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On spin scale-discretised wavelets on the sphere for the analysis of CMB polarisation

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Abstract. A new spin wavelet transform on the sphere is proposed to analyse the polarisation of the cosmic microwave background (CMB), a spin ± 2 signal observed on the celestial sphere. The scalar directional scale-discretised wavelet transform on the sphere is extended to analyse signals of arbitrary spin. The resulting spin scale-discretised wavelet transform probes the directional intensity of spin signals. A procedure is presented using this new spin wavelet transform to recover E- and B-mode signals from partial-sky observations of CMB polarisation.

Keywords. cosmology: cosmic microwave background, cosmology: observations, cosmology: early universe, methods: data analysis, techniques: image processing

1. Introduction

The polarisation of the cosmic microwave background (CMB) is a powerful probe of the physics of inflation (Spergel & Zaldarriaga 1997) and the reionisation history of the Universe (Zaldarriaga 1997). Numerous experiments have now measured CMB polarisation (some of the more recent include: Hanson *et al.* 2013; Naess *et al.* 2014; BICEP2 Collaboration 2014). Although the Planck satellite also measured CMB polarisation, polarisation data were not included in the Planck 2013 release (Planck Collaboration I 2013) but are anticipated later this year.

Since different physical processes often exhibit different symmetries, their signatures in observables like CMB polarisation may behave differently under a parity transform. CMB polarisation can be separated into parity even and parity odd components, so called E- and B-mode components, respectively (Zaldarriaga & Seljak 1997). Density perturbations in the early Universe provide no mechanism to generate B-mode polarisation in the CMB, whereas gravitational waves can induce both E- and B-mode components. The detection of primordial B-mode polarisation would thus provide evidence for gravitational waves and would provide a powerful probe of the physics of inflation.

In these proceedings we outline a new spin wavelet transform on the sphere to analyse observations of CMB polarisation, a spin ± 2 signal observed on the celestial sphere. In addition, we describe a simple technique based on this wavelet framework to separate

E- and B-mode CMB polarisation components from partial-sky observations. We present a preliminary discussion only; further details of these methods, fast implementations, and a rigorous evaluation of their performance will be given in a series of forthcoming articles.

2. Spin scale-discretised wavelets on the sphere

Scalar wavelets on the sphere (e.g. Antoine & Vandergheynst 1998, 1999; Baldi *et al.* 2009; McEwen *et al.* 2006a, 2007a, 2013; Marinucci *et al.* 2008; Narcowich *et al.* 2006; Starck *et al.* 2006; Wiaux *et al.* 2005, 2006, 2008; Leistedt *et al.* 2013) have proved an effective tool for analysing the temperature anisotropies of the CMB (e.g. Vielva *et al.* 2004; Vielva *et al.* 2006; McEwen *et al.* 2005, 2006b, 2008a, 2006c, 2007b, 2008b; Pietrobon *et al.* 2006; Faÿ *et al.* 2008; Feeney *et al.* 2011a,b; Bobin *et al.* 2013; Planck Collaboration XII 2013; Planck Collaboration XXIII 2013; Planck Collaboration XXIV 2014; Planck Collaboration XXV 2013). For a somewhat dated review see McEwen *et al.* (2007c). Spin wavelets to analyse the polarisation of the CMB have been constructed by Geller *et al.* (2008) and Starck *et al.* (2009). However, a spin wavelet transform on the sphere capable of probing the directional intensity of signals does not yet exist.† We propose such a transform here by extending the directional scale-discretised wavelet transform of Wiaux *et al.* (2008) to signals of arbitrary spin on the sphere.

Spin scale-discretised wavelets ${}_s\Psi^{(j)} \in \mathrm{L}^2(\mathbb{S}^2)$ can be constructed on the sphere $\mathbb{S}^2$ in an analogous manner to the scalar wavelet construction (Wiaux *et al.* 2008; Leistedt *et al.* 2013; McEwen *et al.* 2013), that is simply by defining the spin harmonic coefficients of the wavelets in the factorised form:

$$ {}_s\Psi^{(j)}_{\ell m} \equiv \sqrt{\frac{2\ell+1}{8\pi^2}}\, \kappa^{(j)}(\ell)\, \zeta_{\ell m} \, , \tag{2.1} $$

where ${}_s\Psi^{(j)}_{\ell m} = \langle {}_s\Psi^{(j)},\, {}_sY_{\ell m} \rangle$ are the spin $s \in \mathbb{Z}$ spherical harmonic coefficients of the wavelets, with ${}_sY_{\ell m}$ denoting the spherical harmonic functions and $\ell \in \mathbb{N}_0$, $m \in \mathbb{Z}$, such that $|s| \leqslant \ell$ and $|m| \leqslant \ell$. The *kernel* $\kappa^{(j)} \in \mathrm{L}^2(\mathbb{R}^+)$ controls the angular localisation of the wavelets, while their directional properties are controlled by the *directionality component* $\zeta \in \mathrm{L}^2(\mathbb{S}^2)$, with harmonic coefficients $\zeta_{\ell m} = \langle \zeta,\, Y_{\ell m} \rangle$. The wavelet scale $j \in \mathbb{N}_0$ encodes the angular localisation of $\Psi^{(j)}$. The kernel and directionality component are defined as in the scalar setting (Wiaux *et al.* 2008; McEwen *et al.* 2013).

The wavelet transform of a spin signal ${}_sf \in \mathrm{L}^2(\mathbb{S}^2)$ on the sphere is defined by the directional convolution of ${}_sf$ with the wavelet ${}_s\Psi^{(j)} \in \mathrm{L}^2(\mathbb{S}^2)$. The wavelet coefficients $W^{{}_s\Psi^{(j)}} \in \mathrm{L}^2(\mathrm{SO}(3))$ thus read

$$ W^{{}_s\Psi^{(j)}}(\rho) \equiv ({}_sf \star {}_s\Psi^{(j)})(\rho) \equiv \langle {}_sf,\, \mathcal{R}_\rho\, {}_s\Psi^{(j)} \rangle = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) {}_sf(\omega) (\mathcal{R}_\rho\, {}_s\Psi^{(j)})^*(\omega) \, , \tag{2.2} $$

where $\omega = (\theta, \varphi) \in \mathbb{S}^2$ denotes spherical coordinates with colatitude $\theta \in [0, \pi]$ and longitude $\varphi \in [0, 2\pi)$, $\mathrm{d}\Omega(\omega) = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ is the usual rotation invariant measure on the sphere, and $\cdot^*$ denotes complex conjugation. The rotation operator is defined by

$$ (\mathcal{R}_\rho\, {}_s\Psi^{(j)})\, (\omega) \equiv {}_s\Psi^{(j)}(\mathbf{R}_\rho^{-1} \cdot \omega) \, , \tag{2.3} $$

where $\mathbf{R}_\rho$ is the three-dimensional rotation matrix corresponding to $\mathcal{R}_\rho$. Rotations are

† Spin curvelets (Starck *et al.* 2009) could be used for a directional analysis however these are constructed on the base pixels of Healpix (Górski *et al.* 2005) and so do not live naturally on the sphere.

specified by elements of the rotation group SO(3), parameterised by the Euler angles $\rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3)$, with $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$ and $\gamma \in [0, 2\pi)$. Note that the wavelet coefficients are a scalar signal defined on the rotation group SO(3). The wavelet transform of Eqn. (2.2) thus probes the directional intensity of the signal of interest ${}_s f$.

Provided the wavelets satisfy an admissibility property analogous to the scalar setting, the original signal can be synthesised exactly from its wavelet coefficients by

$$ {}_s f(\omega) = \sum_{j=J_0}^{J} \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W^{{}_s\Psi^j}(\rho) (\mathcal{R}_\rho \, {}_s\Psi^j)(\omega) \, , \tag{2.4} $$

where $\mathrm{d}\varrho(\rho) = \sin\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma$ is the usual invariant measure on SO(3) and J_0 and J are the minimum and maximum wavelet scales considered, respectively, i.e. $J_0 \leqslant j \leqslant J$. Throughout this description we have neglected to include a scaling function, which must be introduced to capture the low-frequency content of the analysed signal ${}_s f$.

3. E- and B-mode separation

CMB experiments measure the scalar Stoke parameters $I, Q, U \in \mathrm{L}^2(\mathbb{S}^2)$, where I encodes the intensity and Q and U the linear polarisation of the incident CMB radiation (the circular polarisation component of the four Stokes parameters $V \in \mathrm{L}^2(\mathbb{S}^2)$ is zero). The linear polarisation signal that is observed depends on the choice of local coordinate frame. The component $Q \pm \mathrm{i}U$ transforms under a rotation of the local coordinate frame by $\chi \in [0, 2\pi)$ as $(Q \pm \mathrm{i}U)'(\omega) = \exp(\mp \mathrm{i}2\chi)(Q \pm \mathrm{i}U)(\omega)$ and is thus a spin ± 2 signal on the sphere (Zaldarriaga & Seljak 1997). The quantity $Q \pm \mathrm{i}U$ can be decomposed into parity even and odd components by $\tilde{E}(\omega) = -\frac{1}{2}\left[\bar{\eth}^2(Q + \mathrm{i}U)(\omega) + \eth^2(Q - \mathrm{i}U)(\omega)\right]$ and $\tilde{B}(\omega) = \frac{\mathrm{i}}{2}\left[\bar{\eth}^2(Q + \mathrm{i}U)(\omega) - \eth^2(Q - \mathrm{i}U)(\omega)\right]$ respectively, where $\tilde{E}, \tilde{B} \in \mathrm{L}^2(\mathbb{S}^2)$ and $\eth$ and $\bar{\eth}$ are spin raising and lowering operators, respectively (Zaldarriaga & Seljak 1997). Recovering E- and B-modes from full-sky observations is relatively straightforward, however in practice we observe the CMB over only part of the sky, since microwave emissions from our Galaxy obscure our view. A number of techniques have been developed to recover E- and B-modes from Q and U maps observed on the partial-sky (e.g. Lewis *et al.* 2002; Bunn *et al.* 2003; Kim 2011; Bowyer *et al.* 2011). Here we propose a simple alternative approach using the spin scale-discretised wavelet transform described above (a similar approach using needlets has been proposed by Geller *et al.* (2008), however there are some minor differences since spin needlets yield spin and not scalar wavelet coefficients).

First, consider the wavelet coefficients of the observable $Q + \mathrm{i}U$ signal computed by a *spin* wavelet transform: $W^{{}_2\Psi^{(j)}}_{Q+\mathrm{i}U}(\rho) \equiv \langle Q + \mathrm{i}U, \, \mathcal{R}_\rho \, {}_2\Psi^{(j)} \rangle$. Second, consider the wavelet coefficients of the unobservable $\tilde{E}$ and $\tilde{B}$ signals computed by a *scalar* wavelet transform: $W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) \equiv \langle \tilde{E}, \, \mathcal{R}_\rho \, {}_0\tilde{\Psi}^j \rangle$ and $W^{{}_0\tilde{\Psi}^j}_{\tilde{B}}(\rho) \equiv \langle \tilde{B}, \, \mathcal{R}_\rho \, {}_0\tilde{\Psi}^j \rangle$. If the wavelet used in the scalar wavelet transform is a spin lowered version of the wavelet used in the spin wavelet transform, i.e. ${}_0\tilde{\Psi}^j = \bar{\eth}^2 \, {}_2\Psi^j$, then the wavelet coefficients of $\tilde{E}$ and $\tilde{B}$ are simply related to the wavelet coefficients of $Q + \mathrm{i}U$ by $W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) = -\mathrm{Re}\left[W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right]$ and $W^{{}_0\tilde{\Psi}^j}_{\tilde{B}}(\rho) = -\mathrm{Im}\left[W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right]$, respectively.

This leads to an elegant procedure to recover E- and B-modes from Q and U maps observed over the partial-sky. Firstly, compute the spin wavelet transform of $Q + \mathrm{i}U$. Secondly, mitigate the impact of the partial sky coverage in wavelet space, where signal content (and thus the influence of the mask) is localised in scale and position simultaneously. Thirdly, reconstruct $\tilde{E}$ and $\tilde{B}$ maps by inverse scalar wavelet transforms of the real and imaginary components, respectively, of the processed spin wavelet coefficients.

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