \mathbf{m}\neq\mathbf{0} \text{ and } \mathbf{m}\cdot\mathbf{r}=0\}.$$

Then, provided F is a continuous fuhction which does not vanish on $\mathcal{T}^n$,

$$\lim_{\mu(\mathbf{r})\to\infty} M(F(z^{r_1},\ldots,z^{r_n}))=M(F(z_1,\ldots,z_n)). \tag{11}$$

This easily leads to the following useful result valid for all continuous F:

$$\limsup_{\mu(\mathbf{r})\to\infty} M(F(z^{r_1},\ldots,z^{r_n}))\le M(F(z_1,\ldots,z_n)). \tag{12}$$

The results (11) and (12) are proved in [9]. Lawton [18] has announced a result which seems to imply (11) for all polynomials, even those with zeros on $\mathcal{T}^n$, but the only copy of his paper which I have seen does not contain a proof. (For clarification, see section 7(2).)

Returning now to the main theme, we see that it is unreasonable to expect the set L to be closed in light of Theorem 1 and examples such as (8). (One would of course prefer a *proof* that $L\neq L^{\#}$). However, it does seem plausible to me that:

CONJECTURE 1. $L^{\#}$ is a closed set.

CONJECTURE 2. $L_0^{\#}$ is a closed set.

A proof of Conjecture 1 would answer Lehmer's question in the negative. To see this, recall that if 1 is a limit point of L, then L is dense in $[1,\infty)$ and hence so is $L^{\#}$. But if $L^{\#}$ is closed then it cannot be dense in $[1,\infty)$ since it is countable. Of course this is not likely to be the easiest way to solve Lehmer's problem, but it does indicate that $L^{\#}$ is a natural object of study.

A proof of Conjecture 2 would not have the same consequences. It is conceivable that it might be more accessible than Conjecture 1 although the methods used in [28] to show that $\inf L_0\ge\theta_0$ do not seem to be immediately applicable.

4. **Limit points of** L. Let us return now to our original problem of characterizing the limit points of L. Certainly some of these are of the form

$M(F(z_1, \ldots, z_n))$, and if Conjecture 1 is correct then all limit points are of this form.

However, not all such numbers need be limit points. If we concentrate on polynomials in two variables then (9) only shows that $M(F(z_1, z_2))$ is a limit point if we also have $M(F(z, z^n)) \neq M(F(z_1, z_2))$ for infinitely many n. To see how this might fail, consider $F(z_1, z_2) = z_1^a z_2^b G(z_1^c z_2^d)$ where G is a polynomial in one variable and a, b, c, d are integers. Then, for any integers $r_1 \neq 0$, $r_2 \neq 0$ we have

$$M(F(z^{r_1}, z^{r_2})) = M(G(z)) = M(F(z_1, z_2)), \tag{13}$$

so even (11) does not show $M(F)$ is a limit point, as is to be expected since not all $M(F)$ could be limit points or else L would be perfect and hence uncountable.

Another example based on a different principle was suggested to me by Smyth. Consider $F(z_1, z_2) = 2 + z_1 + z_2$. The polynomial $F(z, z^n) = 2 + z + z^n$ has no zeros in $|z| < 1$ since $2 > |z + z^n|$ for such z, so $M(F(z, z^n)) = 2$ for all $n \geq 1$. However, in this case if we consider instead $M(F(z, z^{-n})) = M(z^{n+1} + 2z^n + 1)$, we see by Rouché's theorem that $z^{n+1} + 2z^n + 1$ has n zeros in $|z| < 1$ and hence, one, say $-\theta_n$ in $|z| \geq 1$. Clearly θ_n is real and $\theta_n \to 2$ as $n \to \infty$, but $\theta_n \neq 2$. Hence $M(F(z_1, z_2)) \neq M(F(z, z^{-n}))$ for $n \geq 1$, and $2 \in L^{(1)}$. Of course we already knew that $2 \in S^{(2)} \subset T^{(3)} \subset L^{(3)}$ in this case.

An interesting limit point which was mentioned in [8] is

$$\beta = M(1 + z_1 + z_2) = 1.38135\ldots \tag{14}$$

The indicated value of β was computed in [5] from a quickly convergent series for $\log \beta$. Smyth has recently shown (see Appendix 1) that

$$\log \beta = \frac{3\sqrt{3}}{4\pi} L(2, \chi_3) = \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{1}{n^2}. \tag{15}$$

This makes it virtually inconceivable that β is algebraic.

In view of the above discussion, one may question whether β is really a limit point or whether in fact $\beta = M(1 + z + z^n)$ for all $n \geq n_0$. This would mean that β is algebraic, but although this seems unlikely, it is also unlikely that a proof will be found in the near future. Fortunately, we can show directly (Appendix 2) that

$$\log M(1 + z + z^n) = \log M(1 + z_1 + z_2) + \frac{c(n)}{n^2} + 0\left(\frac{1}{n^3}\right) \tag{16}$$

where $c(n) = -\sqrt{3}\pi/6$ if $n \equiv 2 \pmod 3$ and $c(n) = \sqrt{3}\pi/18$ if $n \equiv 0$ or $1 \pmod 3$, so β is indeed a limit point of L.

The interest in β is that it seems to be the smallest limit point of L_0. However, we do not even know whether $\min L_0' > \min L_0 = \theta_0$. A proof of even this would be extremely significant. (More *is* known, see section 7(3).)

Once one allows reciprocal polynomials, smaller limit points are to be expected. In fact every point in S is in $T^{(1)} \subset L^{(1)}$, by Salem's result quoted earlier. Salem's construction also shows that each element in L_0 is a limit point of L, modulo some of the difficulties which we met in the case of $2+z_1+z_2$. One simply observes that, by Jensen's formula,

$$M(wF(z_1,\ldots,z_n)+F(z_1^{-1},\ldots,z_n^{-1}))=M(F(z_1,\ldots,z_n)) \tag{17}$$

since $|F(z_1^{-1},\ldots,z_n^{-1})/F(z_1,\ldots,z_n)|=1$ on the torus. Then one applies (10) to $wF(z_1,\ldots,z_n)+F(z_1^{-1},\ldots,z_n^{-1})$, (which is within a monomial of being a polynomial). If F is reciprocal then (17) expresses the trivial fact that $M(w+z_1^{a_1},\ldots,z_n^{a_n})=1$, but if F is non-reciprocal, one expects that (10) and (17) will be enough to show that $M(F(z_1,\ldots,z_n))$ is a limit point.

Some smaller "limit points" than $\theta_0=\min L_0$ were exhibited in [8], and we list these below along with the only other known example with $M(F(z_1,z_2))<\theta_0$ which is not of the type (13). A result analogous to (16) is expected to be true but we have not carried out a calculation to rule out $M(F(z,z^n))=M(F(z_1,z_2))$ for $n\geq n_0$, so the appellation "limit point" used in [8] carries an element of hope:

$$\alpha_1=M(xy+y+x+1+x^{-1}+y^{-1}+x^{-1}y^{-1})=1.25542\ldots$$

$$\alpha_2=M(x+y+1+y^{-1}+x^{-1})=1.28573\ldots$$

$$\alpha_3=M(xy+y+x^{-1}y+x^2+x+1+x^{-1}+x^{-2}+xy^{-1}+y^{-1}+x^{-1}y^{-1})=1.31566\ldots$$

5. **Polynomials with measure 1.** One's attention naturally turns to $F(z_1,\ldots,z_n)$ for which $M(F)=1$, since one might expect to be able to use these in (10) or (11) to answer Lehmer's question in the affirmative, unless it should happen that for such F, $M(F(z^{r_1},\ldots,z^{r_n}))=1$ for almost all integer vectors $(r_1,\ldots,r_n)$. Fortunately for Conjecture 1, this is exactly the case. We have recently shown [9] that any F with $M(F)=1$ is a product of a finite number of factors of the form $z_1^{a_1}\cdots z_n^{a_n}$ and $\Phi(z_1^{b_1}\cdots z_n^{b_n})$, where $\Phi(z)$ is cyclotomic and a_i, b_i are integers. Our proof uses (12) together with results of Schinzel [26]. (See also section 7(4).)

Another way of seeing that $\lim M(F(z^{r_1},\ldots,z^{r_n}))=1$ implies $M(F(z^{r_1},\ldots,z^{r_n}))=1$ for $\mu(\mathbf{r})\geq\mu_0$ is to apply a recent result of Dobrowolski [12] which states that if g is the number of non-zero coefficients in $P(z)$, then either $M(P)=1$ or $M(P)\geq\delta(g)>1$, where δ is an explicit function of g only. Since the number of non-zero coefficients in $F(z^{r_1},\ldots,z^{r_n})$ is bounded by the number in $F(z_1,\ldots,z_n)$, we see from (12) that if $M(F)=1$ then $M(F(z^{r_1},\ldots,z^{r_n}))\geq\delta(g)$ cannot hold for arbitrarily large $\mu(n)$. Dobrowolski's result generalizes a result of Schinzel [26] which is in turn a constructive version of a result of Lawton [18]. This latter paper should be consulted by anyone intending to work on the conjecture that $L^{\#}$ is closed, since it contains a number of relevant ideas. (See also section 7(5).)

6. **Restrictions on the support of polynomials with small measure.** Smyth [29] has recently found a different and more direct proof of the characterization of F with $M(F)=1$. His proof utilizes some nice geometric notions and an inequality (18) which has some interesting consequences, as we shall see.

Represent F as a sum of monomials:

$$F(z_1,\ldots,z_n)=\sum_{\mathbf{j}\in J} a(\mathbf{j})z_1^{j_1}\cdots z_n^{j_n},$$

where J is the subset of $\mathscr{Z}^n$ for which $a(j)\neq 0$, called the *support* of F (or more reasonably, the support of the Fourier transform of F as a function on $\mathscr{T}^n$). Now let $\mathscr{C}(F)$ be the convex hull of J in $\mathscr{R}^n$. ($\mathscr{C}(F)$ is what B. Clarke [10] calls the *exponent polytope* of F). If $\mathscr{C}'$ is a face of $\mathscr{C}(F)$ (its intersection with a supporting hyperplane) and $J'=J\cap\mathscr{C}'$, then we call

$$F'(z_1,\ldots,z_n)=\sum_{\mathbf{j}\in J'} a(\mathbf{j})z_1^{j_1}\cdots z_n^{j_n}$$

a *face* of F. If $\mathscr{C}',\mathscr{C}''$ are intersections of $\mathscr{C}(F)$ with parallel supporting hyperplanes then we say F' and F'' are *opposite faces.*

By a change of variables of the form $z_i=w_1^{b_{i1}}\cdots w_n^{b_{im}}$, where the b_{ij} are integers, F' becomes a polynomial in less than n variables multiplied by a monomial $w_1^{c_1}\cdots w_n^{c_n}$. This leaves $M(F')$ unchanged. Using such a change of variables and applying Jensen's inequality, Smyth [29] shows that

$$M(F)\geq M(F'). \tag{18}$$

His proof is easily modified to give the sharper result

$$M(F)\geq M(F'\vee F''), \tag{19}$$

where we write $a\vee b=\max(|a|,|b|)$.

The inequality (18) forms the basis of an inductive argument by which Smyth characterizes F with $M(F)=1$ without appeal to the rather deep results of Schinzel which were needed in [9]. To conclude our paper, we wish to explore briefly some other consequences of (18) and (19).

Let us introduce the notation

$$M^+(F)=M(F\vee 1). \tag{20}$$

Then Jensen's formula shows that

$$M(w+F(z_1,\ldots,z_n))=M^+(F). \tag{21}$$

Let $P(z)$ be a non-constant polynomial. Then two opposite faces of $w+P(z)$ are $w+a_d$ and a_0z^d so (19) shows that

$$\begin{aligned} M^+(P(z)) &= M(w+P(z)) \\ &\geq M((w+a_d)\vee a_0)\geq M((w+1)\vee 1)=M^+(w+1) \end{aligned} \tag{22}$$

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But $M^+(w+1)=M(z+w+1)=\beta$, hence

$$M^+(P(z))\geq\beta=1.38135\ldots \tag{23}$$

We can use (12) to extend this to

$$M^+(F(z_1,\ldots,z_n))\geq\beta, \tag{24}$$

where F can be any non-constant polynomial in any number of variables.

Turning to $n=2$, if $\mathscr{C}(F)$ is a polygon with an odd number of sides, then there must be a pair of opposite faces $\mathscr{C}'$, $\mathscr{C}''$ in which $\mathscr{C}''$ is a single vertex but $\mathscr{C}'$ is not. If F', F'' are the corresponding faces of F then

$$M(F)\geq M(F'\vee F'')\geq M(F'\vee 1)=M^+(F')\geq\beta, \tag{25}$$

by (24).

Moving to higher dimensions, if $\mathscr{C}(F)$ is a polyhedron one of whose two-dimensional faces $\mathscr{C}'$ has an odd number of vertices then, using (18) and (25) (as applied to F'),

$$M(F)\geq M(F')\geq\beta. \tag{26}$$

For example the *octahedron*

$$F(z_1,z_2,z_3)=z_1z_2z_3(z_1+z_2+z_3+z_1^{-1}+z_2^{-1}+z_3^{-1})$$

has $M(F')=\beta$ for all of its two-dimensional faces, and $M(F)=\beta$. By a change of variables this example can be viewed as a special case of the construction used in (17). The equation (10) suggests that β is an element of $L^{(2)}$. Is β the smallest element of $L^{(2)}$?

For the *tetrahedron* $F=1+z_1+z_2+z_3$, all of its two-dimensional faces also have measure β, but

$$M(F)=M^+(1+z_1+z_2)>M(1+z_1+z_2)=\beta. \tag{27}$$

The quantity $M(F)$ is given explicitly in (8).

Of course, since the inequalities (18) and (19) only use information about the boundary of $\mathscr{C}(F)$, they naturally give no information about $M(F)$ in case $M(F'\vee F'')=1$ for every pair of opposite faces. It is thus not surprising that the three examples given at the end of §4 should have this property. The reader is encouraged to verify this by plotting the exponent polytope $\mathscr{C}(F)$ in these cases.

We conclude with one final suggestion. For an irreducible $F(z_1,\ldots,z_n)$, define $\dim(F)$ to be the dimension of the convex set $\mathscr{C}(F)$. Then a change of variables of the type $z_i=w_1^{b_{i1}}\cdots w_n^{b_{in}}$ reduces F to a polynomial in $\dim(F)$ variables multiplied by a monomial. Our characterization of F with $M(F)=1$ shows that $\dim(F)=1$ for such F (assuming still that F is irreducible). Can one show that if $\lambda(m)=\min\{M(F): F \text{ is irreducible and } \dim F=m\}$, then $\lambda(m)\to\infty$ as $m\to\infty$? Given a particular element of L, e.g. 2, this would show that there is a maximum value of $\dim(F)$ for which $2 = M(F)$. Using the proof that 2 is in $T^{(3)}$, one can construct an irreducible F with $\dim(F) = 4$ and $M(F) = 2$, namely

$$F(w, x, y, z) = w + x + y + z + 2wx + 2yz + wxy + xyz + yzw + zwx.$$

Is it possible to do better?

7. **Additional comments.** The following facts were brought to my attention by W. J. Lawton, C. J. Smyth and A. Schinzel after they had read the original version of this paper:

(1) The paper, "The structure of compact connected groups which admit an expansive automorphism", by W. J. Lawton, Lecture Notes in Mathematics, Volume 318, Springer-Verlag, Berlin and New York, 1973, pp. 182–196, contains results related to those of Lind [20].

(2) W. J. Lawton informs me that the Appendix referred to in [18] was never completed so a general proof of (11), valid for all polynomials, is still unavailable.

(3) C. J. Smyth informs me that he proved $\inf L_0^{(1)} > \theta_0 + 10^{-4}$ in his Ph.D. thesis, "Topics in the theory of numbers", Cambridge, 1972. Only part of this appears in his paper [28].

(4) Lawton has independently characterized those F with $M(F) = 1$ in "A generalization of a theorem of Kronecker", Journal of the Science Faculty of Chiangmai University (Thailand), **4** (1977), 15–23.

(5) A. Schinzel informs me that [12], [18] and [26] have been combined into a joint paper, "On a problem of Lehmer", by E. Dobrowolski, W. Lawton and A. Schinzel, to appear in a volume of Acta Math. Acad. Sci. Hung. in memory of Paul Turán.

Appendix 1. Explicit formulas for two measures (results of C. J. Smyth). Let $\beta = M(1 + z_1 + z_2)$ and $\beta_2 = M(1 + z_1 + z_2 + z_3)$. It will be shown here that

$$\log \beta = \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2}, \tag{15}$$

where $\chi(n) = 0$, 1 or -1 according to whether $n \equiv 0$, 1 or 2 (mod 3), and

$$\log \beta_2 = \frac{7}{2\pi^2} \zeta(3) \tag{8}$$

By Jensen's formula $M(1 + z_1 + z_2) = M(\max(|1 + z_1|, 1))$, so that, if $z_1 = e^{it}$,

$$\log \beta = \frac{1}{2\pi} \int_{-2\pi/3}^{2\pi/3} \log |1 + e^{it}| \, dt.$$

Now using

$$\log |1 + e^{it}| = \operatorname{Re} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{int},$$

and

$$\int_{-2\pi/3}^{2\pi/3} e^{int}\,dt = \frac{2}{n}\sin\frac{2n\pi}{3} = \frac{\sqrt{3}}{n}\chi(n),$$

one has

$$\log\beta = \frac{\sqrt{3}}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\chi(n)}{n^2} = \frac{\sqrt{3}}{2\pi}\left\{\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2} - 2\sum_{n=1}^{\infty}\frac{\chi(2n)}{(2n)^2}\right\},$$

which gives (15) after using $\chi(2n)=\chi(2)\chi(n)=-\chi(n)$.

To obtain the result (8), start again with Jensen's formula which gives $M(az_3+b)=|a|\max(|b/a|,1)=\max(|a|,|b|)$. Now write $z_2=zz_3$ and then

$$M(1+z_1+z_2+z_3)=M(1+z_1+z_3(1+z))=M(\max(|1+z_1|,|1+z|))$$

so that

$$\begin{aligned}\log\beta_2 &= \frac{1}{\pi^2}\int_0^{\pi}dt\int_0^{\pi}\max(\log|1+e^{it}|,\log|1+e^{iu}|)\,du\\ &= \frac{2}{\pi^2}\int_0^{\pi}\log|1+e^{it}|\,dt\int_t^{\pi}du\\ &= \frac{2}{\pi^2}\int_0^{\pi}(\pi-t)\log|1+e^{it}|\,dt\\ &= -\frac{2}{\pi^2}\int_0^{\pi}t\log|1+e^{it}|\,dt, \quad \text{since} \quad M(1+z)=1.\end{aligned}$$

Using the expansion of $\log|1+e^{it}|$ as above, an integration by parts reveals that

$$\log\beta_2 = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} = \frac{7}{2\pi^2}\zeta(3).$$

Appendix 2. Proof that β is a genuine limit point. We establish here the formula

(16) $$\log M(1+z+z^n) = \log M(1+z_1+z_2) + \frac{c(n)}{n^2} + 0\left(\frac{1}{n^3}\right),$$

where $c(n)=-\sqrt{3}\pi/6$ if $n\equiv 2 \pmod 3$, and $c(n)=\sqrt{3}\pi/18$ if $n\equiv 0$ or $1 \pmod 3$.

Let $z=e^{it}$ for $|t|\le\pi$, and observe that

(28) $$\begin{aligned}\log(1+z+z^n) &= \log(1+z)+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{z^n}{1+z}\right)^m, \quad \text{if} \quad |t|<2\pi/3\\ &= \log(z^n)+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{1+z}{z^n}\right)^m, \quad \text{if} \quad |t|>2\pi/3.\end{aligned}$$

Thus, if

$$\lambda_n = \log M(1+z+z^n) = \frac{1}{\pi}\int_0^{\pi}\log|1+z+z^n|\,dt,$$

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and

$$\lambda = \log M(1+z_1+z_2) = \frac{1}{\pi}\int_0^{\pi} \log^+ |1+z|\, dt,$$

then we have

(29) $$\lambda_n - \lambda = \frac{1}{\pi} \operatorname{Re} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (c_1(m)+c_2(m)),$$

where

$$c_1(m) = \int_0^{2\pi/3} e^{inmt}(1+e^{it})^{-m}\, dt,$$

and

$$c_2(m) = \int_{2\pi/3}^{\pi} e^{-inmt}(1+e^{it})^{m}\, dt$$

Now write $\omega = \exp(2\pi i/3)$, and integrate by parts four times to obtain

(30) $$c_1(m) = \frac{(-1)^m \omega^{(n-2)m}}{inm} - \frac{1}{inm2^m} + \frac{(-1)^{m+1}\omega^{(n-2)m-1}}{in(nm+1)} - \frac{1}{in(nm+1)2^{m+1}} + \varepsilon_1$$

where $|\varepsilon_1| \le K/n^3 m$.
This last estimate requires the standard estimate

$$\int_0^{2\pi/3} |1+e^{it}|^{-m-3}\, dt = 0(m^{-1}).$$

Similarly, one obtains

(31) $$c_2(m) = \frac{(-1)^m \omega^{-(n-2)m}}{inm} + \frac{(-1)^{m-1}\omega^{-(n-2)m-1}}{in(nm-1)} + \varepsilon_2,$$

where $|\varepsilon_2| \le K/n^3 m$, using

$$\int_{2\pi/3}^{\pi} |1+e^{it}|^{m-3}\, dt = 0(m^{-1}).$$

Adding, and taking real parts, the only surviving terms give

(32) $$\operatorname{Re}(c_1(m)+c_2(m)) = \frac{(-1)^m}{n^2 m}\sqrt{3}\cos((n-2)m2\pi/3) + 0\left(\frac{1}{n^3 m}\right)$$

and hence

(33) $$\lambda_n - \lambda = -\frac{\sqrt{3}}{\pi n^2} \sum_{m=1}^{\infty} \frac{\cos((n-2)m2\pi/3)}{m^2} + 0\left(\frac{1}{n^3}\right).$$

The value of $\cos((n-2)m2\pi/3)$ depends only on $(n-2)m \pmod 3$, and taking into account the two cases $n \equiv 2$ or $n \equiv 0, 1 \pmod 3$, one obtains the result (16).

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Appendix 3. This appendix is devoted to the proof of

$$\lim_{n\to\infty} M(F(z, z^n)) = M(F(z, w)), \tag{9}$$

if F is a polynomial. The proof is exactly as in [5] and is published here for the first time. Note that it is modelled on the explicit estimate obtained in Appendix 2 for the special polynomial $F(z, w) = 1+z+w$.

Write

$$\begin{aligned} F(z, w) &= a_0(w)z^d + \cdots + a_d(w) \\ &= a_0(w)(z - z_1(w)) \cdots (z - z_d(w)), \end{aligned}$$

where the z_k are the branches of the algebraic function defined by $F(z, w) = 0$. If we write $z = e^{it}$, $w = e^{is}$, we then have

$$\int_0^{2\pi} \log |F(z, z^n)| \, dt = \int_0^{2\pi} \log |a_0(z^n)| \, dt + \sum_{k=1}^{d} \int_0^{2\pi} \log |z - z_k(z^n)| \, dt, \tag{34}$$

while, by Jensen's formula,

$$\begin{aligned} &\int_0^{2\pi} ds \int_0^{2\pi} \log |F(z, w)| \, dt \\ &\qquad = 2\pi \int_0^{2\pi} \log |a_0(w)| \, ds + 2\pi \sum_{k=1}^{d} \int_0^{2\pi} \log^+ |z_k(w)| \, ds. \end{aligned} \tag{35}$$

Since, obviously, $\int_0^{2\pi} \log |a_0(z^n)| \, dt = \int_0^{2\pi} \log |a_0(w)| \, ds$, it suffices to show that

$$\lim_{n\to\infty} \int_0^{2\pi} \log |z - g(z^n)| \, dt = \int_0^{2\pi} \log^+ |g(w)| \, ds, \tag{36}$$

where $g(w)$ is any of the algebraic functions $z_k(w)$. To prove (36), we expand

$$\log |e^{it} - g(w)| = \sum_{m=-\infty}^{\infty} c_m(w) e^{imt}, \tag{37}$$

where $c_0(w) = \log^+ |g(w)|$ and

$$c_m(w) = -|2m|^{-1} g(w)^{\pm m} \quad \text{or} \quad -|2m|^{-1} \overline{g(w)}^{\pm m} \quad \text{if} \quad m \neq 0, \tag{38}$$

where the sign is such that $|g(w)^{\pm m}| \leq 1$.

Now write $s = nt$ in the left member of (36) and we have

$$\begin{aligned} \int_0^{2\pi} \log |e^{it} - g(e^{int})| \, dt &= n^{-1} \int_0^{2\pi n} \log |e^{is/n} - g(e^{is})| \, ds \\ &= \int_0^{2\pi} \left\{ n^{-1} \sum_{k=0}^{n-1} \log |e^{i(s+2k\pi)/n} - g(e^{is})| \right\} ds \\ &= \int_0^{2\pi} \left\{ c_0(e^{is}) + \sum_{m\neq 0} c_{mn}(e^{is}) e^{ims} \right\} ds, \end{aligned} \tag{39}$$

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using the expansion (37). We thus will have proved (36) if we can show that

$$\lim_{n\to\infty} \sum_{m\neq 0} \left| \int_0^{2\pi} c_{mn}(e^{is})e^{ims}\, ds \right| = 0 \tag{40}$$

To prove (40), we observe that since $g(w)$ is an algebraic function, it has a convergent expansion $g(w) = g(w_0) + a(w - w_0)^\alpha + \cdots$ in the neighbourhood of any point w_0, where α is a positive rational number.

We divide $[0, 2\pi)$ into a finite number of subintervals I so that in each we have $|g| \leq 1$ or $|g| \geq 1$ with equality at one endpoint at most, or else $|g| = 1$ throughout I. We can assume also that either $g' \neq 0$ throughout I or else $g' = 0$ at one endpoint at most. We then can write

$$|2mn| \int_0^{2\pi} c_{mn}(e^{is})e^{ims}\, ds = \int_0^{2\pi} g(e^{is})^{\pm mn} e^{ims}\, ds \tag{41}$$

as a sum of integrals over the subintervals. The contribution to (41) of each term of this sum is one of the following:

(i) If $|g| \leq b < 1$ or $|g|^{-1} \leq b < 1$ in I, the contribution $0(b^{|mn|})$.

(ii) If $|g| = 1$ at one endpoint only, then a change of variables gives an integral of the sort

$$\int_0^\varepsilon (1 - cu^\alpha + o(u^\alpha))^{|mn|}\, du = 0(|mn|^{-1/\alpha})$$

(iii) If $|g| = 1$ throughout I and $g' \neq 0$ in I, then $g(e^{is}) = e^{ih(s)}$ where h is real-valued and $h' \neq 0$ in I. An integration by parts shows that such an interval contributes $0(|mn|^{-1})$.

(iv) Finally, if $|g| = 1$ throughout I and $g' = 0$ at an endpoint, a change of variables produces an integral of the type

$$\int_0^\varepsilon e^{inmh(u)} e^{imu}\, du,$$

where h is real-valued and $h(u) = cu^\alpha + o(u^\alpha)$ as $u \to 0+$, and $\alpha > 1$. The integral over $0 < u \leq |nm|^{-1/\alpha}$ is $0(|nm|^{-1/\alpha})$ and the remaining integral can be treated as in (iii) to obtain an estimate $0(\max(|nm|^{-1/\alpha}, |m|^{-1}))$. The discussion of the "method of stationary phase" in [15, pp. 51–56] is relevant here.

Combining (i)–(iv), we find that the sum in (40) is $0(n^{-2})$ if $g'(w)$ does not vanish at a point where $|g(w)| = 1$ and is $0(n^{-1-c})$ for some $0 < c < 1$ otherwise. This completes the proof.

Appendix 4. Here we will prove the iterated limit formula

$$\lim_{r_2\to\infty} \cdots \lim_{r_n\to\infty} M(F(z, z^{r_2}, \ldots, z^{r_n})) = M(F(z_1, \ldots, z_n)) \tag{10}$$

where F is a polynomial. For this we need (9) and the

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LEMMA. *Suppose $f(z_1, \ldots, z_n)$ is a continuous function on the torus $\mathcal{T}^n$, then*

$$\lim_{r_2\to\infty} \cdots \lim_{r_n\to\infty} \int_{\mathcal{T}} f(z, z^{r_2}, \ldots, z^{r_n})\, dt = \int_{\mathcal{T}^n} f(z_1, \ldots, z_n)\, dt, \tag{42}$$

where $z = e(t)$, $z_j = e(t_j)$, $\mathbf{t} = (t_1, \ldots, t_n)$ and $e(t) = e^{2\pi i t}$.

Proof. Using the Weierstrass approximation theorem, it suffices to prove (42) for a trigonometric polynomial

$$f(e(t_1), \ldots, e(t_n)) = \sum_{\mathbf{m}\in J} a(\mathbf{m}) e(\mathbf{m}\cdot\mathbf{t})$$

where J is a finite set. We then have

$$\int_{\mathcal{T}} f(z, z^{r_2}, \ldots, z^{r_n})\, dt = \sum \{a(\mathbf{m}) : m_1 + m_2 r_2 + \cdots + m_n r_n = 0\}$$

But if $r_n \to \infty$ then $m_1 + m_2 r_2 \cdots + m_n r_n = 0$ implies $m_n = 0$, so

$$\lim_{r_n\to\infty} \int_{\mathcal{T}} f(z, z^{r_2}, \ldots, z^{r_n})\, dt = \sum \{a(\mathbf{m}) : m_1 + m_2 r_2 + \cdots + m_{n-1} r_{n-1} = 0, m_n = 0\},$$

and by induction,

$$\lim_{r_2\to\infty} \cdots \lim_{r_n\to\infty} \int_{\mathcal{T}} f(z, \ldots, z^{r_n})\, dt = \sum \{a(\mathbf{m}) : m_1 = m_2 = \cdots = m_n = 0\}$$

$$= a(\mathbf{0}) = \int_{\mathcal{T}^n} f(z_1, \ldots, z_n)\, d\mathbf{t}.$$

Proof of (10). Write $F(z_1, \ldots, z_n) = F_0(z_1, \ldots, z_{n-1}) \prod_{k=1}^d (z_n - h_k(z_1, \ldots, z_{n-1}))$, where the h_k are continuous functions. By Jensen's formula,

$$M(F(z_1, \ldots, z_n)) = M(F_0(z_1, \ldots, z_{n-1})) \prod_{k=1}^d M^+(h_k(z_1, \ldots, z_{n-1})). \tag{43}$$

By Appendix 3, and then Jensen's formula again,

$$\lim_{r_n\to\infty} M(F(z, z^{r_2}, \ldots, z^{r_n})) = M(F(z, \ldots, z^{r_{n-1}}, w)) \tag{44}$$

$$= M(F_0(z, \ldots, z^{r_{n-1}})) \prod_{k=1}^d M^+(h_k(z, \ldots, z^{r_{n-1}})),$$

Now $\log^+ |h_k(z_1, \ldots, z_{n-1})|$ is continuous on the torus $\mathcal{T}^{n-1}$ so, by the Lemma,

$$\lim_{r_2\to\infty} \cdots \lim_{r_{n-1}\to\infty} M^+(h_k(z, \ldots, z^{r_{n-1}})) = M^+(h_k(z_1, \ldots, z_{n-1})). \tag{45}$$

Using induction, we may assume (10) for F_0, then putting together (43) through (45) proves (10) for F.

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