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ON PRIME RINGS WITH ASCENDING CHAIN CONDITION ON ANNIHILATOR RIGHT IDEALS AND NONZERO INJECTIVE RIGHT IDEALS

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If I is a right ideal of a ring R, I is said to be an *annihilator* right ideal provided that there is a subset S in R such that

$$I = \{r \in R \mid sr = 0, \quad \forall s \in S\}.$$

I is said to be injective if it is injective as a submodule of the right regular R-module R_R. The purpose of this note is to prove that a prime ring R (not necessarily with 1) which satisfies the ascending chain condition on annihilator right ideals is a simple ring with descending chain condition on one sided ideals if R contains a nonzero right ideal which is injective.

LEMMA 1. *Let M and T be right R-modules such that M is injective and T has zero* singular submodule [4] *and no nonzero injective submodule. Then* $\operatorname{Hom}_R(M, T)=\{0\}$.

Proof. Suppose $f \in \operatorname{Hom}_R(M, T)$ such that $f\neq 0$. Let K be the kernel of f. Then K is a proper submodule of M and there exists $m \in M$ such that $f(m)\neq 0$. Let $(K{:}m)=\{r \in R \mid mr \in K\}$. Since the singular submodule of T is zero and $f(m)(K{:}m)=\{0\}$ the right ideal $(K{:}m)$ has zero intersection with some nonzero right ideal J in R. Then $mJ\neq\{0\}$ and $K \cap mJ=\{0\}$. Let $m\hat{J}$ be the injective hull of mJ. Since M is injective, $m\hat{J}$ is a submodule of M. $m\hat{J} \cap K=\{0\}$ since mJ has nonzero intersection with each submodule which has nonzero intersection with $m\hat{J}$ (See [4, p. 712]). Hence f restricted to $m\hat{J}$ is a monomorphism and $f(m\hat{J})$ is an injective submodule of T. This is a contradiction.

The following lemma is a consequence of [4, Theorem 1.1].

LEMMA 2. *Let R be a prime ring with zero (right) singular ideal. Then there is a prime ring R_u with* 1 *in which R is a two-sided ideal such that R_u is a prime ring with zero singular ideal and every nonzero submodule of R_u, as (right) R-module, has nonzero intersection with R. Furthermore, if I is a nonzero right ideal of R such that I is injective, then I is an annihilator right ideal of R.*

Proof. In view of [4, Theorem 1.1], it needs only to be shown that R_u is a prime ring and I is an annihilator right ideal of R. Let S_1, S_2 be right ideals of R_u such that $S_1S_2=\{0\}$. If $S_i\neq\{0\}$, $i=1, 2$, then $S_i \cap R\neq\{0\}$ for all $i=1, 2$. Since $S_i \cap R$ is a nonzero right ideal in R for each $i=1, 2$, and R is a prime ring, it must be true that either $S_1=\{0\}$ or $S_2=\{0\}$. It is easy to show that if I is an injective right ideal of R then I is an injective right ideal of R_u. Thus there exists a right ideal K in R_u

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such that $R_u = I \oplus K$ by [1, Theorem 1]. Since $1 \in R_u$, there must exist an idempotent $e \in I$ such that $I = eI = eR$. Let $L = R(1-e)$. Since R is a two-sided ideal in R_u, $L \subseteq R$. Let $t \in R$ such that $Lt = \{0\}$. Then $(1-e)t = 0$ since R_u is a prime ring and R is a two-sided ideal in R_u. Thus $t = et$ and $I = \{r \in R \mid lr = 0, \forall l \in L\}$.

THEOREM. *The following two statements are equivalent*:

(a) *R is a simple ring with descending chain condition on right ideals.*

(b) *R is a prime ring with ascending chain condition on annihilator right ideals and R contains a nonzero right ideal which is injective.*

Proof. (a) ⇒ (b). R is certainly a prime ring and R satisfies the ascending chain condition on right ideals by [3, p. 48, Theorem 15]. Furthermore, R is injective by [2, p. 11, Theorem 4.2].

(b) ⇒ (a). Let I_0 be a nonzero right ideal of R such that I_0 is injective. By [5, Lemma 2.1], the singular ideal of R is zero. If $I_0 = R$ then R is an injective R_u-module where R_u is the ring given in Lemma 2. Hence there must exist a R_u-module T in $R_{u_{R_u}}$ such that $R \oplus T = R_u$ by [1, Theorem 1]. T is also an R-module. Hence by Lemma 2, if T were not zero then $T \cap R \neq \{0\}$. Thus $R = R_u$. If $I_0 \neq R$, then there must exist a nonzero right ideal K in R_u such that $R = I_0 \oplus K$. Since, for each $k \in K$, the left multiplication by k is an R_u-homomorphism of I_0 into K and $KI_0 \neq 0$, by Lemma 1 it must be true that K contains a nonzero right ideal K which is injective. Let $I_1 = I_0 \oplus K_1$. Then I_1 is an injective right ideal of R. Inductively we construct the sequences of injective right ideals $\{I_i\}$ and $\{K_{i+1}\}$ such that $I_{k+1} = I_i \oplus K_{i+1}$ for all $i = 0, 1, 2, \ldots$. By Lemma 2, I_i is an annihilator right ideal of R for all $i = 0, 1, 2, \ldots$. Since $I_i \subset I_{i+1}$ for $i = 0, 1, 2, \ldots$ and R satisfies the ascending chain condition on annihilator right ideals, there must exist a positive integer n such that $R = I_n \oplus K_{n+1}$ and K_{n+1} does not contain any nonzero injective right ideal of R. Since in this case $\mathrm{Hom}_{R_u}(I_n, K_{n+1}) = \{0\}$ by Lemma 1, and each element of K_{n+1} determines a homomorphism of I_n into K_{n+1}, $K_{n+1}I_n = \{0\}$. Since R_u is a prime ring, this implies $K_{n+1} = \{0\}$ and $I_n = R = R_u$. Now by [5, Theorem 1] (a) is true.

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