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NEW INFINITE FAMILIES OF CONGRUENCES MODULO 4 AND 8 FOR 1-SHELL TOTALLY SYMMETRIC PLANE PARTITIONS

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Abstract

In 2012, Blecher ['Geometry for totally symmetric plane partitions (TSPPs) with self-conjugate main diagonal', *Util. Math.* **88** (2012), 223–235] introduced a special class of totally symmetric plane partitions, called 1-shell totally symmetric plane partitions. Let $f(n)$ denote the number of 1-shell totally symmetric plane partitions of weight n. More recently, Hirschhorn and Sellers ['Arithmetic properties of 1-shell totally symmetric plane partitions', *Bull. Aust. Math. Soc.* to appear. Published online 27 September 2013] discovered a number of arithmetic properties satisfied by $f(n)$. In this paper, employing some results due to Cui and Gu ['Arithmetic properties of l-regular partitions', *Adv. Appl. Math.* **51** (2013), 507–523], and Hirschhorn and Sellers, we prove several new infinite families of congruences modulo 4 and 8 for 1-shell totally symmetric plane partitions. For example, we find that, for $n \geq 0$ and $\alpha \geq 1$,

$$f(8 \times 5^{2\alpha} n + 39 \times 5^{2\alpha-1}) \equiv 0 \pmod{8}.$$

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1. Introduction

The aim of this paper is to establish some new infinite families of congruences modulo 4 and 8 for 1-shell totally symmetric plane partitions by using some results due to Cui and Gu [4] and Hirschhorn and Sellers [5].

Recall that a plane partition is a two-dimensional array of integers $\pi_{i,j}$ that are weakly decreasing and that add up to a given number n. In other words, $\pi_{i,j} \geq \pi_{i+1,j}$, $\pi_{i,j} \geq \pi_{i,j+1}$ and $\sum \pi_{i,j} = n$. Plane partitions invariant under any permutations of the three axes are called totally symmetric plane partitions (TSPPs). (For more details about TSPPs, see, for example, Andrews *et al.* [1] and Stembridge [6]). Blecher [3] gave a definition of a special class of TSPPs, called 1-shell TSPPs. As defined by Blecher, a TSPP is called a 1-shell TSPP if this partition has a self-conjugate first

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row/column (as an ordinary partition) and all other entries are 1. For example, the following is a 1-shell TSPP:

$$\begin{matrix} 4 & 4 & 2 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 1 & & \\ 2 & 1 & & \end{matrix}$$

Let $f(n)$ denote the number of 1-shell TSPPs of weight n; this means that the parts of the TSPP sum to n. Blecher [3] found the generating function of $f(n)$. He proved that

$$\sum_{n=0}^{\infty} f(n)q^n = 1 + \sum_{n=1}^{\infty} q^{3n-2} \prod_{i=0}^{n-2} (1 + q^{6i+3}).$$

Recently, Hirschhorn and Sellers [5] proved a number of arithmetic properties satisfied by $f(n)$ by employing elementary generating function manipulations and some well-known results due to Ramanujan and Watson. They proved that, for $n \geq 1$,

$$f(3n) = f(3n-1) = 0, \tag{1.1}$$

$$f(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } 3 \nmid n \text{ and } n = k^2 \text{ for some integer } k, \\ 0 \pmod 2 & \text{otherwise} \end{cases}$$

and

$$f(10n-5) \equiv 0 \pmod 5. \tag{1.2}$$

At the end of their paper [5], Hirschhorn and Sellers said: 'it appears that $f(n)$ satisfies congruences in arithmetic progression modulo 4 and 8 based on the computational evidence available. It would be desirable to see proofs of these results'. The objective of this paper is to prove some new congruences modulo 4 and 8 satisfied by $f(n)$ by employing some results given by Cui and Gu [4] and Hirschhorn and Sellers [5]. Our main results can be stated as follows.

THEOREM 1.1. *For all $n \geq 0$,*

$$f(8n+3) \equiv 0 \pmod 4. \tag{1.3}$$

THEOREM 1.2. *For any prime $p \equiv -1 \pmod 6$, $\alpha \geq 1$, $i = 1, 2, \ldots, p-1$ and $n \geq 0$,*

$$f(8p^{2\alpha}n + (24i+7p)p^{2\alpha-1}) \equiv 0 \pmod 4, \tag{1.4}$$

$$f(8p^{2\alpha}n + (24i+5p)p^{2\alpha-1}) \equiv 0 \pmod 4 \tag{1.5}$$

and

$$f(8p^{2\alpha}n + (24i+3p)p^{2\alpha-1}) \equiv 0 \pmod 8. \tag{1.6}$$

EXAMPLE 1.3. Setting $p = 5$ and $i = 1$ in (1.6), we find that, for $n \geq 0$ and $\alpha \geq 1$,

$$f(8 \times 5^{2\alpha}n + 39 \times 5^{2\alpha-1}) \equiv 0 \pmod 8.$$

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This paper is organised as follows. In Section 2 we recall some notation and terminology on q-series and three dissection formulas due to Ramanujan [2] and Cui and Gu [4]. In Section 3 we give a proof of Theorem 1.1 by using a 2-dissection formula given by Ramanujan [2]. In Section 4, employing p-dissection formulas of Ramanujan's theta functions $\psi(q)$ and f_1 established by Cui and Gu [4], we present a proof of Theorem 1.2.

2. Preliminary results

To prove Theorems 1.1 and 1.2, we need three dissection formulas due to Ramanujan [2] and Cui and Gu [4]. Let us begin with some notation and terminology on q-series. In this paper, we adopt the common notation

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

where $|q| < 1$. Recall that the Ramanujan theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{2.1}$$

where $|ab| < 1$. Two special cases of (2.1) are

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \tag{2.2}$$

and

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

For any positive integer k, we use f_k to denote $f(-q^k)$, that is,

$$f_k = (q^k;q^k)_\infty = \prod_{n=1}^{\infty}(1-q^{nk}).$$

The following relation is a consequence of dissection formulas of Ramanujan collected in Entry 25 in Berndt's book [2, page 40].

Theorem 2.1. *The following identity holds:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}. \tag{2.3}$$

Recently, Cui and Gu [4] established p-dissection formulas for $\psi(q)$ and f_1. They proved the following two theorems.

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THEOREM 2.2. *For any odd prime p,*

$$\psi(q) = \sum_{k=0}^{(p-3)/2} q^{(k^2+k)/2} f(q^{(p^2+(2k+1)p)/2}, q^{(p^2-(2k+1)p)/2}) + q^{(p^2-1)/8}\psi(q^{p^2}).$$

THEOREM 2.3. *For any prime $p \geq 5$,*

$$f_1 = \sum_{\substack{k=(1-p)/2,\\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2},$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \dfrac{p-1}{6} & \text{if } p \equiv 1 \pmod 6, \\ \dfrac{-p-1}{6} & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1.

To prove (1.2), Hirschhorn and Sellers [5] proved that, for $n \geq 1$,

$$f(3n-2) = h(n), \tag{3.1}$$

where $h(n)$ is defined by

$$\sum_{n=1}^{\infty} h(n)q^n = \sum_{n=1}^{\infty} q^n \prod_{i=0}^{n-2}(1+q^{2i+1}).$$

Employing some well-known results of Ramanujan and Watson, they proved that

$$\sum_{n=0}^{\infty} h(2n+1)q^n = \prod_{n=1}^{\infty}(1+q^n)^3(1-q^n). \tag{3.2}$$

Using the notation f_k, we can rewrite (3.2) as

$$\sum_{n=0}^{\infty} h(2n+1)q^n = \frac{f_2^3}{f_1^2}. \tag{3.3}$$

By Theorem 2.1 and (3.3), we are led to generating functions of $h(8n+1)$, $h(8n+3)$, $h(8n+5)$ and $h(8n+7)$.

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Lemma 3.1. *We have*

$$\sum_{n=0}^{\infty} h(8n+1)q^n = \frac{f_2^5 f_4^3}{f_1^5 f_8^2},$$

$$\sum_{n=0}^{\infty} h(8n+3)q^n = 2\frac{f_4^7}{f_1^3 f_2 f_8^2}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} h(8n+5)q^n = 2\frac{f_2^7 f_8^2}{f_1^5 f_4^3} \tag{3.5}$$

and

$$\sum_{n=0}^{\infty} h(8n+7)q^n = 4\frac{f_2 f_4 f_8^2}{f_1^3}. \tag{3.6}$$

Proof. Substituting (2.3) into (3.3),

$$\sum_{n=0}^{\infty} h(2n+1)q^n = f_2^3\Big(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\Big) = \frac{f_8^5}{f_2^2 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^2 f_8},$$

which yields

$$\sum_{n=0}^{\infty} h(4n+1)q^n = \frac{f_4^5}{f_1^2 f_8^2} \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} h(4n+3)q^n = 2\frac{f_2^2 f_8^2}{f_1^2 f_4}. \tag{3.8}$$

Substituting (2.3) into (3.7) and (3.8),

$$\sum_{n=0}^{\infty} h(4n+1)q^n = \frac{f_4^5}{f_8^2}\Big(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\Big) = \frac{f_4^5 f_8^3}{f_2^5 f_{16}^2} + 2q\frac{f_4^7 f_{16}^2}{f_2^5 f_8^3} \tag{3.9}$$

and

$$\sum_{n=0}^{\infty} h(4n+3)q^n = 2\frac{f_2^2 f_8^2}{f_4}\Big(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\Big) = 2\frac{f_8^7}{f_2^3 f_4 f_{16}^2} + 4q\frac{f_4 f_8 f_{16}^2}{f_2^3}. \tag{3.10}$$

Lemma 3.1 follows from (3.9) and (3.10). This completes the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Thanks to (3.6), for $n \geq 0$,

$$h(8n+7) \equiv 0 \pmod 4. \tag{3.11}$$

Replacing n by $8n+7$ in (3.1) and using (3.11), for $n \geq 0$,

$$f(24n+19) \equiv 0 \pmod 4. \tag{3.12}$$

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From (1.1),

$$f(24n + 3) = f(24n + 11) = 0. \tag{3.13}$$

Congruence (1.3) follows from (3.12) and (3.13). The proof is complete. □

4. Proof of Theorem 1.2

By the binomial theorem, it is easy to see that, for all positive integers k and m,

$$f_m^{2k} \equiv f_{2m}^k \pmod 2. \tag{4.1}$$

By (4.1),

$$\frac{f_4^7}{f_1^3 f_2 f_8^2} \equiv \frac{f_8}{f_1} \pmod 2. \tag{4.2}$$

In view of (3.4) and (4.2), for $n \geq 0$,

$$h(8n + 3) \equiv 2b_8(n) \pmod 4, \tag{4.3}$$

where $b_8(n)$ is the number of 8-regular partitions of n and the generating function of $b_8(n)$ is

$$\sum_{n=0}^{\infty} b_8(n)q^n = \frac{f_8}{f_1}.$$

Cui and Gu [4] found some congruences modulo 2 for $b_8(n)$. They proved that, for any prime $p \equiv -1 \pmod 6$, $\alpha \geq 1$ and $n \geq 0$,

$$b_8\Big(p^{2\alpha}n + \frac{(24i + 7p)p^{2\alpha-1} - 7}{24}\Big) \equiv 0 \pmod 2, \quad i = 1, 2, \ldots, p - 1. \tag{4.4}$$

Replacing n by $p^{2\alpha}n + ((24i + 7p)p^{2\alpha-1} - 7)/24$ $(i = 1, 2, \ldots, p - 1)$ in (4.3) and using (4.4),

$$h\Big(8p^{2\alpha}n + \frac{(24i + 7p)p^{2\alpha-1} + 2}{3}\Big) \equiv 0 \pmod 4, \quad i = 1, 2, \ldots, p - 1. \tag{4.5}$$

Replacing n by $8p^{2\alpha}n + ((24i + 7p)p^{2\alpha-1} + 2)/3$ $(i = 1, 2, \ldots, p - 1)$ in (3.1) and using (4.5), we see that, for $n \geq 0$ and $\alpha \geq 1$,

$$f(24p^{2\alpha}n + (24i + 7p)p^{2\alpha-1}) \equiv 0 \pmod 4, \quad i = 1, 2, \ldots, p - 1. \tag{4.6}$$

By (1.1),

$$f(8p^{2\alpha}(3n + 1) + (24i + 7p)p^{2\alpha-1}) = f(8p^{2\alpha}(3n + 2) + (24i + 7p)p^{2\alpha-1}) = 0. \tag{4.7}$$

Congruence (1.4) follows from (4.6) and (4.7).

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Next, we prove (1.5). By (2.2) and (4.1), it is easy to check that

$$\frac{f_2^7 f_8^2}{f_1^5 f_4^3} \equiv f_1 \psi(q^4) \pmod{2}. \tag{4.8}$$

Let $a(n)$ be defined by

$$\sum_{n=0}^{\infty} a(n) q^n = f_1 \psi(q^4). \tag{4.9}$$

Combining (3.5), (4.8) and (4.9), we deduce that, for $n \geq 0$,

$$h(8n+5) \equiv 2a(n) \pmod{4}. \tag{4.10}$$

For any prime $p \equiv -1 \pmod{6}$, employing (4.9) and Theorems 2.2 and 2.3,

$$\begin{aligned}\sum_{n=0}^{\infty} a(n) q^n = \Bigg(& \sum_{\substack{m=(1-p)/2, \\ m \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^m q^{(3m^2+m)/2} f(-q^{(3p^2+(6m+1)p)/2}, -q^{(3p^2-(6m+1)p)/2}) \\ & + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2} \Bigg) \\ & \times \Bigg(\sum_{k=0}^{(p-3)/2} q^{2(k^2+k)} f(q^{2(p^2+(2k+1)p)}, q^{2(p^2-(2k+1)p)}) + q^{(p^2-1)/2} \psi(q^{4p^2}) \Bigg).\end{aligned}$$

Now we consider the congruence

$$\frac{3m^2+m}{2} + 2(k^2+k) \equiv \frac{13(p^2-1)}{24} \pmod{p}, \tag{4.11}$$

where $-(p-1)/2 \leq m \leq (p-1)/2$ and $0 \leq k \leq (p-1)/2$. We can rewrite (4.11) as

$$(6m+1)^2 + 3(4k+2)^2 \equiv 0 \pmod{p}. \tag{4.12}$$

Since $p \equiv -1 \pmod{6}$ and -3 is a quadratic nonresidue modulo p, (4.12) yields

$$6m+1 \equiv 4k+2 \equiv 0 \pmod{p}.$$

Hence, $m = (-p-1)/6$ and $k = (p-1)/2$. The fact that (4.11) has only one solution $(m,k) = ((-p-1)/6, (p-1)/2)$ implies that

$$\sum_{n=0}^{\infty} a\Big(pn + \frac{13(p^2-1)}{24}\Big) q^{pn+13(p^2-1)/24} = (-1)^{(-p-1)/6} q^{13(p^2-1)/24} f_{p^2} \psi(q^{4p^2}). \tag{4.13}$$

Dividing by $q^{13(p^2-1)/24}$ on both sides of (4.13) and then replacing q^p by q,

$$\sum_{n=0}^{\infty} a\Big(pn + \frac{13(p^2-1)}{24}\Big) q^n = (-1)^{(-p-1)/6} f_p \psi(q^{4p}),$$

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which implies that

$$\sum_{n=0}^{\infty} a\Big(p^2 n + \frac{13(p^2-1)}{24}\Big) q^n = (-1)^{(-p-1)/6} f_1 \psi(q^4) \tag{4.14}$$

and, for $n \geq 0$,

$$a\Big(p(pn+i) + \frac{13(p^2-1)}{24}\Big) = 0, \quad i = 1, 2, \ldots, p-1. \tag{4.15}$$

In view of (4.9) and (4.14),

$$a\Big(p^2 n + \frac{13(p^2-1)}{24}\Big) \equiv a(n) \pmod 2. \tag{4.16}$$

By (4.16) and mathematical induction, we see that, for $n \geq 0$ and $\alpha \geq 0$,

$$a\Big(p^{2\alpha} n + \frac{13(p^{2\alpha}-1)}{24}\Big) \equiv a(n) \pmod 2. \tag{4.17}$$

Replacing n by $p(pn+i) + 13(p^2-1)/24$ $(i = 1, 2, \ldots, p-1)$ in (4.17) and employing (4.15), for $n \geq 0$ and $\alpha \geq 1$,

$$a\Big(p^{2\alpha} n + \frac{(24i+13p)p^{2\alpha-1} - 13}{24}\Big) \equiv 0 \pmod 2, \quad i = 1, 2, \ldots, p-1. \tag{4.18}$$

Replacing n by $p^{2\alpha} n + ((24i+13p)p^{2\alpha-1} - 13)/24$ $(i = 1, 2, \ldots, p-1)$ in (4.10) and using (4.18), for $n \geq 0$ and $\alpha \geq 1$,

$$h\Big(8p^{2\alpha} n + \frac{(24i+13p)p^{2\alpha-1} + 2}{3}\Big) \equiv 0 \pmod 4, \quad i = 1, 2, \ldots, p-1. \tag{4.19}$$

Replacing n by $8p^{2\alpha} n + ((24i+13p)p^{2\alpha-1} + 2)/3$ $(i = 1, 2, \ldots, p-1)$ in (3.1) and using (4.19), for $n \geq 0$ and $\alpha \geq 1$,

$$f(24p^{2\alpha} n + (24i+13p)p^{2\alpha-1}) \equiv 0 \pmod 4, \quad i = 1, 2, \ldots, p-1. \tag{4.20}$$

Thanks to (1.1),

$$f(24p^{2\alpha} n + (24i+5p)p^{2\alpha-1}) = f(24p^{2\alpha} n + (24i+21p)p^{2\alpha-1}) = 0. \tag{4.21}$$

Congruence (1.5) follows from (4.20) and (4.21).

To conclude this section, we give a proof of (1.6). By (2.2) and (4.1), it is easy to check that

$$\frac{f_2 f_4 f_8^2}{f_1^3} \equiv f_{16} \psi(q) \pmod 2. \tag{4.22}$$

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Let $c(n)$ be defined by

$$\sum_{n=0}^{\infty} c(n)q^n = f_{16}\psi(q). \tag{4.23}$$

In view of (3.6), (4.22) and (4.23), for $n \geq 0$,

$$h(8n+7) \equiv 4c(n) \pmod 8. \tag{4.24}$$

We consider the congruence

$$16 \times \frac{3m^2+m}{2} + \frac{k^2+k}{2} \equiv \frac{19(p^2-1)}{24} \pmod p, \tag{4.25}$$

where $-(p-1)/2 \leq m \leq (p-1)/2$ and $0 \leq k \leq (p-1)/2$. For any prime $p \equiv -1 \pmod 6$, (4.25) holds if and only if $m = (-p-1)/6$ and $k = (p-1)/2$. Using (4.23) and Theorems 2.2 and 2.3,

$$\sum_{n=0}^{\infty} c\Big(pn + \frac{19(p^2-1)}{24}\Big)q^n = (-1)^{(-p-1)/6} f_{16p}\psi(q^p),$$

which yields

$$\sum_{n=0}^{\infty} c\Big(p^2n + \frac{19(p^2-1)}{24}\Big)q^n = (-1)^{(-p-1)/6} f_{16}\psi(q) \tag{4.26}$$

and, for $n \geq 0$,

$$c\Big(p(pn+i) + \frac{19(p^2-1)}{24}\Big) = 0, \quad i = 1, 2, \ldots, p-1. \tag{4.27}$$

Combining (4.23) and (4.26),

$$c\Big(p^2n + \frac{19(p^2-1)}{24}\Big) \equiv c(n) \pmod 2. \tag{4.28}$$

By (4.28) and mathematical induction, for $n \geq 0$ and $\alpha \geq 0$,

$$c\Big(p^{2\alpha}n + \frac{19(p^{2\alpha}-1)}{24}\Big) \equiv c(n) \pmod 2. \tag{4.29}$$

Replacing n by $p(pn+i) + 19(p^2-1)/24$ $(i = 1, 2, \ldots, p-1)$ in (4.29) and using (4.27), for $n \geq 0$ and $\alpha \geq 1$,

$$c\Big(p^{2\alpha}n + \frac{(24i+19p)p^{2\alpha-1}-19}{24}\Big) \equiv 0 \pmod 2, \quad i = 1, 2, \ldots, p-1. \tag{4.30}$$

Replacing n by $p^{2\alpha}n + ((24i+19p)p^{2\alpha-1}-19)/24$ $(i = 1, 2, \ldots, p-1)$ in (4.24) and employing (4.30), for $n \geq 0$ and $\alpha \geq 1$,

$$h\Big(8p^{2\alpha}n + \frac{(24i+19p)p^{2\alpha-1}+2}{3}\Big) \equiv 0 \pmod 8, \quad i = 1, 2, \ldots, p-1. \tag{4.31}$$

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Replacing n by $8p^{2\alpha}n + ((24i + 19p)p^{2\alpha-1} + 2)/3$ ($i = 1, 2, \ldots, p-1$) in (3.1) and using (4.31), for $n \geq 0$ and $\alpha \geq 1$,

$$f(24p^{2\alpha}n + (24i + 19p)p^{2\alpha-1}) \equiv 0 \pmod{8}, \quad i = 1, 2, \ldots, p-1. \tag{4.32}$$

By (1.1),

$$f(24p^{2\alpha}n + (24i + 3p)p^{2\alpha-1}) = f(24p^{2\alpha}n + (24i + 11p)p^{2\alpha-1}) = 0. \tag{4.33}$$

Congruence (1.6) follows from (4.32) and (4.33). This completes the proof. □

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