For practical workings with the formula (8) a useful variant is obtained by employing not standard deviations but "variances" V defined by

$$n\sigma^2 = c^2 V,$$

where c is the "class interval." We then have

$$n_1 \sigma_{1x}^2 = c_{1x}^2 V_{1x}, \text{ etc.},$$

$$n_1 r_1 \sigma_{1x} \sigma_{1y} = r_1 c_{1x} c_{1y} \sqrt{(V_{1x} V_{1y})}.$$

The reader may be left to make the substitution, which takes a specially useful form when, as is normally the case, the class intervals for both groups in x, as well as in y, are the same.

The Probability Distribution of a Bridge Hand

By J. B. MARSHALL.

The probability distribution of a bridge hand affords a good example of drawings without replacement from a limited stock.

Let n drawings be made from such a stock. Let p_{rs} and q_{rs} be the probabilities of success and failure after there have been r drawings with s successes, and let the probabilities in successive drawings be connected by the relation

$$p_{rs}\, q_{r+1,\, s+1} = q_{rs}\, p_{r+1,\, s}. \tag{1}$$

[This relation is easily seen to hold in the case of a bridge hand. For if b is the number of cards left in the pack after r drawings, and if a is the number which will give a successful result, then

$$p_{rs} = a/b, \quad p_{r+1,\, s} = a/(b-1), \quad p_{r+1,\, s+1} = (a-1)/(b-1),$$

whence

$$p_{rs}\, q_{r+1,\, s+1} = \frac{a}{b} \times \frac{(b-1)-(a-1)}{b-1}$$

$$= \frac{b-a}{b} \times \frac{a}{b-1}$$

$$= q_{rs}\, p_{r+1,\, s}.]$$

Let us, in the usual manner, construct a generating function (G.F.) by introducing a variable t, the powers of which will enumerate

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the successes, the coefficient being the associated probability. Then for the first drawing the G.F. is

$$p_{00}\,t + q_{00}.$$

For the first two drawings the G.F. is

$$\begin{aligned} &p_{00}\,t\,(p_{11}\,t + q_{11}) + q_{00}\,(p_{10}\,t + q_{10}) \\ &= p_{00}\,p_{11}\,t^2 + 2p_{00}\,q_{11}\,t + q_{00}\,q_{10}, \qquad \text{by (1).} \end{aligned}$$

For three drawings the G.F. is

$$\begin{aligned} &p_{00}\,p_{11}\,t^2\,(p_{22}\,t + q_{22}) + 2p_{00}\,q_{11}\,t\,(p_{21}\,t + q_{21}) + q_{00}\,q_{10}\,(p_{20}\,t + q_{20}) \\ &= p_{00}\,p_{11}\,p_{22}\,t^3 + 3p_{00}\,p_{11}\,q_{22}\,t^2 + 3p_{00}\,q_{11}\,q_{21}\,t + q_{00}\,q_{10}\,q_{20}, \quad \text{by (1).} \end{aligned}$$

The form of the general result now begins to appear, and we can in fact show, by an induction from n to $n+1$ based on (1), that the G.F. for n drawings is

$$\begin{aligned} &p_{00}\,p_{11}\,\ldots\,p_{n-1,\,n-1}\,t^n + np_{00}\,p_{11}\ldots\,p_{n-2,\,n-2}\,q_{n-1,\,n-1}\,t^{n-1} \\ &+ \binom{n}{2}p_{00}\,p_{11}\,\ldots\,p_{n-3,\,n-3}\,q_{n-2,\,n-2}\,q_{n-1,\,n-2}\,t^{n-2} + \ldots + q_{00}\,q_{10}\ldots\,q_{n-1,\,0}. \end{aligned}$$

The "moment" G.F. is obtained, as usual, by putting $t = e^{a}$ in the above and expanding in powers of a. The first or constant term in this expansion will be

$$p_{00}\,p_{11}\,\ldots\,p_{n-1,\,n-1} + np_{00}\,p_{11}\,\ldots\,p_{n-2,\,n-2}\,q_{n-1,\,n-1} + \ldots,$$

which from the meaning of a moment G.F. must equal unity, as can also be proved by means of relation (1). The coefficient of a is

$$np_{00}\,[p_{11}\,\ldots\,p_{n-1,\,n-1} + (n-1)\,p_{11}\,\ldots\,p_{n-2,\,n-2}\,q_{n-1,\,n-1} + \ldots].$$

The part within the bracket here is of exactly the same form as the expression given above for the first term, with an initial probability of p_{11} instead of p_{00}, and correspondingly $n-1$ for n. Hence it also must equal unity, and so the mean of the distribution under view is np_{00}.

The coefficient of $a^2/2!$, which gives the second moment or mean square, is

$$\begin{aligned} &np_{00}\,[np_{11}\,\ldots\,p_{n-1,\,n-1} + (n-1)^2\,p_{11}\,\ldots\,p_{n-2,\,n-2}\,q_{n-1,\,n-1} \\ &+ \binom{n-1}{2}(n-2)\,p_{11}\ldots p_{n-3,\,n-3}\,q_{n-2,\,n-2}\,q_{n-1,\,n-2} + (n-1)\,2p_{11}\,q_{22}\ldots q_{n-1,\,2} \\ &+ q_{11}\,\ldots\,q_{n-1,\,1}]. \end{aligned}$$

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The part within the bracket may be written

$$\{1 \quad + (n-1) \quad\} p_{11} p_{22} \ldots p_{n-1, n-1}$$
$$+ \{(n-1) + (n-1)(n-2)\} p_{11} \ldots p_{n-2, n-2} q_{n-1, n-1}$$
$$+ \left\{\binom{n-1}{2} + (n-1)\binom{n-2}{2}\right\} p_{11} \ldots p_{n-3, n-3} q_{n-2, n-2} q_{n-1, n-2}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \{(n-1) + (n-1)\} p_{11} q_{22} \ldots q_{n-1, 2}$$
$$+ \{1 \quad + \quad 0\} q_{11} \ldots q_{n-1, 1}.$$

Adding vertically we get two expressions, exactly similar to the first term, the second expression being multiplied by $(n-1) p_{11}$. Hence the part within the brackets is equal to $1 + (n-1) p_{11}$, and so the coefficient of $\alpha^2/2!$ in the moment G.F. is

$$np_{00} [1 + (n-1) p_{11}].$$

Transferring to the mean as origin by subtracting $(np_{00})^2$, we derive the second moment μ_2 about the mean, or squared standard deviation σ^2, as

$$\sigma^2 = np_{00} q_{00} - n(n-1) p_{00} (p_{00} - p_{11}).$$

Thus the mean and standard deviation of this particular distribution are given in simple terms.

Example. For the distribution of one suit in a bridge hand, we have

$$p_{00} = \frac{1}{4}, \quad q_{00} = \frac{3}{4}, \quad n = 13, \quad p_{11} = \frac{12}{51} = \frac{4}{17}.$$

Hence $\mu_2 = \sigma^2 = \frac{39}{16} - \frac{39}{68}$

$$= \frac{507}{272} = 1{\cdot}86.$$

and so $\sigma = 1{\cdot}36.$

A Problem in Combinations

By A. C. Aitken.

1. If there are n individuals $A_1, A_2, \ldots, A_n$, in how many ways can they be put into groups? For example, if there are three individuals A, B, C, they may be grouped as

$A + B + C$; $A + (B + C)$, $B + (C + A)$, $C + (A + B)$; $(A + B + C)$,

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