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ON ODD PERFECT NUMBERS (II), MULTIPERFECT NUMBERS AND QUASIPERFECT NUMBERS

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Abstract

Let N be a positive integer. This paper is concerned with obtaining bounds for $\sum_{p|N} 1/p$ (p prime), when N is an odd perfect number, a multiperfect number, or a quasiperfect number, under assumptions on the existence of such numbers (where none is known) and whether 3 and 5 are divisors. We argue that our new lower bounds in the case of odd perfect numbers are not likely to be significantly improved further. Triperfect numbers are investigated in some detail, and it is shown that an odd triperfect number must have at least nine distinct prime factors.

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1. Introduction

Let $\sigma(N)$ denote as usual the sum of the positive divisors of an integer N. We say N is perfect when $\sigma(N) = 2N$. No odd perfect numbers have been found, but many necessary conditions for their existence have been established. For example, bounds have been obtained for

$$\Sigma \equiv \sum_{p|N} \frac{1}{p},$$

where the sum is over the distinct prime divisors p of N, under the assumption that N is an odd perfect number. Improved bounds can be given under further assumptions on specific prime divisors of N, and it has become usual to consider the four cases dependent on whether 3 and 5 are divisors.

In Cohen (1978), we considered upper bounds for Σ. Prior to the appearance of that paper, the best known bounds were described by Suryanarayana and Hagis (1970), and we repeat their table here (Table 1), giving the results to six decimal places.

In this paper, we shall obtain improved lower bounds for Σ. Incorporating the results in Cohen (1978), the present best bounds (to six decimal places) are given in the first two columns of Table 2, the row order being the same as in Table 1.

TABLE 1

	Lower bound	Upper bound	Range
$3 \nmid N, 5 \mid N$	0.647387	0.678036	0.030649
$3 \nmid N, 5 \nmid N$	0.667450	0.693148	0.025698
$3 \mid N, 5 \mid N$	0.596063	0.673770	0.077707
$3 \mid N, 5 \nmid N$	0.603831	0.657304	0.053473

A number N such that $\sigma(N) = 2N+1$ is called quasiperfect. No such numbers are known, but necessary properties of them are described in detail by Abbott *et al.* (1973). They showed that we must have $N > 10^{20}$ and that N must have at least five distinct prime divisors. Kishore (1978) has since shown that N must have at least six distinct prime divisors. We shall give bounds for Σ when N is a quasiperfect number. The lower bounds are the same as for odd perfect numbers; the upper bounds to six decimal places are given in Table 2, column headed QP.

TABLE 2

Lower bound	Upper bound	Range	Upper bound QP	Range QP
0.647649	0.677637	0.029988	0.670017	0.022368
0.667472	0.693148	0.025676	0.693148	0.025676
0.596063	0.673634	0.077571	0.602009	0.005946
0.604707	0.657304	0.052597	0.625140	0.020433

When $\sigma(N) = kN$, for some integer $k \geqslant 2$, N is more generally called multiperfect. No odd multiperfect numbers are known. We shall give bounds for Σ under the assumption that N is a multiperfect number (even or odd), and in particular will improve a result of Krawczyk (1972) that $\Sigma > (\log k)/2$. In fact, our Theorem 3 will imply that $\Sigma > (\log k)/(2 \log 2)$, and this is clearly the best possible result of its kind, given today's knowledge of even perfect numbers. The case $k = 3$ will be investigated in more detail: interesting restrictions on the divisors of N when N is even will be given (Theorem 4), and better bounds for Σ when N is odd will be obtained. The latter are given to six decimal places in Table 3, the row order being the same as in Table 1. We will show finally that if $\sigma(N) = 3N$ and N is odd, then N has at least nine distinct prime factors.

TABLE 3

Lower bound	Upper bound	Range
1.048742	1.082999	0.034257
1.071445	1.098613	0.027168
0.980982	1.045807	0.064825
1.000806	1.061421	0.060615

2. Notation and statement of results

The following notation is used henceforth.

The letters p, q and r (with and without subscripts) will always denote primes. By $p,\dots,q$ (whether or not p and q are given values), we shall mean, unless we indicate otherwise, all primes and only primes in $[p,q]$. The positive integer N will always have prime factor decomposition

$$N = \prod_{i=1}^{t} p_i^{a_i},$$

where we assume further that $p_1 < \dots < p_t$. The letter k always denotes an integer greater than 1.

We write, for any s primes,

$$R(q_1,\dots,q_s) = \sum_{i=1}^{s} \frac{1}{q_i},$$

and, for any $x \geqslant 2$,

$$L_k(q_1,\dots,q_s;\, x) = -\log\left\{k\prod_{i=1}^{s}(1-q_i^{-1})\right\}\Big/ x\log(1-x^{-1}),$$

$$S_k(q_1,\dots,q_s;\, x) = R(q_1,\dots,q_s)+L_k(q_1,\dots,q_s;\, x).$$

We shall always write Σ for $R(p_1,\dots,p_t)$, the sum of the reciprocals of the prime factors of N.

For any positive integer M, we define

$$\lambda_k(M) = \sum_{p\mid M}\frac{1}{p}+\log\frac{kM}{\sigma(M)} \quad (M>1), \quad \lambda_k(1) = \log k;$$

$$\Pi(M) = \frac{\sigma(M)}{M}.$$

The lower bounds for Σ when N is an odd perfect number or a quasiperfect number are included in

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THEOREM 1. *Suppose* $\sigma(N) \geqslant 2N$. *Then*
(i) *if* $p_1 = 5$, $\Sigma > S_2(5, \ldots, 19, 41, 907; 911)$;
(ii) *if* $p_1 \geqslant 7$, $\Sigma > S_2(7, \ldots, 59, 307; 311)$;
(iii) *if* $p_1 = 3$, $p_2 = 5$, $\Sigma > S_2(3, 5, 17, 257, 65537; 65539)$;
(iv) *if* $p_1 = 3$, $p_2 \geqslant 7$, $\Sigma > S_2(3, 7, 11, 29, 331; 337)$.

We remark that the lower bounds given in Suryanarayana and Hagis (1970) for Σ when N is an odd perfect number are, respectively in the four cases, $S_2(5, \ldots, 19; 23)$ $S_2(7, \ldots, 59; 61)$, $S_2(3, 5, 17; 257)$ and $S_2(3, 7, 11; 13)$. It will also be shown following the proof of this theorem that, until new (non-obvious) restrictions are found for certain prime divisors of N or upper bounds are found for certain exponents a_i in the prime factor decomposition of N, the lower bounds of Theorem 1 cannot be further improved except at best beyond the sixth decimal place.

For completeness, we state here the best known upper bounds for Σ when N is an odd perfect number, and give references for their proofs. These are, in the order of Theorem 1: $\lambda_2(5^2 31^4 61)$ (Cohen (1978)), $\lambda_2(1)$ ($= \log 2$) (Suryanarayana (1963), Cohen (1978)), $\lambda_2(3^2 13^2 61^4 5)$ (Cohen (1978)), and $\lambda_2(3^2 13^4 17)$ (Suryanarayana and Hagis (1970)).

Upper bounds for Σ when N is quasiperfect are given by

THEOREM 2. *Suppose* $\sigma(N) = 2N+1$. *Then*
(i) *if* $p_1 = 5$, $\Sigma < \lambda_2(5^6)$;
(ii) *if* $p_1 \geqslant 7$, $\Sigma < \lambda_2(1) = \log 2$;
(iii) *if* $p_1 = 3$, $p_2 = 5$, $\Sigma < \lambda_2(3^4 5^6)$;
(iv) *if* $p_1 = 3$, $p_2 \geqslant 7$, $\Sigma < \lambda_2(3^4)$.

In Abbott *et al.* (1973) it is shown that no quasiperfect number is divisible by $3 \cdot 5 \cdot p$ for $p = 7$, 11 or 13, or by $3 \cdot 5 \cdot 17 \cdot p$ for $p = 19$, 23, 29 or 31. We improve this in

COROLLARY 1. *No quasiperfact number is divisible by* $3 \cdot 5 \cdot 17 \cdot p$ *for* $p = 19, \ldots, 101$.

It will become clear that further results of this nature are easily obtained. For example, no quasiperfect number is divisible by $3 \cdot 7 \cdot 11 \cdot p$ for $p = 13$ or 17.

For multiperfect numbers, we have

THEOREM 3. *Suppose* $\sigma(N) = kN$. *Then*
(i) *if N is odd,*

$$\frac{\log k}{3 \log 3/2} < \Sigma < \begin{cases} \log k & \textit{if } k \textit{ is odd,} \\ \dfrac{\log k}{3 \log 4/3} & \textit{if } k \textit{ is even;} \end{cases}$$

(ii) *if N is even,*

$$\frac{1}{2}+\frac{\log k/2}{3\log 3/2}<\Sigma<\frac{1}{2}+\frac{\log 2k/3}{3\log 4/3}.$$

We shall concentrate on the case $k=3$ (such numbers then being called triperfect), since then some use can be made of results concerning odd perfect numbers. The next theorem and its corollary give some interesting divisor properties of even triperfect numbers, which (apart from (i) of the Theorem, which is obvious) do not appear to have been noticed previously.

THEOREM 4. *Suppose $\sigma(N)=3N$.*
(i) *If $2\|N$, then $N/2$ is an odd perfect number.*
(ii) *If $2^2\cdot 3\cdot 5\mid N$, then $N=120$.*
(iii) *If $2^3\cdot 3\cdot 7\mid N$, then $N=672$.*

COROLLARY 2. *Suppose $\sigma(N)=3N$.*
(i) *If $2^a\|N$ and $3\mid N$, then $a\not\equiv 3\ (\mathrm{mod}\ 4)$ (except if $N=120$); if $2^a\|N$, then $a\not\equiv 5\ (\mathrm{mod}\ 6)$ (except if $N=672$).*
(ii) *If $3^b\|N$, then $b\not\equiv 3\ (\mathrm{mod}\ 4)$ and $b\not\equiv 5\ (\mathrm{mod}\ 6)$.*
(iii) *If $5^c\|N$ and $2^2\mid N$, then $c\not\equiv 5\ (\mathrm{mod}\ 6)$.*

The following improvement of Theorem 3 for odd triperfect numbers will be proved.

THEOREM 5. *Suppose $\sigma(N)=3N$. Then*
(i) *if $p_1=5$,*

$$S_3(5,\ldots,139;\ 149)<\Sigma<\lambda_3(5^2\,31^2\,331^4);$$

(ii) *if $p_1\geqslant 7$,*

$$S_3(7,\ldots,523;\ 541)<\Sigma<\lambda_3(1)=\log 3;$$

(iii) *if $p_1=3$, $p_2=5$,*

$$\tfrac{1}{3}+S_2(5,\ldots,19,41,907;\ 911)<\Sigma<\lambda_3(3^2\,5^2\,13^2\,31^2\,61^4\,331^4);$$

(iv) *if $p_1=3$, $p_2\geqslant 7$,*

$$\tfrac{1}{3}+S_2(7,\ldots,59,307;\ 311)<\Sigma<\lambda_3(3^2\,13^2\,61^6).$$

Finally, we will prove

THEOREM 6. *If $\sigma(N)=3N$ and N is odd, then N has at least nine distinct prime factors.*

3. The lemmas

The following two lemmas are basic in deriving lower bounds for Σ. The first is a simple generalization of Lemma 1 in Suryanarayana and Hagis (1970), but we prove it here for completeness.

LEMMA 1. *Suppose $\sigma(N)\geqslant kN$. Then*

$$\Sigma > S_k(p_1,\dots,p_s;\,q), \tag{1}$$

where $1\leqslant s\leqslant t-1$, and $p_s<q\leqslant p_{s+1}$.

PROOF. We have

$$k\prod_{i=1}^{t} p_i^{a_i}\leqslant \prod_{i=1}^{t}\frac{p_i^{a_i+1}-1}{p_i-1},$$

and so

$$k\leqslant \prod_{i=1}^{t}\frac{p_i-p_i^{-a_i}}{p_i-1}<\prod_{i=1}^{t}\frac{p_i}{p_i-1}=\prod_{i=1}^{t}\frac{1}{1-p_i^{-1}}.$$

Thus

$$k\prod_{i=1}^{s}(1-p_i^{-1})<\prod_{i=s+1}^{t}(1-p_i^{-1})^{-1}.$$

So, taking logarithms,

$$\begin{aligned}\log\left\{k\prod_{i=1}^{s}(1-p_i^{-1})\right\} &< -\sum_{i=s+1}^{t}\log(1-p_i^{-1})=\sum_{i=s+1}^{t}\sum_{j=1}^{\infty}\frac{1}{jp_i^j}\\ &=\sum_{j=1}^{\infty}\frac{1}{j}\sum_{i=s+1}^{t}\frac{1}{p_i^j}\leqslant\sum_{i=s+1}^{t}\frac{1}{p_i}+\sum_{j=2}^{\infty}\frac{1}{jq^{j-1}}\sum_{i=s+1}^{t}\frac{1}{p_i}\\ &=q\left(\frac{1}{q}+\sum_{j=2}^{\infty}\frac{1}{jq^j}\right)\sum_{i=s+1}^{t}\frac{1}{p_i}\\ &=-q\log(1-q^{-1})\left(\Sigma-\sum_{i=1}^{s}\frac{1}{p_i}\right).\end{aligned}$$

Rearranging this, we obtain (1), as required.

Note that the proof of Lemma 1 is readily adjusted to show that if $\sigma(N)\geqslant kN$ and $q\leqslant p_1$, then

$$\Sigma>-\frac{\log k}{q\log(1-q^{-1})}. \tag{2}$$

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LEMMA 2. *The function f, where*

$$f(x) = x\log(1-x^{-1}),$$

is increasing on $[2,\infty)$.

This is essentially Lemma 2 in Suryanarayana and Hagis (1970), and is easily proved.

It follows that if $r<q$, then

$$\frac{1}{r}-\frac{\log(1-r^{-1})}{q\log(1-q^{-1})}<0. \tag{3}$$

Hence if $r_1,\ldots,r_u$ are any primes less than q and, in Lemma 1, $\{p_1,\ldots,p_s\}$ is a subset of $\{r_1,\ldots,r_u\}$, then

$$S_k(r_1,\ldots,r_u;\, q)\leqslant S_k(p_1,\ldots,p_s;\, q)<\Sigma.$$

This will be a common means of applying (3).

The next two lemmas will be used in obtaining upper bounds for Σ.

LEMMA 3. *Suppose* $\sigma(N) = zN$, $z>1$, *and let* $M<N$ *be a divisor of* N. *If* N *is an odd perfect square, then*

$$\Sigma<\sum_{p\mid M}\frac{1}{p}+\log\frac{zM}{\sigma(M)}-\frac{1}{4}\sum_{p\mid N,\, p\nmid M}\frac{1}{p^2}$$

(*where an empty sum on the right, for example when* $M=1$, *is defined to be zero*).

PROOF. An empty product in this proof is defined to be 1. Put

$$M=\prod_{i=1}^{t}p_i^{b_i}$$

for integers $b_1,\ldots,b_t$ with $0\leqslant b_i\leqslant a_i$ for each i. Then

$$\begin{aligned} z=\frac{\sigma(N)}{N}&=\prod_{i=1}^{t}\left(1+\frac{1}{p_i}+\ldots+\frac{1}{p_i^{a_i}}\right)\\ &\geqslant\prod_{i=1,\, p_i\mid M}^{t}\left(1+\frac{1}{p_i}+\ldots+\frac{1}{p_i^{b_i}}\right)\prod_{i=1,\, p_i\nmid M}^{t}\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}\right), \end{aligned}$$

since N is a perfect square. In Cohen (1978), it is shown that

$$1+x+x^2>\exp(x+\tfrac{1}{4}x^2)\quad \text{if } 0<x\leqslant\tfrac{1}{3},$$

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so

$$z > \frac{\sigma(M)}{M} \prod_{i=1,\, p_i \nmid M}^{t} \exp\left(\frac{1}{p_i} + \frac{1}{4p_i^2}\right).$$

Taking logarithms,

$$\log z > \log \frac{\sigma(M)}{M} + \Sigma - \sum_{p|M} \frac{1}{p} + \frac{1}{4} \sum_{p|N,\, p \nmid M} \frac{1}{p^2},$$

and the result follows.

LEMMA 4. *If* $0 \leqslant x \leqslant \frac{1}{3}$, *then*

$$1 + x \geqslant \exp(3x \log 4/3).$$

PROOF. Put $g(x) = 1 + x - \exp(3x \log 4/3)$. Then $g(0) = g(\frac{1}{3}) = 0$. Since $g''(x) < 0$ for all x, we have $g(x) \geqslant 0$ for $0 \leqslant x \leqslant \frac{1}{3}$, as required.

Our final lemma is used in proving Theorems 4 and 6.

LEMMA 5. *If* $\sigma(N) = kN$ *and* M *is any divisor of* N, *then* $\Pi(M) \leqslant k$.

PROOF. We have $k = \Pi(N) = \Sigma_{d|N}\, 1/d \geqslant \Pi(M)$.

4. Proofs of the theorems

PROOF OF THEOREM 1. (i) Put $S_2(5, \ldots, 19, 41, 907; 911) = A_1$. We will consider a mutually exclusive and exhaustive set of possibilities for the prime factors $p_1, \ldots, p_8$ of N, showing in each case that $\Sigma > A_1$. We are given that $p_1 = 5$.

Suppose $p_2 \geqslant 11$. Using (1) and (3), we have

$$\Sigma > S_2(5, 11, \ldots, r; q)$$

for any $q \geqslant 13$, and where r is the greatest prime less than q. Computations show that the right-hand expression increases with q until $q = 43$ and is again less when $q = 47$. We observe that the numerator in the expression for $L_2(5, 11, \ldots, 43; 47)$ is negative. Thus, using Lemma 2,

$$S_2(5, 11, \ldots, 43; q') < S_2(5, 11, \ldots, 43; 47),$$

for any prime $q' > 47$, so that, by (3),

$$S_2(5, 11, \ldots, r'; q') < S_2(5, 11, \ldots, 43; 47),$$

where r' is the greatest prime less than q'. Hence $S_2(5, 11, \ldots, r; q)$ is greatest when $q = 43$ and $r = 41$. Since $S_2(5, 11, \ldots, 41; 43) > A_1$, we need only consider the possibility $p_2 = 7$.

We now repeat this form of argument, in a much abbreviated but obvious fashion, to show in turn that we need only consider the possibilities

$$p_3 = 11, \ldots, p_6 = 19.$$

$$p_3 \geqslant 13 \Rightarrow \Sigma > \max S_2(5, 7, 13, \ldots, r; q) = S_2(5, 7, 13, \ldots, 31; 37) > A_1.$$

$$p_4 \geqslant 17 \Rightarrow \Sigma > \max S_2(5, 7, 11, 17, \ldots, r; q) = S_2(5, 7, 11, 17, \ldots, 29; 31) > A_1.$$

$$p_5 \geqslant 19 \Rightarrow \Sigma > \max S_2(5, \ldots, 13, 19, \ldots, r; q) = S_2(5, \ldots, 13, 19, 23, 29; 31) > A_1.$$

$$p_6 \geqslant 23 \Rightarrow \Sigma > \max S_2(5, \ldots, 17, 23, \ldots, r; q) = S_2(5, \ldots, 17, 23, 29; 31) > A_1.$$

If p_7 is either of $23, \ldots, 37$, then

$$\Sigma > R(5, \ldots, 19, 37) > A_1;$$

if $p_7 \geqslant 43$, then, by (1),

$$\Sigma > S_2(5, \ldots, 19; 43) > A_1.$$

Hence we need only consider the possibility $p_7 = 41$.

If p_8 is either of $43, \ldots, 887$, then $\Sigma > R(5, \ldots, 19, 41, 887) > A_1$; if $p_8 \geqslant 911$, then, by (1), $\Sigma > S_2(5, \ldots, 19, 41; 911) > A_1$. The only remaining possibility is $p_8 = 907$. In that case, by (1),

$$\Sigma > S_2(5, \ldots, 19, 41, 907; 911) = A_1,$$

and the proof is complete.

(ii) Put $S_2(7, \ldots, 59, 307; 311) = A_2$. The proof is similar to that above, and the scheme used there is used here to show in turn that we need only consider the possibilities $p_1 = 7, \ldots, p_{13} = 53$.

$$p_1 \geqslant 11 \Rightarrow \Sigma > \max S_2(11, \ldots, r; q) = S_2(11, \ldots, 113; 127) > A_2.$$

$$p_2 \geqslant 13 \Rightarrow \Sigma > \max S_2(7, 13, \ldots, r; q) = S_2(7, 13, \ldots, 89; 97) > A_2.$$

$$p_3 \geqslant 17 \Rightarrow \Sigma > \max S_2(7, 11, 17, \ldots, r; q) = S_2(7, 11, 17, \ldots, 79; 83) > A_2.$$

$$p_4 \geqslant 19 \Rightarrow \Sigma > \max S_2(7, 11, 13, 19, \ldots, r; q) = S_2(7, 11, 13, 19, \ldots, 73, 79) > A_2.$$

$$p_5 \geqslant 23 \Rightarrow \Sigma > \max S_2(7, \ldots, 17, 23, \ldots, r; q) = S_2(7, \ldots, 17, 23, \ldots, 71; 73) > A_2.$$

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$$p_6 \geqslant 29 \Rightarrow \Sigma > \max S_2(7, \ldots, 19, 29, \ldots, r; q) = S_2(7, \ldots, 19, 29, \ldots, 71; 73) > A_2.$$

$$p_7 \geqslant 31 \Rightarrow \Sigma > \max S_2(7, \ldots, 23, 31, \ldots, r; q) = S_2(7, \ldots, 23, 31, \ldots, 67; 71) > A_2.$$

$$p_8 \geqslant 37 \Rightarrow \Sigma > \max S_2(7, \ldots, 29, 37, \ldots, r; q) = S_2(7, \ldots, 29, 37, \ldots, 67; 71) > A_2.$$

$$p_9 \geqslant 41 \Rightarrow \Sigma > \max S_2(7, \ldots, 31, 41, \ldots, r; q) = S_2(7, \ldots, 31, 41, \ldots, 61; 67) > A_2.$$

$$p_{10} \geqslant 43 \Rightarrow \Sigma > \max S_2(7, \ldots, 37, 43, \ldots, r; q) = S_2(7, \ldots, 37, 43, \ldots, 61; 67) > A_2.$$

$$p_{11} \geqslant 47 \Rightarrow \Sigma > \max S_2(7, \ldots, 41, 47, \ldots, r; q) = S_2(7, \ldots, 41, 47, \ldots, 61; 67) > A_2.$$

$$p_{12} \geqslant 53 \Rightarrow \Sigma > \max S_2(7, \ldots, 43, 53, \ldots, r; q) = S_2(7, \ldots, 43, 53, 59, 61; 67) > A_2.$$

$$p_{13} \geqslant 59 \Rightarrow \Sigma > \max S_2(7, \ldots, 47, 59, \ldots, r; q) = S_2(7, \ldots, 47, 59, 61; 67) > A_2.$$

If $p_{14} \geqslant 71$, then, by (1),

$$\Sigma > S_2(7, \ldots, 53; 71) > A_2;$$

if $p_{14} = 67$, then, by (1),

$$\Sigma > S_2(7, \ldots, 53, 67; 71) > A_2;$$

if $p_{14} = 61$ and $p_{15} \geqslant 137$, then, by (1),

$$\Sigma > S_2(7, \ldots, 53, 61; 137) > A_2;$$

if $p_{14} = 61$ and p_{15} is either of $67, \ldots, 131$, then

$$\Sigma > R(7, \ldots, 53, 61, 131) > A_2.$$

Hence we need only consider the possibility $p_{14} = 59$.

If p_{15} is either of $61, \ldots, 293$, then $\Sigma > R(7, \ldots, 59, 293) > A_2$; if $p_{15} \geqslant 311$, then, by (1), $\Sigma > S_2(7, \ldots, 59; 311) > A_2$. There remains the possibility $p_{15} = 307$, in which case, by (1),

$$\Sigma > S_2(7, \ldots, 59, 307; 311) = A_2,$$

and we are finished.

(iii) Our improvement over the lower bound in Suryanarayana and Hagis (1970) for this case is in the eighth decimal place only. The proof is similar to those of the other parts of this theorem, and is omitted.

(iv) Put $S_2(3, 7, 11, 29, 331; 337) = A_4$. We are given that $p_1 = 3$. If $p_2 \geqslant 11$, then, by (1), $\Sigma > S_2(3; 11) > A_4$, so we need only consider the possibility $p_2 = 7$.

If $p_3 \geqslant 17$, then, by (1), $\Sigma > S_2(3, 7; 17) > A_4$; if $p_3 = 13$, then, by (1), $\Sigma > S_2(3, 7, 13; 17) > A_4$. Hence we need only consider $p_3 = 11$.

If p_4 is either of $13, \ldots, 23$, then $\Sigma > R(3,7,11,23) > A_4$; if $p_4 \geqslant 37$, then, by (1), $\Sigma > S_2(3,7,11;37) > A_4$; if $p_4 = 31$, then, by (1), $\Sigma > S_2(3,7,11,31;37) > A_4$. Hence we need only consider $p_4 = 29$.

If p_5 is either of $31, \ldots, 317$, then $\Sigma > R(3,7,11,29,317) > A_4$; if $p_5 \geqslant 337$, then, by (1), $\Sigma > S_2(3,7,11,29;337) > A_4$. The only remaining possibility is $p_5 = 331$, in which case, by (1),

$$\Sigma > S_2(3,7,11,29,331;337) = A_4.$$

This completes the proof of Theorem 1.

Since these proofs involve a complete enumeration of possibilities, it follows, in (i) for example, that the lower bound for Σ cannot be increased beyond $S_2(5,\ldots,19,41,907;\infty)$, the limit of the increasing function $S_2(5,\ldots,19,41,907;x)$ as $x \to \infty$, unless it can be shown either that not all of $5,\ldots,19,41,907$ can be divisors of N, or that the exponents on these primes in the prime factor decomposition of N can be restricted in a way to allow an improvement of Lemma 1. We observe that $x\log(1-x^{-1}) \to -1$ as $x \to \infty$; so a calculation of

$$S_2(5,\ldots,19,41,907;\infty),$$

and comparison with $S_2(5,\ldots,19,41,907;911)$, shows that, with the proviso above, our lower bound for Σ in (i) can be improved at best in the eighth decimal place.

With corresponding provisos, calculating $S_2(7,\ldots,59,307;\infty)$ allows an improvement in the seventh decimal place at best for the lower bound in (ii), calculating $S_2(3,5,17,257,65537;\infty)$ allows an improvement in the eighth decimal place at best for the lower bound in (iii), and calculating $S_2(3,7,11,29,331;\infty)$ allows an improvement in the seventh decimal place at best for the lower bound in (iv).

Proof of Theorem 2. The following results from Abbott *et al.* (1973) are required. If N is a quasiperfect number, then N is an odd perfect square, with at least five distinct prime factors, and such that the smallest exponents on 3 and 5, if factors of N, are 4 and 6 respectively.

We prove (ii) first. Since N is an odd perfect square, we may apply Lemma 3, with $z = 2+1/N$ and $M = 1$. We obtain:

$$\Sigma < \log 2\left(1+\frac{1}{2N}\right) - \frac{1}{4}\sum_{p\mid N}\frac{1}{p^2}$$

$$< \log 2 + \frac{1}{2N} - \frac{1}{4p_1^2} < \log 2,$$

since $N > 2p_1^2$.

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(i) We know that $5^6|N$. Applying Lemma 3, with $z = 2+1/N$ and $M = 5^6$, we have

$$\Sigma < \lambda_2(5^6)+\log\left(1+\frac{1}{2N}\right)-\frac{1}{4}\sum_{i=2}^{t}\frac{1}{p_1^2}$$

$$< \lambda_2(5^6)+\frac{1}{2N}-\frac{1}{4p_2^2} < \lambda_2(5^6),$$

since $N > 2p_2^2$.

(iii) As above, since $3^4 5^6|N$ and $N > 2p_3^2$, Lemma 3 implies $\Sigma < \lambda_2(3^4 5^6)$, as required.

(iv) Since $3^4|N$ and $N > 2p_2^2$, we similarly obtain here $\Sigma < \lambda_2(3^4)$, as required.

PROOF OF COROLLARY 1. If N is quasiperfect and divisible by $3\cdot 5\cdot 17\cdot p$, where p is one of $19, \ldots, 101$, then

$$\Sigma > R(3,5,17,101) > \lambda_2(3^4 5^6),$$

contradicting Theorem 2(iii).

PROOF OF THEOREM 3. (i) If k is odd, then $\sigma(N)$ is odd and it is clear that N must be a perfect square. In this case, Lemma 3, with $z = k$ and $M = 1$, gives

$$\Sigma < \log k - \frac{1}{4}\sum_{p|N}\frac{1}{p^2} < \log k.$$

If k is even, then, using Lemma 4,

$$k = \prod_{i=1}^{t}\left(1+\frac{1}{p_i}+\ldots+\frac{1}{p_i^{\alpha_i}}\right) \geqslant \prod_{i=1}^{t}\left(1+\frac{1}{p_i}\right) > \prod_{i=1}^{t}\exp\left(\frac{3}{p_i}\log\frac{4}{3}\right),$$

so

$$\log k > 3\log\tfrac{4}{3}\cdot\Sigma,$$

as required.

The lower bound is given by (2), with $q = 3$.

(ii) Using Lemma 4,

$$k \geqslant \left(1+\frac{1}{2}\right)\prod_{i=2}^{t}\left(1+\frac{1}{p_i}\right) > \frac{3}{2}\prod_{i=2}^{t}\exp\left(\frac{3}{p_i}\log\frac{4}{3}\right).$$

Hence

$$\log k > \log\tfrac{3}{2}+3\log\tfrac{4}{3}\cdot(\Sigma-\tfrac{1}{2}),$$

leading to the required upper bound for Σ.

Also,

$$k < \prod_{i=1}^{t}(1-p_i^{-1})^{-1} = 2\prod_{i=2}^{t}(1-p_i^{-1})^{-1},$$

and it is clear that we may use (2), with $q = 3$ and $k/2$ in place of k, to give

$$\sum_{i=2}^{t}\frac{1}{p_i} > -\frac{\log k/2}{3\log 2/3}.$$

The given lower bound for Σ then follows.

This completes the proof of Theorem 3.

We will verify here the remark in the Introduction that Theorem 3 implies that $\Sigma > (\log k)/(2\log 2)$. When N is odd, this is immediate, since $3\log 3/2 < 2\log 2$. When N is even, we have, since $k \geqslant 2$,

$$\frac{1}{2}+\frac{\log k/2}{3\log 3/2} = \frac{1}{3\log 3/2}\left(1-\frac{\log 32/27}{2\log k}\right)\log k$$

$$\geqslant \frac{1}{3\log 3/2}\left(1-\frac{\log 32/27}{2\log 2}\right)\log k = \frac{\log k}{2\log 2}.$$

Proof of Theorem 4. (i) Put $N = 2M$, so $(2, M) = 1$. Then

$$\sigma(N) = \sigma(2)\,\sigma(M) = 3\sigma(M).$$

But $\sigma(N) = 3N = 6M$, so $\sigma(M) = 2M$.

(ii) We are given that $3 \mid N$ and $5 \mid N$. We cannot have $2^2 \| N$, for then $\sigma(2^2) = 7 \mid N$ and $\Pi(2^2\cdot 3\cdot 5\cdot 7) > 3$, contradicting Lemma 5. Suppose $2^3 \| N$. If both $3 \| N$ and $5 \| N$, then $N = 120M$, with $(120, M) = 1$. We can only have $M = 1$ (whence $N = 120$, a solution of $\sigma(N) = 3N$), since if there were a prime divisor p of M, then $\Pi(2^3\cdot 3\cdot 5\cdot p) = \Pi(2^3\cdot 3\cdot 5)\,\Pi(p) = 3\Pi(p) > 3$, and Lemma 5 is contradicted. We cannot have $2^3 \| N$ and either $3^2 \mid N$ or $5^2 \mid N$, since $\Pi(2^3\,3^2\,5) > \Pi(2^3\,5^2\,3) > 3$, and we cannot have $2^4 \mid N$, since $\Pi(2^4\cdot 3\cdot 5) > 3$. These final assertions complete the proof of (ii).

(iii) If $2^3 \| N$, then $\sigma(2^3) = 15 \mid 3N$, so $5 \mid N$; we are given that $3 \mid N$, so, since we know also that $7 \mid N$, we have a contradiction, by (ii). Also, we cannot have $2^4 \| N$, for then $\sigma(2^4) = 31 \mid N$, and $\Pi(2^4\cdot 3\cdot 7\cdot 31) > 3$, contradicting Lemma 5. Suppose $2^5 \| N$. If both $3 \| N$ and $7 \| N$, then $N = 672$, a solution of $\sigma(N) = 3N$, as in the corresponding part of the proof of (ii). We cannot have $2^5 \| N$ and either $3^2 \mid N$ or $7^2 \mid N$, for $\Pi(2^4\,3^2\,7) > \Pi(2^4\,7^2\,3) > 3$, and we cannot have $2^6 \mid N$, for $\Pi(2^6\cdot 3\cdot 7) > 3$. These assertions complete the proof of (iii), and of Theorem 4.

PROOF OF COROLLARY 2. (i) Suppose $2^a \| N$ and $3|N$, and suppose $a \equiv 3 \pmod 4$. Put $a = 4n-1$ $(n \geqslant 1)$. Then $\sigma(2^a) = 2^{4n}-1 = 16^n-1$, so $16^n-1 | 3N$. But $15 | 16^n-1$, so $5|N$, and, by Theorem 4(ii), we have a contradiction unless $a = 3$ and $N = 120$. Next, suppose $2^a \| N$, where $a \equiv 5 \pmod 6$. Put $a = 6m-1$ $(m \geqslant 1)$. Then $\sigma(2^a) = 2^{6m}-1 = 64^m-1 | 3N$. But $63 | 64^m-1$, so $3|N$ and $7|N$, and, by Theorem 4(iii), we have a contradiction unless $a = 5$ and $N = 672$.

(ii) Suppose $3^b \| N$, where $b \equiv 3 \pmod 4$. Put $b = 4n-1$ $(n \geqslant 1)$. Then

$$\sigma(3^b) = (3^{4n}-1)/2 = (81^n-1)/2 \,|\, 3N.$$

But $80 | 81^n-1$, so $40|N$. Since $b \geqslant 3$, we have a contradiction of Theorem 4(ii). Next, suppose $3^b \| N$, where $b \equiv 5 \pmod 6$. Put $b = 6m-1$ $(m \geqslant 1)$. Then $\sigma(3^b) = (3^{6m}-1)/2 = (729^m-1)/2 | 3N$. But $728 | 729^m-1$, and $728 = 2^3 \cdot 7 \cdot 13$, so $2^2 \cdot 7 \cdot 13 | N$. Since $b \geqslant 5$, we obtain a contradiction via Lemma 5, since $\Pi(2^2 \cdot 3^5 \cdot 7 \cdot 13) > 3$.

(iii) Suppose $5^c \| N$ and $2^2 | N$, and suppose $c \equiv 5 \pmod 6$. Put $c = 6n-1$ $(n \geqslant 1)$. Then $\sigma(5^c) = (5^{6n}-1)/4 = (15625^n-1)/4 | 3N$. But $15624 | 15625^n-1$, and

$$15624 = 2^3 \cdot 3^2 \cdot 7 \cdot 31.$$

Hence $2^2 \cdot 3 \cdot 5^5 | N$, and we have a contradiction, by Theorem 4(ii).

PROOF OF THEOREM 5. We observe, as in the proof of Theorem 3(i), that N must be a perfect square. The upper bounds will be obtained in each part, except (ii), by finding a mutually exclusive and exhaustive set of possibilities for certain divisors of N, and stating, as a consequence of Lemma 3, that $\Sigma < \lambda_3(M)$ for each such divisor M of N.

(i) Put $B_1 = \lambda_3(5^2 31^2 331^4)$. If $5^4 | N$, then $\Sigma < \lambda_3(5^4) < B_1$. If $5^2 \| N$, then $\sigma(5^2) = 31 | N$, so $31^2 | N$. In this case, either $31^4 | N$, and then $\Sigma < \lambda_3(5^2 31^4) < B_1$, or $31^2 \| N$. If $5^2 31^2 \| N$, then $331^2 | N$ since $\sigma(31^2) = 3 \cdot 331$; since $3 | \sigma(331^2)$, but $3 \nmid N$, we must in fact have $331^4 | N$. Then $\Sigma < \lambda_3(5^2 31^2 331^4) = B_1$, and we have obtained the desired upper bound.

For the lower bound, we use (1) and (3):

$$\Sigma > \max S_3(5, \ldots, r; q) = S_3(5, \ldots, 139; 149)$$

(in the same fashion as in the proof of Theorem 1).

(ii) The upper bound is given by Theorem 3(i). For the lower bound, by (1) and (3),

$$\Sigma > \max S_3(7, \ldots, r; q) = S_3(7, \ldots, 523; 541).$$

(iii) Put $B_3 = \lambda_3(3^2\,5^2\,13^2\,31^2\,61^4\,331^4)$. If $3^4|N$, then $\Sigma < \lambda_3(3^4) < B_3$; if $5^4|N$, then $\Sigma < \lambda_3(5^4) < B_3$. Hence we suppose $3^2\,5^2\|N$. Then $13^2\,31^2|N$, since $\sigma(3^2) = 13$, $\sigma(5^2) = 31$. In that case, if $13^4|N$, then $\Sigma < \lambda_3(3^2\,5^2\,13^4\,31^2) < B_3$, and if $31^4|N$, then $\Sigma < \lambda_3(3^2\,5^2\,13^2\,31^4) < B_3$. So we suppose in addition that $13^2\,31^2\|N$. Then $61^2|N$ since $\sigma(13^2) = 3\cdot 61$, and $331^2|N$ since $\sigma(31^2) = 3\cdot 331$. However, we cannot have both $61^2\|N$ and $331^2\|N$ since $3|\sigma(61^2)$ and $3|\sigma(331^2)$, and we would have $3^3|N$. If $61^2\|N$, then $97^2|N$ since $97|\sigma(61^2)$; but $97^2\nparallel N$ since $3|\sigma(97^2)$. Then $97^4|N$, and $\Sigma < \lambda_3(3^2\,5^2\,13^2\,31^2\,61^2\,97^4\,331^4) < B_3$. If $331^2\|N$, then $7^2|N$ since $7|\sigma(331^2)$; but $7^2\nparallel N$ since $3|\sigma(7^2)$. Then $7^4|N$, and $\Sigma < \lambda_3(3^2\,5^2\,7^4) < B_3$. Finally, if $61^4\,331^4|N$, then $\Sigma < \lambda_3(3^2\,5^2\,13^2\,31^2\,61^4\,331^4) = B_3$. This establishes our upper bound.

For the lower bound, we simply observe that

$$L_3(3, \ldots, r;\, x) = L_2(5, \ldots, r;\, x)$$

for any $r \geqslant 5$, and the result follows from Theorem 1(i).

(iv) Put $B_4 = \lambda_3(3^2\,13^2\,61^6)$. If $3^4|N$, then $\Sigma < \lambda_3(3^4) < B_4$. Otherwise, $3^2\|N$, in which case $13^2|N$. If then $13^4|N$, then $\Sigma < \lambda_3(3^2\,13^4) < B_4$, so suppose that $3^2\,13^2\|N$. Then $61^2|N$ since $61|\sigma(13^2)$. We cannot have $61^4\|N$ since $5|\sigma(61^4)$ but $5\nmid N$; if $61^6|N$, then $\Sigma < \lambda_3(3^2\,13^2\,61^6) = B_4$. Hence we suppose also that $61^2\|N$. But then $97^2|N$ since $97|\sigma(61^2)$, and $\Sigma < \lambda_3(3^2\,13^2\,61^2\,97^2) < B_4$, establishing our upper bound.

For the lower bound, the result follows, as in (iii), from Theorem 1(ii).

The proof of Theorem 5 is complete.

Proof of Theorem 6. We must show that $t \geqslant 9$. Certainly, $t \geqslant 8$, for if $t \leqslant 7$, then $\Sigma \leqslant R(3, \ldots, 19) < 0.96$, contradicting Theorem 5. We now suppose $t = 8$, and will show that this also leads to a contradiction.

The primes $3, \ldots, 19$ must divide N, for otherwise $p_7 \geqslant 23$ and

$$\Sigma \leqslant R(3, \ldots, 17, 23, 29) < 0.9809,$$

contradicting Theorem 5. Also, we cannot have $p_8 \geqslant 41$, since $R(3, \ldots, 19, 41) < 0.98$, so $p_8 = 23$, 29, 31 or 37, and we may write

$$N = 3^{2b_1}\,5^{2b_2}\,7^{2b_3}\,11^{2b_4}\,13^{2b_5}\,17^{2b_6}\,19^{2b_7}\,p_8^{2b_8},$$

as N is a perfect square. Straightforward computation shows that if $b_1 > 2$, then in fact $b_1 \geqslant 11$, for each of $\sigma(3^{2i})$, $3 \leqslant i \leqslant 10$, has at least one prime divisor exceeding 37. Similarly, either $b_2 = 1$ or $b_2 \geqslant 8$, either $b_3 = 1$ or $b_3 \geqslant 6$, $b_5 \geqslant 5$, $b_6 \geqslant 5$, $b_7 \geqslant 4$, and $b_8 \geqslant 4$ (for each possible p_8).

We cannot have $b_2 = 1$, since then $p_8 = \sigma(5^2) = 31$, and

$$\Pi(N) < \frac{3}{2}\cdot\frac{5-5^{-2}}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{31}{30} < 3;$$

but of course $\Pi(N) = 3$. Similarly, we cannot have $b_1 = 1$, for then

$$\Pi(N) < \frac{3-3^{-2}}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{23}{22} < 3,$$

so we suppose for now that $b_1 = 2$. Then $p_8 \neq 37$, since

$$\frac{3-3^{-4}}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{37}{36}<3.$$

If $b_3 = 1$, then $p_8 = 23$ or 29, since

$$\frac{3-3^{-4}}{2}\cdot\frac{5}{4}\cdot\frac{7-7^{-2}}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{31}{30}<3.$$

But then $\Pi(3^4\,5^{16}\,7^2\,11^2\,13^{10}\,17^{10}\,19^8\,p_8^8)>3$, contradicting Lemma 5. Hence $b_3 \geqslant 6$ (when $b_1 = 2$). However, this too is impossible, since

$$\Pi(3^4\,5^{16}\,7^{12}\,11^2\,13^{10}\,17^{10}\,19^8\,p_8^8)>3,$$

whether $p_8 = 23$, 29 or 31. Hence the assumption $b_1 = 2$ is untenable.

Thus $b_1 \geqslant 11$. Since $\Pi(3^{22}\,5^{16}\,7^2\,11^2\,13^{10}\,17^{10}\,19^8\,p_8^8)>3$ for $p_8 = 23$, 29 or 31, so $p_8 = 37$. We cannot have $b_3 = 1$, since

$$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7-7^{-2}}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{37}{36}<3,$$

so $b_3 \geqslant 6$. We then have the final contradiction, which proves the theorem: $\Pi(3^{22}\,5^{16}\,7^{12}\,11^2\,13^{10}\,17^{10}\,19^8\,37^8)>3$.

ADDED IN PROOF: Since this paper was prepared for publication, it has come to my notice that Theorem 6 was proved by McDaniel (Wayne McDaniel (1970), 'On odd multiply perfect numbers', *Boll. Un. Mat. Ital.* (4) 3, 185–190). Also relevant is that certain divisor properties of N when $\sigma(N)=3N$ were investigated by Steuerwald (Rudolf Steuerwald (1954), 'Ein Satz über natürliche Zahlen N mit $\sigma(N)=3N$', *Archiv der Math.* **5**, 449–451).

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