THE ORDER CYCLE SYSTEM OF STOCK CONTROL

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1. Introduction

The order cycle system of stock control can be formulated as follows: demands for stock occur according to some pattern which we call the demand process and specify in detail later in this section. At fixed intervals of time orders are placed to replenish stock. Let orders for stock be placed at the instants $jN, j = 1, 2, \cdots$; the interval $[(j-1)N, jN)$ which we suppose closed at its lower end-point and open at its upper end-point, is called the j-th order cycle and N the length of this interval is called the order cycle period. We suppose that an order placed at time jN is delivered into stock at time $jN + l_j$ where $\{l_j\}$ is a sequence of non-negative random variables independent of the demand process and the order cycle period, we suppose also that the l_j are mutually independent and identically distributed with common distribution function $L(x)$ with $L(0+) = 0$ and finite expectation $l = \int_0^\infty x dL(x)$. The quantity l_j is called the lead time of the j-th order, that is of the order placed at jN and is supposed independent of the amount ordered. That portion of a demand, if any, which cannot be satisfied immediately is satisfied from future deliveries, thus every demand is satisfied eventually and a negative inventory or back orders can be held. We shall suppose that all order cycle periods under consideration are multiples of some fixed time interval of length τ which we shall take as our unit of time. For example, the interval of length τ could be one day and we would consider order cycle periods which were integral multiples of days. For convenience we take $\tau = 1$ and suppose that the order cycle period N is an integer. In this paper we shall consider the following two demand processes.

(i) Steady Deterministic Demand.

The demand A_n in the time interval $[(n-1), n)$ is a constant A. The demand in the j-th order cycle period is then AN. This demand process is a particular case of

(ii) General Independent Demand

The demand in the time interval $[(n-1), n)$ is A_n where $\{A_n\}$ is a sequence of identically and independently distributed random variables assuming non-negative integral values only. Write $a_j = P(A_n = j)$ and $a(z) = \sum_{j=0}^{\infty} a_j z^j$, $|z| \leqq 1$. The demand $A_n(N)$ in the n-th order cycle is given by

$$A_n(N) = \sum_{m=1}^{N} A_{(n-1)N+m}.$$

If $a_j(N) = P(A_n(N) = j)$ then we have

$$a(z, N) = \sum_{j=0}^{\infty} a_j(N) z^j = \{a(z)\}^N.$$

We suppose that $A = a'(1) = \sum jaj < \infty$ so that $E(A_n) = A$ exists.

Although a steady deterministic demand is a particular case of general independent demand it is convenient to treat the two cases separately. We remark that if we are interested in the stock level only at the beginning or end of an order cycle then we need not specify the demand within a unit time interval. In such cases the methods used below remain valid if the distribution of the $A_n(N)$ only is specified.

Throughout this paper we suppose that the amount of the order placed at the end of the j-th order cycle is equal to the demand within that order cycle.

The order cycle system has been considered by Pitt (2) who assumed a Poisson Demand and constant lead time and Gani (1) who considered a similar system with finite capacity, that is one in which a negative inventory cannot be held.

There are two practical limitations in the model discussed in this paper. Firstly the assumption of independent lead times is often unrealistic because it permits orders placed later than jN to arrive before that placed at jN. This limitation will be removed in a later paper in which we generalise the results cbtained below to a class of dependent lead times. Secondly the assumption of independent demand is not sufficiently wide to cover most practical situations. In a further paper we shall generalise some of the results obtained below to the case when the demand is a discrete weakly stationary process.

2. Purpose of the Analysis of this Paper

Let S_j denote the stock level at the instant $jN + 0$. For the purposes of this section we shall use S_j as a measure of the stock level in the j-th order cycle. Let B_j be the amount on order at the instant $jN + 0$, then it is not difficult to see that

$$S_j + B_j = S_0. \tag{2.1}$$

Write $P(n, j) = P(B_j = n)$, $n \geqq 0$, $j \geqq 0$ so that $P(S_j = n) = P(S - n, j)$ for $-\infty < n \leqq S = S_0$. Suppose that there exists a cost function $c(k)$

specifying the cost of holding k units of stock for unit time, then using S_j as a measure of the stock level during the j-th order cycle the expectation of cost per unit time of stock holding during the j-th order cycle is

$$\sum_{n=0}^{\infty} c(S-n)P(n,j). \tag{2.2}$$

We shall later see that $P(n) = \lim_{j\to\infty} P(n,j)$ exists and that (2.2) converges to

$$\sum_{n=0}^{\infty} c(S-n)P(n). \tag{2.3}$$

The distribution $P(n)$ does not depend on S and if we suppose that N, the order cycle period, is fixed we seek to choose S such that (2.3) is a minimum. For example, if $c(k) = kc_1$, $k > 0$, $c(0) = 0$ and $c(k) = |k|c_2$, $k < 0$ we choose S to minimise

$$c_1 \sum_{n=0}^{S-1} (S-n)P(n) + c_2 \sum_{n=S+1}^{\infty} (n-S)P(n).$$

It is easily verified that the required value of S is that for which it first happens that

$$\sum_{n=0}^{S} P(n) > c_2(c_1+c_2)^{-1}.$$

In general there will be an ordering cost associated with each order and the order cycle period N will be a variable quantity also and in order to perform a minimisation of costs we require the distribution of stock level in terms of the two parameters S and N. In the sections below we show how to obtain the distribution of stock size at time m after the beginning of the j-th order cycle and examine this distribution as $j \to \infty$.

3. The Distribution of the Number of Outstanding Orders

Let $\xi_{j,m}$, $j \geqq 0$, $m = 0, 1, 2, \cdots, N-1$, be the number of outstanding orders at time $jN+m$, that is at time m after the start of the $(j+1)$-th order cycle. We suppose that initially $\xi_{0,m} = 0$, $m = 0, 1, \cdots, N-1$, and emphasise that we are concerned only with the number of outstanding orders and not in the amounts of these orders. Introduce a function $f(x, y)$ defined by

$$f(x,y) = \begin{cases} 1 & \text{if} \quad 0 \leqq x < y, \\ 0 & \quad\text{otherwise.} \end{cases}$$

It is easy to see that

$$\xi_{j,m} = \sum_{n=0}^{j-1} f(nN+m, l_{j-n}). \tag{3.1}$$

Note that $\sum_{n=0}^{j-2} f(nN+m, l_{j-n})$ has the same distribution as $\xi_{j-1,m}$ and is independent of l_1. Note also that $f(jN-N+m, l_1) = 1$ if $l_1 > jN-N+m$ and is zero otherwise. Thus from (3.1) we have

$$\xi_{j,m} = \begin{cases} \xi^*_{j-1,m} & \text{if } \ l_1 \leqq jN-N+m, \\ \xi^*_{j-1,m}+1 & \text{if } \ l_1 > jN-N+m, \end{cases} \tag{3.2}$$

where $\xi^*_{j-1,m}$ has the same distribution as $\xi_{j-1,m}$ and is independent of l_1. Write $R^j_{m,n} = P(\xi_{j,m} = n)$, $j \geqq 0$, $n \geqq 0$, $0 \leqq m < N$ and introduce the generating function

$$U^j_m(z) = \sum_{n=0}^{\infty} R^j_{m,n} z^n, \qquad |z| \leqq 1.$$

We prove the following theorem.

THEOREM (3.1). The generating function $U^j_m(z)$ is given by

$$U^j_m(z) = \prod_{i=1}^{j} \{L_{i-1,m} + z(1-L_{i-1,m})\}, \tag{3.3}$$

where

$L_{i,m} = L(iN+m)$, $i \geqq 0$, $0 \leqq m < N$. Further $U_m(z) = \lim_{j\to\infty} U^j_m(z)$

exists, is a generating function and is given by

$$U_m(z) = \prod_{i=1}^{\infty} \{L_{i-1,m} + z(1-L_{i-1,m})\}. \tag{3.4}$$

There exists a limiting distribution $R_{m,n} = \lim_{j\to\infty} R^j_{m,n}$ and

$$U_m(z) = \sum_{n=0}^{\infty} R_{m,n} z^n, \qquad |z| \leqq 1. \tag{3.5}$$

PROOF. From (3.2) we obtain

$$R^j_{m,n} = R^{j-1}_{m,n} L_{j-1,m} + R^{j-1}_{m,n-1}(1-L_{j-1,m}), \tag{3.6}$$

where $j \geqq 1$, $0 \leqq m < N$, $n \geqq 0$ and $R^{j-1}_{m,-1} \equiv 0$. Hence we have

$$U^j_m(z) = \{L_{j-1,m} + z(1-L_{j-1,m})\} U^{j-1}_m(z).$$

Since $U^0_m(z) \equiv 1$, $0 \leqq m < N$ we obtain (3.3). In order to prove that $U_m(z)$ exists we note that

$$U_m(z) = \prod_{i=1}^{\infty} \{1-(1-z)(1-L_{i-1,m})\}. \tag{3.7}$$

We shall show that $\sum_{i=1}^{\infty}(1-L_{i,0}) < \infty$. Since $1-L_{i,m} < 1-L_{i,0}$ it will follow that $\sum_{i=1}^{\infty}(1-L_{i-1,m}) < \infty$ and hence that the infinite product in (3.7) is absolutely convergent for all values of z and thus $U_m(z)$ exists. To prove that $\sum_{i=1}^{\infty}(1-L_{i,0}) < \infty$ note that

$$\sum_{i=1}^{j}(1-L_{i,0}) \leqq N^{-1}\int_0^{jN}(1-L(x))dx \leqq N^{-1}\int_0^{\infty}(1-L(x))dx$$
$$= N^{-1}\int_0^{\infty} x dL(x) < \infty.$$

Since $U_m(1) = 1$ it follows from the continuity theorem for generating functions that limiting probabilities $R_{m,n}$ exist and that (3.5) holds.

COROLLARY.

Let M_m^j, V_m^j be respectively the mean and variance of the distribution $\{R_{m,n}^j\}$ then from (3.3) we obtain

$$M_m^j = \sum_{i=1}^{j}(1-L_{i-1,m}),$$
$$V_m^j = \sum_{i=1}^{j} L_{i-1,m}(1-L_{i-1,m}).$$

Similarly if M_m, V_m are respectively the mean and variance of the distribution $\{R_{m,n}\}$

$$M_m = \sum_{i=1}^{\infty}(1-L_{i-1,m}),$$
$$V_m = \sum_{i=1}^{\infty} L_{i-1,m}(1-L_{i-1,m}).$$

We remark that in practice it is usually the case that $L(x) = 1$ for $x \geqq kN$ where k is an integer. If k is reasonably small, for example, $k = 5$ then the infinite product (3.4) contains effectively only a finite number of factors and it is easy to obtain explicit expressions for the probabilities $R_{m,n}^j$, $R_{m,n}$ from the generating functions $U_m^j(z)$, $U_m(z)$.

When the distribution of lead time is exponential, that is when

$$L(x) = 1 - e^{-\mu x}, \qquad x \geqq 0. \tag{3.8}$$

It is possible to obtain the limiting distribution explicitly, namely we have

THEOREM (3.2). If $L(x)$ is given by (3.8) then the limiting distribution $\{R_{m,n}\}$ is given by

$$R_{m,n} = \sum_{i=n}^{\infty}(-)^{i-n}\binom{i}{n} C_{m,i}, \tag{3.9}$$

where

$$C_{m,i} = e^{i(N-m)\mu}\prod_{j=1}^{i} e^{-jN\mu}(1-e^{-jN\mu})^{-1}, \quad i \geqq 1, \quad 0 \leqq m < N, \tag{3.10}$$

and $C_{m,0} \equiv 1$, $0 \leqq m < N$.

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PROOF. From (3.4) we obtain

$$(3.11) \qquad U_m(z) = (1 - e^{-m\mu} + ze^{-m\mu})U_m(1 - e^{-N\mu} + ze^{-N\mu}).$$

Introduce the binomial moments $C_{m,i}$ of the distribution $\{R_{m,n}\}$ defined by

$$C_{m,i} = (i!)^{-1}[(d^i/dz^i)U_m(z)]_{z=1}, \qquad i \geqq 0.$$

From (3.11) we obtain

$$C_{m,i} = e^{-iN\mu}C_{m,i} + e^{-(iN-N+m)\mu}C_{m,i-1}, \qquad i \geqq 1.$$

Hence we obtain (3.10). In order to prove (3.9) note that $U_m(z) = \sum_{i=0}^{\infty} C_{m,i} (z-1)^i$ is convergent for every z since $\lim_{i\to\infty} C_{m,i}/C_{m,i-1} = 0$. Thus

$$R_{m,n} = (n!)^{-1}[(d^n/dz^n)U_m(z)]_{z=0} = \sum_{i=n}^{\infty} (-)^{i-n}\binom{i}{n}C_{m-i}.$$

4. Steady Deterministic Demand

In this section we assume a steady demand A stock units per unit time and suppose that an amount AN is ordered at the end of each order cycle. Let $S_{j,m}$ be the stock level at time $jN + m$, $j \geqq 0$, $0 \leqq m < N$ and suppose that initially $S_{0,0} = S$ and $\xi_{0,0} = 0$. Then we have

$$(4.1) \qquad S_{j,m} = S_{j,0} + AN(\xi_{j,0} - \xi_{j,m}) - mA,$$

and

$$(4.2) \qquad S_{j,0} = S - AN\xi_{j,0}.$$

Hence

$$(4.3) \qquad S_{j,m} = S - AN\xi_{j,m} - mA.$$

It is more convenient to work with the stock deficit $B_{j,m} = S - S_{j,m}$ than with the stock level $S_{j,m}$. From (4.3) we have

$$(4.4) \qquad B_{j,m} = A(N\xi_{j,m} + m), \quad j \geqq 0, \quad 0 \leqq m < N.$$

Write $P^j_{m,n} = P(B_{j,m} = n)$, $n \geqq 0$, introduce the generating function $G^j_m(z) = \sum_{n=0}^{\infty} P_{m,n}z^n$. Then from (4.4) we obtain easily

$$(4.5) \qquad G^j_m(z) = z^{mA}U^j_m(z^{NA}), \qquad 0 \leqq m < N,$$

where $U^j_m(z)$ is given by (3.3). Hence using Theorem (3.1) $G_m(z) = \lim_{j\to\infty} G^j_m(z)$ exists and is given by

$$(4.6) \qquad G_m(z) = z^{mA}U_m(z^{NA}), \qquad 0 \leqq m < N,$$

where $U_m(z)$ is given by (3.4). From (4.4) and the corollary to Theorem (3.1) we have

$$E(B_{j,m}) = mA + NA \sum_{i=1}^{j} (1 - L_{i-1,m}),$$

$$Var(B_{j,m}) = N^2A^2 \sum_{i=1}^{j} L_{i-1,m}(1 - L_{i-1,m}).$$

It is easily verified that the mean and variance of the limiting distribution $P_{m,n} = \lim_{j\to\infty} P^j_{m,n}$ are given by similar expressions with the finite sums replaced by the corresponding infinite series.

5. General Independent Demand

We suppose now that the demand process is the general independent demand defined in section 1 and that the amount ordered at the end of each order cycle is equal to the demand in that cycle. As in section 4 we write $S_{j,m}$ for the stock level at $jN + m$ and write $B_{j,m} = S - S_{j,m}$ where $S = S_{0,0}$. We suppose also that $\xi_{0,0} = 0$. We define probabilities $P^j_{m,n}$ and the generating function $G^j_m(z)$ as in the previous section and prove the following theorem.

THEOREM (5.1). The generating function $G^j_m(z)$ is given by

$$G^j_m(z) = \{a(z)\}^m U^j_m\{(a(z))^N\}, \qquad 0 \leqq m < N, \tag{5.1}$$

where $U^j_m(z)$ is given by (3.3) and $a(z)$ is the generating function of the demand per unit time. Limiting probabilities $P_{m,n} = \lim_{j\to\infty} P^j_{m,n}$ exist and

$$\sum_{n=0}^{\infty} P_{m,n} z^n = G_m(z), \tag{5.2}$$

where

$$G_m(z) = \{a(z)\}^m U_m\{(a(z))^N\}, \qquad 0 \leqq m < N, \tag{5.3}$$

and $U_m(z)$ is given by (3.4).

PROOF.

Corresponding to equation (4.3) we have

$$S_{j,m} = S - \sum_{k=1}^{\xi_{j,m}} A_{k'}(N) - \sum_{i=1}^{m} A_{jN+i}, \tag{5.4}$$

where A_{jN+i} is the demand in $[jN+i-1, jN+i)$ and the $A_{k'}(N)$ are independent of the A_{jN+i}, are mutually independent and identically distributed with the common distribution of the total demand in an order cycle. Thus we have

$$B_{j,m} = \sum_{k=1}^{\xi_{j,m}} A_{k'}(N) + \sum_{i=1}^{m} A_{jN+i}. \tag{5.5}$$

By section 1 the generating function of the probability distribution of $\sum_{i=1}^{m} A_{jN+i}$ is just $\{a(z)\}^m$ and the generating function of the probability distribution of $\sum_{k=1}^{n} A_{k'}(N)$ is $\{a(z)\}^{Nh}$. Thus the generating function of the distribution of the first term of (5.5) is $U_m^j\{(a(z))^N\}$ where $U_m^j(z)$ is given by (3.3) since the $A_{k'}(N)$ are independent of $\xi_{j,m}$. The two terms of (5.5) are independent, hence we obtain (5.1). The existence of the limiting distribution $P_{m,n}$ and equations (5.2), (5.3) follow readily from (5.1) and theorem (3.1).

Note that (4.5) is a particular case of (5.1) for when the demand is deterministic and steady at the A per unit time $a(z) = z^A$.

COROLLARY.

The mean and variance of the random variable $B_{j,m}$ are given by

(5.6) $$E(B_{j,m}) = mA + NA \sum_{i=1}^{j} (1 - L_{i-1,m}),$$

(5.7) $$Var(B_{j,m}) = A^2N^2 \sum_{i=1}^{j} L_{i-1,m}(1 - L_{i-1,m}) + NV(A) \sum_{i=1}^{j} (1 - L_{i-1,m}).$$

where A, $V(A)$ are the mean and variance respectively of the demand per unit time.

Equations (5.6), (5.7), follow immediately from (5.1) and the Corollary to theorem (3.1). Similarly it follows from (5.3) that the mean and variance of the limiting distribution are given by replacing the finite sums in (5.6) and (5.7) by the corresponding infinite series.

References

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