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Symbolic dynamics for Hénon maps near the boundary of the horseshoe locus

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Abstract. Bedford and Smillie [A symbolic characterization of the horseshoe locus in the Hénon family. *Ergod. Th. & Dynam. Sys.* **37**(5) (2017), 1389–1412] classified the dynamics of the Hénon map $f_{a,b} : (x, y) \mapsto (x^2 - a - by, x)$ defined on $\mathbb{R}^2$ in terms of a symbolic dynamics when (a, b) is close to the boundary of the horseshoe locus. The purpose of the current article is to generalize their results for all $b \neq 0$ (including the case $b < 0$ as well). The method of the proof is first to regard $f_{a,b}$ as a complex dynamical system in $\mathbb{C}^2$ and second to introduce the new Markov-like partition in $\mathbb{R}^2$ constructed by us [On parameter loci of the Hénon family. *Comm. Math. Phys.* **361**(2) (2018), 343–414].

Contents

1. *Introduction and the main results*

Consider the Hénon family

$$f_{a,b} : (x, y) \longmapsto (x^2 - a - by, x)$$

defined on $\mathbb{R}^2$, where $(a, b) \in \mathbb{R} \times \mathbb{R}^\times$. Let $\Omega(f_{a,b})$ be the non-wandering set of $f_{a,b}$. We say that $f_{a,b}$ is a *hyperbolic horseshoe on* $\mathbb{R}^2$ if $f_{a,b} : \Omega(f_{a,b}) \to \Omega(f_{a,b})$ is topologically conjugate to the full shift with two symbols and $\Omega(f_{a,b})$ is a hyperbolic set for $f_{a,b}$. Denote by $h_{\mathrm{top}}(f_{a,b})$ the topological entropy of $f_{a,b} : \Omega(f_{a,b}) \to \Omega(f_{a,b})$. We know that $0 \le h_{\mathrm{top}}(f_{a,b}) \le \log 2$ for any $(a, b) \in \mathbb{R} \times \mathbb{R}^\times$. Note also that if $f_{a,b}$ is a hyperbolic horseshoe, then $f_{a,b}$ attains the maximal entropy on $\mathbb{R}^2$, that is, $h_{\mathrm{top}}(f_{a,b}) = \log 2$.

In [**AI**], we have defined the *hyperbolic horseshoe locus* as

$$\mathcal{H}_{\mathbb{R}} \equiv \{(a, b) \in \mathbb{R} \times \mathbb{R}^\times : f_{a,b} \text{ is a hyperbolic horseshoe on } \mathbb{R}^2\}$$

and the *maximal entropy locus* as

$$\mathcal{M}_{\mathbb{R}} \equiv \{(a, b) \in \mathbb{R} \times \mathbb{R}^\times : f_{a,b} \text{ attains the maximal entropy on } \mathbb{R}^2\}.$$

Let us set $\mathcal{H}_{\mathbb{R}}^{\pm} \equiv \mathcal{H}_{\mathbb{R}} \cap \{\pm b > 0\}$. Based on previous work [**BS$_{\mathbb{R}}$2**], it has been shown in [**AI**] that there exists a real analytic function $a_{\mathrm{tgc}} : \mathbb{R}^\times \to \mathbb{R}$ from the b-axis to the a-axis of the parameter space $\mathbb{R} \times \mathbb{R}^\times$ for the Hénon family $f_{a,b}$ with $\lim_{b\to 0} a_{\mathrm{tgc}}(b) = 2$ so that:

- $(a, b) \in \mathcal{H}_{\mathbb{R}}$ if and only if $a > a_{\mathrm{tgc}}(b)$;
- $(a, b) \in \mathcal{M}_{\mathbb{R}}$ if and only if $a \ge a_{\mathrm{tgc}}(b)$.

Moreover, it has been shown that:

- when $(a, b) \in \partial\mathcal{H}_{\mathbb{R}}^{+}$ holds (that is, $a = a_{\mathrm{tgc}}(b)$ and $b > 0$), $f_{a,b}$ has exactly one orbit of homoclinic tangencies of $W^s(p_1)$ and $W^u(p_1)$;
- when $(a, b) \in \partial\mathcal{H}_{\mathbb{R}}^{-}$ holds (that is, $a = a_{\mathrm{tgc}}(b)$ and $b < 0$), $f_{a,b}$ has exactly one orbit of heteroclinic tangencies of $W^s(p_1)$ and $W^u(p_3)$,

where p_1 (respectively p_3) is the unique saddle fixed point in the first (respectively third) quadrant. These results extend the previous assertions in [**BS$_{\mathbb{R}}$2**] for the case $|b| < 0.05$ to all $b \neq 0$. We note that their proofs (and the ones in [**AI**] as well) rely on the profound theory of *quasi-hyperbolicity* [**BS$_{\mathbb{C}}$8**] combined with a detailed analysis of Hénon maps with maximal entropy [**BS$_{\mathbb{R}}$1**].

In their subsequent paper [**BS$_{\mathbb{R}}$3**], Bedford and Smillie classified the dynamics of $f_{a,b}$ in terms of a symbolic dynamics when (a, b) is close to the boundary $\partial\mathcal{H}_{\mathbb{R}}^{+}$. However, their result holds only for $0 < b < 0.4$ approximately because their construction is based on Yoccoz puzzle pieces for the complex one-dimensional map $p(z) = z^2 - 1$ (see the last paragraph of [**AI**, Appendix B]).

The purpose of the current article is to generalize the results in [**BS$_{\mathbb{R}}$3**] for all $b \neq 0$ (including the case $b < 0$ as well) by applying the new Markov-like partition constructed in [**AI**]. To present our results, let us first recall that we have defined in [**AI**] a neighborhood $\mathcal{F}_{\mathbb{R}}^{\pm}$ of $\partial\mathcal{H}_{\mathbb{R}}^{\pm}$ so that $f_{a,b}$ satisfies the crossed mapping condition with respect to a certain family of projective polydisks $\{\mathcal{B}_i^{\pm}\}_{i\in\Sigma^{\pm}}$ for $(a, b) \in \mathcal{F}_{\mathbb{R}}^{\pm}$ (see [**AI**, Theorem 2.12(iii)]). Moreover, their real sections $\mathcal{B}_i^{\pm} \cap \mathbb{R}^2$ cover the set $K_{\mathbb{R}}$ of points in $\mathbb{R}^2$ whose both forward

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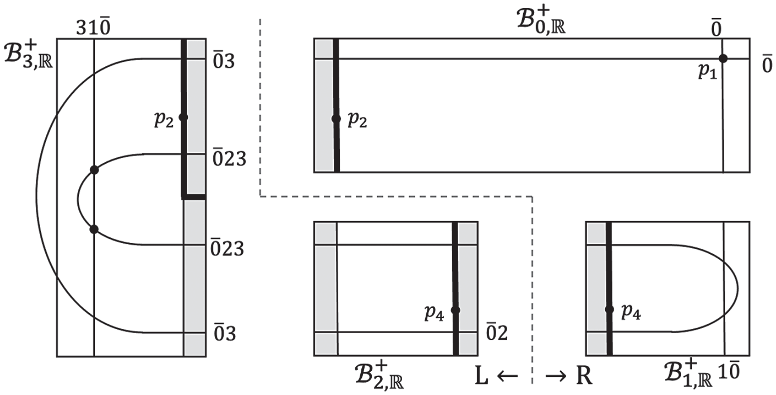

FIGURE 1. $\{L, R\}$-partition for $(a, b) \in \mathcal{E}^+_{\mathbb{R}}$.

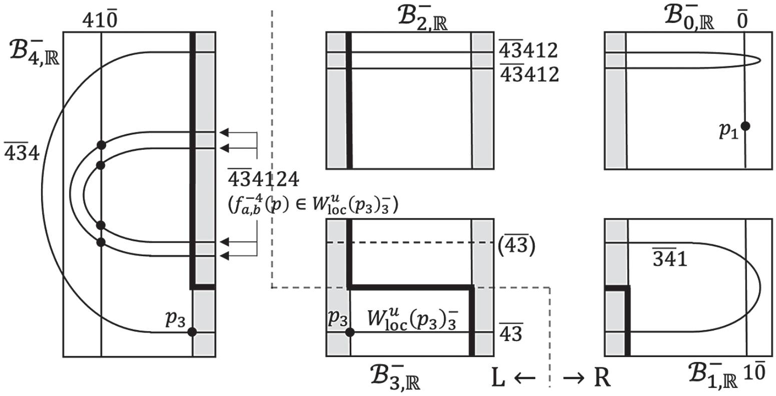

FIGURE 2. $\{L, R\}$-partition for $(a, b) \in \mathcal{E}^-_{\mathbb{R}}$.

and the backward orbits by $f_{a,b}$ are bounded. Let $a^{\pm}_{\mathrm{aprx}}$ be the piecewise linear function approximating $\partial\mathcal{H}^{\pm}_{\mathbb{R}}$ as in [**AI**] and set

$$\mathcal{E}^{\pm}_{\mathbb{R}} \equiv \{(a, b) \in \mathbb{R} \times \{\pm b > 0\} : |a - a^{\pm}_{\mathrm{aprx}}(b)| \leq 0.05\}.$$

One can easily see that $\mathcal{E}^{\pm}_{\mathbb{R}} \subset \mathcal{F}^{\pm}_{\mathbb{R}}$. Moreover, in Appendix A, we show that $\mathcal{E}^{\pm}_{\mathbb{R}}$ forms a neighborhood of $\partial\mathcal{H}^{\pm}_{\mathbb{R}}$ (see Proposition A.1).

Let $p_i \in \mathbb{R}^2$ be the unique saddle fixed point for $i = 1, 3$ and the unique saddle periodic point of period two for $i = 2, 4$ in the ith quadrant. In Theorem 4.3 (respectively Theorem 4.6), we obtain a partition of $K_{\mathbb{R}} \setminus (W^s(p_2) \cup W^s(p_4))$ (respectively $K_{\mathbb{R}} \setminus W^s(p_3)$) into two parts, say, the left part and the right part for $(a, b) \in \mathcal{E}^+_{\mathbb{R}}$ (respectively for $(a, b) \in \mathcal{E}^-_{\mathbb{R}}$). Figures 1 and 2 describe the shape of the partition pieces. This partition defines a coding with the alphabet $\{L, R\}$ by assigning L to the left part and R to the right part. (Our coding is different from that in [**BS$_{\mathbb{R}}$3**] when $b > 0$ is close to 0.) As pointed out in [**BS$_{\mathbb{R}}$3**], this coding has both advantages and disadvantages; an advantage is that

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it applies to Hénon maps degenerated from the horseshoe and a disadvantage is that the associated coding map is no more one-to-one.

Given a word w over an alphabet, $\overline{w}$ denotes either the left-infinite repetition $\cdots www$ or the right-infinite repetition $www\cdots$ (depending on the context). Let $\cdot$ be the 'decimal point' of a bi-infinite symbol sequence $\cdots \varepsilon_{-1} \cdot \varepsilon_0\varepsilon_1 \cdots$. Our first main result concerns [**BS$_{\mathbb{R}}$3**, Theorem 1].

THEOREM 1.1. *We have the following.*

(1) *When $(a, b) \in \mathcal{E}^+_{\mathbb{R}}$, there are at most two points in $K_{\mathbb{R}} \setminus (W^s(p_2) \cup W^s(p_4))$ with $\{L, R\}$-coding $\overline{R}L \cdot L\overline{R} \in \{L, R\}^{\mathbb{Z}}$ counted without multiplicity. Moreover:*
  - (a) *$f_{a,b}$ is a hyperbolic horseshoe on $\mathbb{R}^2$ if and only if there are two such points;*
  - (b) *$f_{a,b}$ has a quadratic tangency but $h_{\mathrm{top}}(f_{a,b}) = \log 2$ if and only if there is one such point;*
  - (c) *$h_{\mathrm{top}}(f_{a,b}) < \log 2$ if and only if there are no such points.*

(2) *When $(a, b) \in \mathcal{E}^-_{\mathbb{R}}$, there are at most four points in $K_{\mathbb{R}} \setminus W^s(p_3)$ with $\{L, R\}$-coding $\overline{L}RR \cdot L\overline{R} \in \{L, R\}^{\mathbb{Z}}$ counted without multiplicity. Moreover:*
  - (a) *$f_{a,b}$ is a hyperbolic horseshoe on $\mathbb{R}^2$ if and only if there are four such points;*
  - (b) *$f_{a,b}$ has a quadratic tangency but $h_{\mathrm{top}}(f_{a,b}) = \log 2$ if and only if there are three such points;*
  - (c) *$h_{\mathrm{top}}(f_{a,b}) < \log 2$ if and only if there are at most two such points.*

Obviously the map in either case (a) or (b) of Theorem 1.1 cannot be topologically conjugate to the map in case (c) because they have different values of topological entropy. M. Asaoka (private communication) pointed out that the map in case (b) is not expansive on its non-wandering set. Hence, the map in case (a) is not topologically conjugate to the map in case (b), that is, the map $f_{a,b}$ with $(a, b) \in \mathcal{H}^{\pm}_{\mathbb{R}}$ cannot be topologically conjugate to the map $f_{a,b}$ with $(a, b) \in \partial\mathcal{H}^{\pm}_{\mathbb{R}}$.

We found that the paper [**BS$_{\mathbb{R}}$3**] does not contain a proof of its theorem 1 (which corresponds to our Theorem 1.1). Since our Theorem 1.1 originates from [**BS$_{\mathbb{R}}$3**, Theorem 1], we describe the proof of Theorem 1.1 in detail in §4.2. We also remark that the paper [**BS$_{\mathbb{R}}$3**] did not treat the case $b < 0$ since their partition cannot directly apply to this case. Indeed, it can be shown that the number of points in $W^s(p_1) \cap W^u(p_3)$ having an appropriate $\{L, R\}$-coding with respect to the partition in [**BS$_{\mathbb{R}}$3**] is infinite and cannot provide a classification like [**BS$_{\mathbb{R}}$3** Theorem 1] (see Remark 4.10).

Next we analyze case (b) of Theorem 1.1 in detail. Denote by $\sigma : \{\alpha, \beta\}^{\mathbb{Z}} \to \{\alpha, \beta\}^{\mathbb{Z}}$ the shift map on the space of bi-infinite sequences with two symbols. Our second main result concerns [**BS$_{\mathbb{R}}$3**, Theorem 2].

THEOREM 1.2 *We have the following.*

(1) *If $(a, b) \in \partial\mathcal{H}^+_{\mathbb{R}}$, then $f_{a,b} : \Omega(f_{a,b}) \to \Omega(f_{a,b})$ is topologically conjugate to the factor map $\sigma/_{\sim} : \{\alpha, \beta\}^{\mathbb{Z}}/_{\sim} \to \{\alpha, \beta\}^{\mathbb{Z}}/_{\sim}$, where we define $\sim$ as*

$$\sigma^n(\overline{\alpha}\beta \cdot \beta\beta\overline{\alpha}) \sim \sigma^n(\overline{\alpha}\beta \cdot \alpha\beta\overline{\alpha})$$

for all $n \in \mathbb{Z}$.

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(2) *If* $(a, b) \in \partial\mathcal{H}_{\mathbb{R}}^-$, *then* $f_{a,b} : \Omega(f_{a,b}) \to \Omega(f_{a,b})$ *is topologically conjugate to the factor map* $\sigma/_\sim : \{\alpha, \beta\}^{\mathbb{Z}}/_\sim \to \{\alpha, \beta\}^{\mathbb{Z}}/_\sim$, *where we define* $\sim$ *as*

$$\sigma^n(\overline{\beta}\alpha \cdot \beta\beta\overline{\alpha}) \sim \sigma^n(\overline{\beta}\alpha \cdot \alpha\beta\overline{\alpha})$$

for all $n \in \mathbb{Z}$.

Note that, in [BS$_{\mathbb{R}}$3], the alphabet $\{a, b\}$ is used instead of $\{\alpha, \beta\}$.

It would be interesting to generalize our results to:

(1) a real Hénon map on $\mathbb{R}^2$, where (a, b) is taken near the boundary of a hyperbolic component (see [A1] for hyperbolic components in the real parameter space);

(2) a complex Hénon map on $\mathbb{C}^2$, where (a, b) is taken near the boundary of the *complex* horseshoe locus (compare [BD, Theorem 3.6] for a related result which claims that a topological horseshoe is a hyperbolic horseshoe in $\mathbb{C}^2$).

The structure of this paper is as follows. In §2, we discuss some properties of symbolic dynamics associated with a family of boxes as well as its refinement. In §3, we characterize local stable/unstable manifolds in terms of the symbolic dynamics above. In §4, we construct the $\{L, R\}$-partition and prove Theorem 1.1. In §5, we construct the $\{\alpha, \beta\}$-partition and prove Theorem 1.2. Some statements in this article are proved by Arai [A2] with the help of computer assistance (see Appendix A for details).

2. *Symbolic codings and refinements*

In [AI, §2.1], we constructed a complex neighborhood $\mathcal{F}^\pm$ of the boundary $\partial\mathcal{H}_{\mathbb{R}}^\pm$. When $(a, b) \in \mathcal{F}^\pm \cap \{b \neq 0\}$, we can regard $f_{a,b}$ as a complex dynamical system defined on $\mathbb{C}^2$. Denote by $K_{a,b}$ the set of points in $\mathbb{C}^2$ whose both forward and backward orbits by $f_{a,b}$ are bounded in $\mathbb{C}^2$.

Let us write $\mathcal{F}_{\mathbb{R}}^\pm \equiv \mathcal{F}^\pm \cap \mathbb{R}^2$. When $(a, b) \in \mathcal{F}_{\mathbb{R}}^\pm$ is a real parameter, the restriction $f_{a,b}|_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ is well defined. We denote it by $f_{\mathbb{R}}$ when we insist it has real dynamics and write $K_{\mathbb{R}} \equiv K_{a,b} \cap \mathbb{R}^2$.

2.1. *Symbolic codings.*

Given a finite set called an *alphabet* Σ and a subset $\mathfrak{T} \subset \Sigma \times \Sigma$ called the set of *allowed transitions*, we define

$$\mathfrak{S}_{\mathrm{fwd}}(\mathfrak{T}) \equiv \{(i_n)_{n\geq 0} \in \Sigma^{\mathbb{N}} : (i_n, i_{n+1}) \in \mathfrak{T} \text{ for all } n \geq 0\}$$

and call its element a *forward admissible sequence* with respect to $\mathfrak{T}$. Also we define

$$\mathfrak{S}_{\mathrm{bwd}}(\mathfrak{T}) \equiv \{(i_n)_{n\leq 0} \in \Sigma^{-\mathbb{N}} : (i_{n-1}, i_n) \in \mathfrak{T} \text{ for all } n \leq 0\}$$

and call its element a *backward admissible sequence* with respect to $\mathfrak{T}$. Finally, we set

$$\mathfrak{S}(\mathfrak{T}) \equiv \{(i_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}} : (i_n, i_{n+1}) \in \mathfrak{T} \text{ for all } n \in \mathbb{Z}\}$$

and call its element a *bi-infinite admissible sequence* with respect to $\mathfrak{T}$.

Below, we set $\Sigma^+ \equiv \{0, 1, 2, 3\}$ and $\Sigma^- \equiv \{0, 1, 2, 3, 4\}$. Choose a subset of $\Sigma^+ \times \Sigma^+$ as

$$\mathfrak{T}^+ \equiv \{(0, 0), (0, 2), (0, 3), (1, 0), (2, 2), (2, 3), (3, 1)\}$$

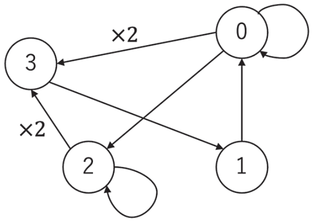

FIGURE 3. Transition diagram for $(a, b) \in \mathcal{F}_{\mathbb{R}}^{+} \cap \{b > 0\}$.

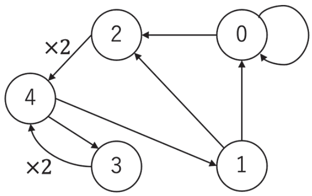

FIGURE 4. Transition diagram for $(a, b) \in \mathcal{F}_{\mathbb{R}}^{-} \cap \{b < 0\}$.

and a subset of $\Sigma^{-} \times \Sigma^{-}$ as

$$\mathfrak{T}^{-} \equiv \{(0, 0), (0, 2), (1, 0), (1, 2), (2, 4), (3, 4), (4, 1), (4, 3)\}.$$

We then write $\mathfrak{S}_{\mathrm{fwd}}^{\pm} \equiv \mathfrak{S}_{\mathrm{fwd}}(\mathfrak{T}^{\pm})$, $\mathfrak{S}_{\mathrm{bwd}}^{\pm} \equiv \mathfrak{S}_{\mathrm{bwd}}(\mathfrak{T}^{\pm})$ and $\mathfrak{S}^{\pm} \equiv \mathfrak{S}(\mathfrak{T}^{\pm})$.

The *transition diagram* for $\mathfrak{S}^{+}$ (respectively $\mathfrak{S}^{-}$) is a directed graph whose vertex set is Σ^{+} (respectively Σ^{-}) and the arrow set is $\mathfrak{T}^{+}$ (respectively $\mathfrak{T}^{-}$). See Figures 3 and 4.

Let D_u and D_v be two topological disks in $\mathbb{C}$. A *projective polydisk* (or a *projective box*, or simply a *box*) $\mathcal{B} = D_u \times_{\mathrm{pr}} D_v$ is the product sets of D_u and D_v with respect to certain projective coordinates in $\mathbb{C}^2$ (see [**I**, §4.3] as well as [**AI**, §2.2] for more detail).

Definition 2.1. Let $\mathcal{B} = D_u \times_{\mathrm{pr}} D_v$ (respectively $\mathcal{B}' = D'_u \times_{\mathrm{pr}} D'_v$) be a projective polydisk, and let $\pi_u : \mathcal{B} \to D_u$ and $\pi_v : \mathcal{B} \to D_v$ (respectively $\pi'_u : \mathcal{B}' \to D'_u$ and $\pi'_v : \mathcal{B}' \to D'_v$) be the projections. We say that $f : \mathcal{B} \cap f^{-1}(\mathcal{B}') \to \mathcal{B}'$ satisfies the crossed mapping condition (CMC) of degree d if it satisfies conditions (1) and (2) below, where $\iota : \mathcal{B} \cap f^{-1}(\mathcal{B}') \to \mathcal{B}$ denotes the inclusion map.

(1) The map $(\pi'_u \circ f, \pi_v \circ \iota) : \mathcal{B} \cap f^{-1}(\mathcal{B}') \to D'_u \times D_v$ is proper and degree d.

(2) The sets $\pi_u(\mathcal{B} \cap f^{-1}(\mathcal{B}'))$ and $\pi'_v(\mathcal{B}' \cap f(\mathcal{B}))$ are relatively compact in D_u and D'_v, respectively.

See [**AI**, Definition 2.11] as well as [**IS**, Definition 5.1] for the original definition.

Definition 2.2. A triple $(f_{a,b}, \{\mathcal{B}_i^{\pm}\}_{i \in \Sigma^{\pm}}, \mathfrak{T}^{\pm})$ is said to satisfy the *crossed mapping condition* if $f_{a,b} : \mathcal{B}_i^{\pm} \cap f_{a,b}^{-1}(\mathcal{B}_j^{\pm}) \to \mathcal{B}_j^{\pm}$ satisfies the crossed mapping condition for all $(i, j) \in \mathfrak{T}^{\pm}$.

The next proposition is identical to [**AI**, Theorem 2.12(iii)].

THEOREM 2.3. *For* $(a, b) \in \mathcal{F}^{\pm} \cap \{b \neq 0\}$*, there is a family of projective polydisks* $\{\mathcal{B}_i^{\pm}\}_{i\in\Sigma^{\pm}}$ *in* $\mathbb{C}^2$ *so that the triple* $(f_{a,b}, \{\mathcal{B}_i^{\pm}\}_{i\in\Sigma^{\pm}}, \mathfrak{T}^{\pm})$ *satisfies the crossed mapping condition.*

The next fact is proved in [AI] and will be used later.

LEMMA 2.4. *We have:*

(1) $\mathcal{B}_0^+ \cap \mathcal{B}_1^+ \cap K_{a,b} = \mathcal{B}_0^+ \cap \mathcal{B}_2^+ \cap K_{a,b} = \mathcal{B}_1^+ \cap \mathcal{B}_3^+ \cap K_{a,b} = \emptyset$ *for* $(a, b) \in \mathcal{F}^+ \cap \{b > 0\}$*;*

(2) $\mathcal{B}_0^- \cap \mathcal{B}_1^- \cap K_{a,b} = \mathcal{B}_0^- \cap \mathcal{B}_3^- \cap K_{a,b} = \mathcal{B}_0^- \cap \mathcal{B}_4^- \cap K_{a,b} = \mathcal{B}_1^- \cap \mathcal{B}_2^- \cap K_{a,b} = \mathcal{B}_1^- \cap \mathcal{B}_4^- \cap K_{a,b} = \mathcal{B}_2^- \cap \mathcal{B}_3^- \cap K_{a,b} = \emptyset$ *for* $(a, b) \in \mathcal{F}^- \cap \{b < 0\}$.

Proof. For claim (1), see the proof of [AI, Lemma 3.4(i)]. For claim (2), see the proof of [AI, Lemma 3.7(i)]. □

For $(a, b) \in \mathcal{F}^{\pm} \cap \{b \neq 0\}$, define the *orbit space* of $f_{a,b}$ as

$$\Lambda^{\pm} \equiv \{((i_n)_{n\in\mathbb{Z}}, (z_n)_{n\in\mathbb{Z}}) : (i_n)_{n\in\mathbb{Z}} \in \mathfrak{S}^{\pm},\ z_n \in \mathcal{B}_{i_n}^{\pm},\ f(z_n) = z_{n+1}\}.$$

By [AI, Proposition 3.1], we see that

$$\bigcap_{n\in\mathbb{Z}} f^n\bigg(\bigcup_{i\in\Sigma^{\pm}} \mathcal{B}_i^{\pm}\bigg) = K_{a,b}.$$

Therefore, the projection

$$\Phi : \Lambda^{\pm} \ni ((i_n)_{n\in\mathbb{Z}}, (z_n)_{n\in\mathbb{Z}}) \longmapsto z_0 \in K_{a,b}$$

can be defined.

Definition 2.5. A bi-infinite sequence $(i_n)_{n\in\mathbb{Z}} \in \mathfrak{S}^{\pm}$ is called a $\Sigma^{\pm}$*-coding* of $z_0 \in K_{a,b}$ if it satisfies $\Phi((i_n)_{n\in\mathbb{Z}}, (f^n(z_0))_{n\in\mathbb{Z}}) = z_0$.

The next proposition is a restatement of [AI, Propositions 3.3 and 3.6].

PROPOSITION 2.6. *Let* $(a, b) \in \mathcal{F}^{\pm} \cap \{b \neq 0\}$*. Then,* $\Phi : \Lambda^{\pm} \to K_{a,b}$ *is surjective.*

In particular, every point in $K_{a,b}$ has at least one $\Sigma^{\pm}$-coding thanks to Proposition 2.6.

One can also show that the map Φ is almost injective. To do this, below we write $\mathcal{B}_i^{\pm} = D_{u,i}^{\pm} \times_{\mathrm{pr}} D_{v,i}^{\pm}$.

LEMMA 2.7. (Numerical Check 1) *Let* $(a, b) \in \mathcal{E}_{\mathbb{R}}^+$*. Then, there exist projective polydisks* $\mathcal{P}_{0,3}^+$*,* $\mathcal{P}_{1,2}^+$*, and* $\mathcal{P}_{2,3}^+$ *so that:*

(1) $\mathcal{P}_{0,3}^+ \supset \mathcal{B}_0^+ \cap \mathcal{B}_3^+$*,* $\mathcal{P}_{1,2}^+ \supset \mathcal{B}_1^+ \cap \mathcal{B}_2^+$*, and* $\mathcal{P}_{2,3}^+ \supset \mathcal{B}_2^+ \cap \mathcal{B}_3^+$ *hold;*

(2) $f : \mathcal{P}_{0,3}^+ \cap f^{-1}(\mathcal{P}_{1,2}^+) \to \mathcal{P}_{1,2}^+$*,* $f : \mathcal{P}_{1,2}^+ \cap f^{-1}(\mathcal{P}_{0,3}^+) \to \mathcal{P}_{0,3}^+$*, and* $f : \mathcal{P}_{2,3}^+ \cap f^{-1}(\mathcal{P}_{1,2}^+) \to \mathcal{P}_{1,2}^+$ *are crossed mappings of degree one.*

Proof. See Appendix A. □

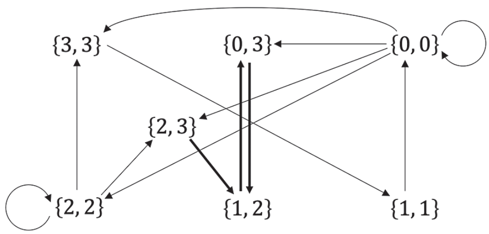

FIGURE 5. Pairwise transition diagram for $(a, b) \in \mathcal{E}^{+}_{\mathbb{R}}$.

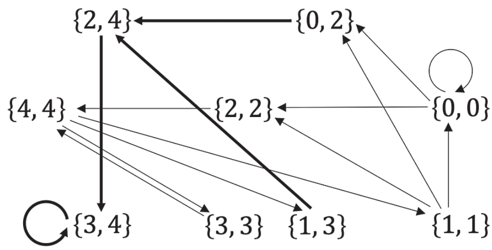

FIGURE 6. Pairwise transition diagram for $(a, b) \in \mathcal{E}^{-}_{\mathbb{R}}$.

LEMMA 2.8. (Numerical Check 2) *Let $(a, b) \in \mathcal{E}^{-}_{\mathbb{R}}$. Then, there exist projective polydisks $\mathcal{P}^{-}_{0,2}$, $\mathcal{P}^{-}_{1,3}$, $\mathcal{P}^{-}_{2,4}$, and $\mathcal{P}^{-}_{3,4}$ so that:*

(1) $\mathcal{P}^{-}_{0,2} \supset \mathcal{B}^{-}_{0} \cap \mathcal{B}^{-}_{2}$, $\mathcal{P}^{-}_{1,3} \supset \mathcal{B}^{-}_{1} \cap \mathcal{B}^{-}_{3}$, $\mathcal{P}^{-}_{2,4} \supset \mathcal{B}^{-}_{2} \cap \mathcal{B}^{-}_{4}$, *and* $\mathcal{P}^{-}_{3,4} \supset \mathcal{B}^{-}_{3} \cap \mathcal{B}^{-}_{4}$ *hold;*

(2) $f : \mathcal{P}^{-}_{0,2} \cap f^{-1}(\mathcal{P}^{-}_{2,4}) \to \mathcal{P}^{-}_{2,4}$, $f : \mathcal{P}^{-}_{1,3} \cap f^{-1}(\mathcal{P}^{-}_{2,4}) \to \mathcal{P}^{-}_{2,4}$, $f : \mathcal{P}^{-}_{2,4} \cap f^{-1}(\mathcal{P}^{-}_{3,4}) \to \mathcal{P}^{-}_{3,4}$, *and* $f : \mathcal{P}^{-}_{3,4} \cap f^{-1}(\mathcal{P}^{-}_{3,4}) \to \mathcal{P}^{-}_{3,4}$ *are crossed mappings of degree one.*

Proof. See Appendix A. □

PROPOSITION 2.9. *We have the following.*

(1) *Let $(a, b) \in \mathcal{E}^{+}_{\mathbb{R}}$. Then, $1 \le \mathrm{card}(\Phi^{-1}(z)) \le 2$ for any $z \in K_{a,b}$ and moreover $\mathrm{card}(\Phi^{-1}(z)) = 1$ if and only if $z \in K_{a,b} \setminus (V^{s}(p_2) \cup V^{s}(p_4))$.*

(2) *Let $(a, b) \in \mathcal{E}^{-}_{\mathbb{R}}$. Then, $1 \le \mathrm{card}(\Phi^{-1}(z)) \le 2$ for any $z \in K_{a,b}$ and moreover $\mathrm{card}(\Phi^{-1}(z)) = 1$ if and only f $z \in K_{a,b} \setminus V^{s}(p_3)$.*

Probably Proposition 2.9 would hold for any $(a, b) \in \mathcal{F}^{\pm} \cap \{b \neq 0\}$, but we restrict ourselves to $(a, b) \in \mathcal{E}^{\pm}_{\mathbb{R}}$ which is sufficient for our purpose. The same would hold for Lemmas 2.10, 2.11, 3.1, and 3.2 below.

To prove this proposition, we need to introduce a finite directed graph called the *pairwise transition diagram* for $\mathfrak{T}^{\pm}$ as follows. A vertex is an unordered pair $\{i, j\} \subset \Sigma^{\pm}$ so that $\mathcal{B}^{\pm}_{i} \cap \mathcal{B}^{\pm}_{j} \cap K_{a,b} \neq \emptyset$ (see Lemma 2.4), and there is an arrow from $\{i, j\}$ to $\{i', j'\}$ if and only if both $f : \mathcal{B}^{\pm}_{i} \cap f^{-1}(\mathcal{B}^{\pm}_{i'}) \to \mathcal{B}^{\pm}_{i'}$ and $f : \mathcal{B}^{\pm}_{j} \cap f^{-1}(\mathcal{B}^{\pm}_{j'}) \to \mathcal{B}^{\pm}_{j'}$ are crossed mappings (by exchanging i' and j', if necessary). See Figures 5 and 6.

Proof. We first prove case (1). Given a point $z \in K_{a,b}$, let us consider the following condition $(*)$:

> there exists $N \in \mathbb{Z}$ so that $f^N(z) \in \mathcal{B}^+_{i_N} \cap \mathcal{B}^+_{j_N}$ holds for some $\{i_N, j_N\} \subset \Sigma^+$ with $i_N \neq j_N$.

Suppose first the case that $z \in K_{a,b}$ does not satisfy condition $(*)$. Since $K_{a,b} \subset \bigcup_{i \in \Sigma^+} \mathcal{B}^+_i$, this means that for any $n \in \mathbb{Z}$, there exists a unique $i_n \in \Sigma^+$ so that $f^n(z) \in \mathcal{B}^+_{i_n}$. This obviously implies that $\mathrm{card}(\Phi^{-1}(z)) = 1$.

Suppose second the case that $z \in K_{a,b}$ satisfies condition $(*)$. Since for every $z \in K_{a,b}$, there exist at most two $i \in \Sigma^+$ so that $z \in \mathcal{B}^+_i$ (see item (1) of Lemma 2.4), such a pair $\{i_N, j_N\}$ is uniquely determined for a given N. We next observe that in the pairwise transition diagram, there exists exactly one out-going arrow from every vertex of the form $\{i, j\}$ with $i \neq j$ (focus on bold arrows in Figure 5) and there are no arrows from such a vertex to a vertex of the form $\{k, k\}$. This implies that there exists a unique path $\{i_N, j_N\} \to \{i_{N+1}, j_{N+1}\} \to \cdots$ in the diagram starting from the vertex $\{i_N, j_N\}$. Since this holds for every N appearing in condition $(*)$, we conclude that $\mathrm{card}(\Phi^{-1}(z)) = 2$. This together with the conclusion in the previous paragraph yields that $z \in K_{a,b}$ satisfies condition $(*)$ if and only if $\mathrm{card}(\Phi^{-1}(z)) = 2$.

Under condition $(*)$, we also observe that the path $\{i_N, j_N\} \to \{i_{N+1}, j_{N+1}\} \to \cdots$ eventually falls into the cycle $\{0, 3\} \to \{1, 2\} \to \{0, 3\} \to \cdots$ of period 2. Now we apply Lemma 2.7. Since $\mathcal{P}^+_{0,3} \supset \mathcal{B}^+_0 \cap \mathcal{B}^+_3$ and $\mathcal{P}^+_{1,2} \supset \mathcal{B}^+_1 \cap \mathcal{B}^+_2$ hold by item (1) of Lemma 2.7, this implies that $f^n(z)$ eventually drops into either $\mathcal{P}^+_{0,3}$ or $\mathcal{P}^+_{1,2}$ and stays there. By item (2) of Lemma 2.7, we see that $f^n(z)$ eventually belongs to either the local stable manifold of p_2 in $\mathcal{P}^+_{0,3}$ or the local stable manifold of p_4 in $\mathcal{P}^+_{1,2}$. Therefore, we may conclude that $z \in V^s(p_2) \cup V^s(p_4)$. Conversely, suppose that $z \in V^s(p_2) \cup V^s(p_4)$. Since $p_2 \in \mathcal{B}^+_0 \cap \mathcal{B}^+_3$ and $p_4 \in \mathcal{B}^+_1 \cap \mathcal{B}^+_2$, there exists $N \in \mathbb{Z}$ so that either $f^N(z) \in \mathcal{B}^+_0 \cap \mathcal{B}^+_3$ or $f^N(z) \in \mathcal{B}^+_1 \cap \mathcal{B}^+_2$ holds. In particular, condition $(*)$ is satisfied. Therefore, we conclude that $z \in K_{a,b}$ satisfies condition $(*)$ if and only if $z \in V^s(p_2) \cup V^s(p_4)$.

The argument above obviously shows $1 \leq \mathrm{card}(\Phi^{-1}(z)) \leq 2$ for all $z \in K_{a,b}$. By combining the conclusions in the previous two paragraphs, we obtain $\mathrm{card}(\Phi^{-1}(z)) = 2$ if and only if $z \in V^s(p_2) \cup V^s(p_4)$, which proves case (1). The proof for case (2) is similarly obtained by using Lemma 2.8. □

2.2. *Refined codings.* In what follows, it is essential to refine the boxes $\mathcal{B}^+_2$, $\mathcal{B}^-_2$ and $\mathcal{B}^-_3$.

LEMMA 2.10. (Numerical Check 3) *When $(a, b) \in \mathcal{E}^+_{\mathbb{R}}$, the crossed mapping*

$$f^4 : \mathcal{B}^+_2 \cap f^{-1}(\mathcal{B}^+_3 \cap f^{-1}(\mathcal{B}^+_1 \cap f^{-1}(\mathcal{B}^+_0 \cap f^{-1}(\mathcal{B}^+_2)))) \longrightarrow \mathcal{B}^+_2$$

of degree two satisfies the off-criticality condition.

Proof. See Appendix A. □

It then follows from [I, Theorem 2.14] that the domain

$$\mathcal{B}^+_2 \cap f^{-1}(\mathcal{B}^+_3 \cap f^{-1}(\mathcal{B}^+_1 \cap f^{-1}(\mathcal{B}^+_0 \cap f^{-1}(\mathcal{B}^+_2))))$$

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consists of two connected components denoted as $\mathcal{B}^+_{2'}$ and $\mathcal{B}^+_{2''}$ with $p_4 \in \mathcal{B}^+_{2'}$. Moreover, both $\mathcal{B}^+_{2'}$ and $\mathcal{B}^+_{2''}$ are biholomorphic to polydisks, and $f^4 : \mathcal{B}^+_{2'} \to \mathcal{B}^+_2$ and $f^4 : \mathcal{B}^+_{2''} \to \mathcal{B}^+_2$ are crossed mappings of degree one (see Step 5 of the proof of [I, Theorem 2.14]).

LEMMA 2.11. (Numerical Check 4) *When $(a, b) \in \mathcal{E}^-_{\mathbb{R}}$, the crossed mapping*

$$f^2 : \mathcal{B}^-_i \cap f^{-1}(\mathcal{B}^-_4 \cap f^{-1}(\mathcal{B}^-_3)) \longrightarrow \mathcal{B}^-_3$$

of degree two satisfies the off-criticality condition for $i = 2, 3$.

Proof. See Appendix A. □

Again, it then follows from [I, Theorem 2.14] that the domain

$$\mathcal{B}^-_i \cap f^{-1}(\mathcal{B}^-_4 \cap f^{-1}(\mathcal{B}^-_3))$$

consists of two connected components denoted as $\mathcal{B}^-_{i'}$ and $\mathcal{B}^-_{i''}$ for $i = 2, 3$ with $p_3 \in \mathcal{B}^-_{3'}$. Moreover, both $\mathcal{B}^-_{i'}$ and $\mathcal{B}^-_{i''}$ are biholomorphic to polydisks, and $f^2 : \mathcal{B}^-_{i'} \to \mathcal{B}^-_3$ and $f^2 : \mathcal{B}^-_{i''} \to \mathcal{B}^-_3$ are crossed mappings of degree one. Note that, *a priori*, we are not able to distinguish the components $\mathcal{B}^-_{i'}$ and $\mathcal{B}^-_{i''}$ (modulo the condition $p_3 \in \mathcal{B}^-_{3'}$) until items (3) and (4) of Corollary 3.6, where we determine $\mathcal{B}^-_{i'}$ and $\mathcal{B}^-_{i''}$ so that $\mathcal{B}^-_{i',\mathbb{R}}$ contains a curve $\gamma_{i,\mathrm{left}}$ on the 'left' and $\mathcal{B}^-_{i'',\mathbb{R}}$ contains a curve $\gamma_{i,\mathrm{right}}$ on the 'right'.

It is useful to consider the corresponding refined symbolic dynamics as follows. We set $\widetilde{\Sigma}^+ \equiv \{0, 1, 2', 2'', 3\}$ and $\widetilde{\Sigma}^- \equiv \{0, 1, 2', 2'', 3', 3'', 4\}$. Choose a subset of $\widetilde{\Sigma}^+ \times \widetilde{\Sigma}^+$ as

$$\widetilde{\mathfrak{T}}^+ \equiv \{(0, 0), (0, 2'), (0, 2''), (0, 3), (1, 0), (2', 3), (2'', 3), (3, 1)\}$$

and a subset of $\widetilde{\Sigma}^- \times \widetilde{\Sigma}^-$ as

$$\widetilde{\mathfrak{T}}^- \equiv \{(0, 0), (0, 2'),(0, 2''), (1, 0), (1, 2'), (1, 2''),\\ (2', 4), (2'', 4), (3', 4), (3'', 4), (4, 1), (4, 3'), (4, 3'')\}.$$

We then write $\widetilde{\mathfrak{S}}^\pm_{\mathrm{fwd}} \equiv \mathfrak{S}_{\mathrm{fwd}}(\widetilde{\mathfrak{T}}^\pm)$, $\widetilde{\mathfrak{S}}^\pm_{\mathrm{bwd}} \equiv \mathfrak{S}_{\mathrm{bwd}}(\widetilde{\mathfrak{T}}^\pm)$, and $\widetilde{\mathfrak{S}}^\pm \equiv \mathfrak{S}(\widetilde{\mathfrak{T}}^\pm)$.

For $(a, b) \in \mathcal{F}^\pm_{\mathbb{R}} \cap \{b \neq 0\}$, define the *refined orbit space* of $f_{a,b}$ as

$$\widetilde{\Lambda}^\pm \equiv \{((i_n)_{n\in\mathbb{Z}}, (z_n)_{n\in\mathbb{Z}}) : (i_n)_{n\in\mathbb{Z}} \in \widetilde{\mathfrak{S}}^\pm,\ z_n \in \mathcal{B}^\pm_{i_n},\ f(z_n) = z_{n+1}\}.$$

Since $\mathcal{B}^+_{2'} \cup \mathcal{B}^+_{2''} \subset \mathcal{B}^+_2$ and $\mathcal{B}^-_{i'} \cup \mathcal{B}^-_{i''} \subset \mathcal{B}^-_i$ $(i = 2, 3)$ hold, we see that

$$\bigcap_{n\in\mathbb{Z}} f^n\bigg(\bigcup_{i\in\widetilde{\Sigma}^\pm} \mathcal{B}^\pm_i\bigg) \subset K_{a,b}.$$

Therefore, the projection

$$\widetilde{\Phi} : \widetilde{\Lambda}^\pm \ni ((i_n)_{n\in\mathbb{Z}}, (z_n)_{n\in\mathbb{Z}}) \longmapsto z_0 \in K_{a,b}$$

can be defined.

Definition 2.12. A bi-infinite sequence $(i_n)_{n\in\mathbb{Z}} \in \widetilde{\mathfrak{S}}^\pm$ is called a $\widetilde{\Sigma}^\pm$-*coding* of $z_0 \in K_{a,b}$ if it satisfies $\widetilde{\Phi}((i_n)_{n\in\mathbb{Z}}, (f^n(z_0))_{n\in\mathbb{Z}}) = z_0$.

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3. *Local stable/unstable manifolds*

3.1. *Dynamics in* $\mathbb{C}^2$. For a real parameter $(a, b) \in \mathcal{F}^{\pm}_{\mathbb{R}} \cap \{b \neq 0\}$, we let $p_i \in \mathbb{R}^2$ be either the unique saddle fixed point (for $i = 1, 3$) or the unique saddle periodic point of period two (for $i = 2, 4$) of $f_{\mathbb{R}}$ in the ith quadrant. For a complex parameter $(a, b) \in \mathcal{F}^{\pm} \cap \{b \neq 0\}$, denote again by $p_i \in \mathbb{C}^2$ its complex continuation. Let $V^{s/u}(p_i)$ be the complex stable/unstable manifold at p_i. Denote by $V^{s/u}_{\mathrm{loc}}(p_i)_j$ the connected component of $V^{u/s}(p_i) \cap \mathcal{B}^{\pm}_j$ containing p_i.

Recall that a bi-infinite $\mathfrak{T}^{\pm}$-admissible sequence $I = (i_n)_{n\in\mathbb{Z}} \in \mathfrak{S}^{\pm}$ is a $\Sigma^{\pm}$-coding of $z \in K_{a,b}$ if and only if $z \in \bigcap_{n\in\mathbb{Z}} f^{-n}(\mathcal{B}^{\pm}_{i_n})$. Based on this observation, we define

$$K^s_I \equiv \bigcap_{n\geq 0} f^{-n}(\mathcal{B}^{\pm}_{i_n})$$

for a forward $\widetilde{\mathfrak{T}}^{\pm}$-admissible sequence $I = (i_n)_{n\geq 0} \in \widetilde{\mathfrak{S}}^{\pm}_{\mathrm{fwd}}$. We also define

$$K^u_J \equiv \bigcap_{n\leq 0} f^{-n}(\mathcal{B}^{\pm}_{j_n})$$

for a backward $\widetilde{\mathfrak{T}}^{\pm}$-admissible sequence $J = (j_n)_{n\leq 0} \in \widetilde{\mathfrak{S}}^{\pm}_{\mathrm{bwd}}$. Given a point $z \in K_{a,b}$, a forward $\widetilde{\mathfrak{T}}^{\pm}$-admissible sequence $I = (i_n)_{n\geq 0} \in \widetilde{\mathfrak{S}}^{\pm}_{\mathrm{fwd}}$ is called a *forward* $\widetilde{\Sigma}^{\pm}$*-coding* of z if z belongs to K^s_I.

We first characterize local stable manifolds in terms of forward $\widetilde{\Sigma}^{\pm}$-coding.

LEMMA 3.1. *Let* $(a, b) \in \mathcal{E}^{+}_{\mathbb{R}}$. *Then, we have the following.*

(1) $V^s_{\mathrm{loc}}(p_1)_0$ *is a vertical disk of degree one in* $\mathcal{B}^+_0$ *and* $V^s_{\mathrm{loc}}(p_1)_0 = K^s_{\overline{0}}$.

(2) $V^s_{\mathrm{loc}}(p_4)_2$ *is a vertical disk of degree one in* $\mathcal{B}^+_2$ *and* $V^s_{\mathrm{loc}}(p_4)_2 = K^s_{\overline{2'310}}$.

(3) $V^s_{\mathrm{loc}}(p_2)_0$ *is a vertical disk of degree one in* $\mathcal{B}^+_0$ *and* $V^s_{\mathrm{loc}}(p_2)_0 = K^s_{\overline{02'31}}$.

(4) $V^s_{\mathrm{loc}}(p_4)_1$ *is a vertical disk of degree one in* $\mathcal{B}^+_1$ *and* $V^s_{\mathrm{loc}}(p_4)_1 = K^s_{\overline{102'3}}$.

(5) $V^s_{\mathrm{loc}}(p_2)_3$ *is a vertical disk of degree one in* $\mathcal{B}^+_3$ *and* $V^s_{\mathrm{loc}}(p_2)_3 = K^s_{\overline{3102'}}$.

(6) $f^{-1}(V^s_{\mathrm{loc}}(p_4)_2) \cap \mathcal{B}^+_2$ *is a vertical disk of degree one in* $\mathcal{B}^+_2$. *Moreover,* $f^{-1}(V^s_{\mathrm{loc}}(p_4)_2) \cap \mathcal{B}^+_2 = K^s_{2''\overline{2'310}}$.

Proof. Claim (1) follows from the fact that $f : \mathcal{B}^+_0 \cap f^{-1}(\mathcal{B}^+_0) \to \mathcal{B}^+_0$ is a crossed mapping of degree one.

It follows from the definition of $\mathcal{B}^+_{2'}$ that $\bigcap_{n\geq 0} f^{-4n}(\mathcal{B}^+_{2'}) = K^s_{\overline{2'310}}$. Since $f^4 : \mathcal{B}^+_{2'} \to \mathcal{B}^+_2$ is a crossed mapping of degree one, $\bigcap_{n\geq 0} f^{-4n}(\mathcal{B}^+_{2'})$ is a vertical disk of degree one in $\mathcal{B}^+_2$ containing the fixed point p_4 and hence is equal to $V^s_{\mathrm{loc}}(p_4)_2$. This proves claim (2).

Claim (3) follows from claim (2) and the fact that $f : \mathcal{B}^+_0 \cap f^{-1}(\mathcal{B}^+_2) \to \mathcal{B}^+_2$ is a crossed mapping of degree one, claim (4) follows from claim (3) and the fact that $f : \mathcal{B}^+_1 \cap f^{-1}(\mathcal{B}^+_0) \to \mathcal{B}^+_0$ is a crossed mapping of degree one, claim (5) follows from claim (4) and the fact that $f : \mathcal{B}^+_3 \cap f^{-1}(\mathcal{B}^+_1) \to \mathcal{B}^+_1$ is a crossed mapping of degree one, and claim (6) follows from claim (2) and the fact that $f : \mathcal{B}^+_{2''} \cap f^{-1}(\mathcal{B}^+_2) \to \mathcal{B}^+_2$ is a crossed mapping of degree one. $\square$

Similarly, we have the following lemma.

LEMMA 3.2. *Let $(a, b) \in \mathcal{E}_{\mathbb{R}}^{-}$. Then, we have the following.*

(1) $V^s_{\mathrm{loc}}(p_1)_0$ *is a vertical disk of degree one in* $\mathcal{B}_0^-$ *and* $V^s_{\mathrm{loc}}(p_1)_0 = K^s_{\overline{0}}$.

(2) $V^s_{\mathrm{loc}}(p_3)_3$ *is a vertical disk of degree one in* $\mathcal{B}_3^-$ *and* $V^s_{\mathrm{loc}}(p_3)_3 = K^s_{\overline{3'4}}$.

(3) $V^s_{\mathrm{loc}}(p_3)_4$ *is a vertical disk of degree one in* $\mathcal{B}_4^-$ *and* $V^s_{\mathrm{loc}}(p_3)_4 = K^s_{\overline{43'}}$.

(4) $f^{-1}(V^s_{\mathrm{loc}}(p_3)_4) \cap \mathcal{B}_2^-$ *consists of two mutually disjoint vertical disks of degree one in* $\mathcal{B}_2^-$ *denoted by* $V_{2'} \subset \mathcal{B}_{2'}^-$ *and* $V_{2''} \subset \mathcal{B}_{2''}^-$. *Moreover,* $V_{2'} = K^s_{2'\overline{43'}}$ *and* $V_{2''} = K^s_{2''\overline{43'}}$.

(5) $f^{-1}(V^s_{\mathrm{loc}}(p_3)_4) \cap \mathcal{B}_3^-$ *consists of two mutually disjoint vertical disks of degree one in* $\mathcal{B}_3^-$ *denoted by* $V_{3'} \subset \mathcal{B}_{3'}^-$ *and* $V_{3''} \subset \mathcal{B}_{3''}^-$. *Moreover,* $V_{3'} = K^s_{3'\overline{43'}}$ *and* $V_{3''} = K^s_{3''\overline{43'}}$.

(6) $f^{-1}(V_{2'}) \cap \mathcal{B}_1^-$ *is a vertical disk of degree one in* $\mathcal{B}_1^-$. *Moreover,* $f^{-1}(V_{2'}) \cap \mathcal{B}_1^- = K^s_{12'\overline{43'}}$.

(7) $f^{-1}(V_{2'}) \cap \mathcal{B}_0^-$ *is a vertical disk of degree one in* $\mathcal{B}_0^-$. *Moreover,* $f^{-1}(V_{2'}) \cap \mathcal{B}_0^- = K^s_{02'\overline{43'}}$.

Proof. Claim (1) follows from the fact that $f : \mathcal{B}_0^- \cap f^{-1}(\mathcal{B}_0^-) \to \mathcal{B}_0^-$ is a crossed mapping of degree one.

It follows from the definition of $\mathcal{B}_{3'}^-$ that $\bigcap_{n\geq 0} f^{-2n}(\mathcal{B}_{3'}^-) = K^s_{\overline{3'4}}$. Since $f^2 : \mathcal{B}_{3'}^- \to \mathcal{B}_3^-$ is a crossed mapping of degree one, $\bigcap_{n\geq 0} f^{-2n}(\mathcal{B}_{3'}^-)$ is a vertical disk of degree one in $\mathcal{B}_3^-$ containing the fixed point p_3 and hence is equal to $V^s_{\mathrm{loc}}(p_3)_3$. This proves claim (2).

Claim (3) follows from claim (2) and the fact that $f : \mathcal{B}_4^- \cap f^{-1}(\mathcal{B}_3^-) \to \mathcal{B}_3^-$ is a crossed mapping of degree one. Claims (6) and (7) follow from the fact in claim (3) that $V_{2'}$ is a vertical disk of degree one in $\mathcal{B}_2^-$ and that $f : \mathcal{B}_1^- \cap f^{-1}(\mathcal{B}_2^-) \to \mathcal{B}_2^-$ and $f : \mathcal{B}_0^- \cap f^{-1}(\mathcal{B}_2^-) \to \mathcal{B}_2^-$ are crossed mappings of degree one.

By Lemma 2.11 and claim (2), we see that $f^{-2}(V^s_{\mathrm{loc}}(p_3)_3) \cap \mathcal{B}_i^-$ consists of two mutually disjoint vertical disks of degree one in $\mathcal{B}_i^-$ for $i = 2, 3$. Since we have $f^{-1}(V^s_{\mathrm{loc}}(p_3)_4) \cap \mathcal{B}_i^- = f^{-2}(V^s_{\mathrm{loc}}(p_3)_3) \cap \mathcal{B}_i^-$, the first half of claims (4) and (5) follow. The second half of claims (4) and (5) easily follows from claim (3). □

For $(a, b) \in \mathcal{F}^+ \cap \{b \neq 0\}$, we know that $f : \mathcal{B}_0^+ \cap f^{-1}(\mathcal{B}_0^+) \to \mathcal{B}_0^+$ is a crossed mapping of degree one. Hence, $V^s_{\mathrm{loc}}(p_1)_0$ is a vertical disk of degree one in $\mathcal{B}_0^+$ and $V^u_{\mathrm{loc}}(p_1)_0$ is a horizontal disk of degree one in $\mathcal{B}_0^+$. For $(a, b) \in \mathcal{F}^- \cap \{b \neq 0\}$, we know that $f : \mathcal{B}_0^- \cap f^{-1}(\mathcal{B}_0^-) \to \mathcal{B}_0^-$ is a crossed mapping of degree one. Hence, $V^s_{\mathrm{loc}}(p_1)_0$ is a vertical disk of degree one in $\mathcal{B}_0^-$. The next fact is stated in [**AI**, Proposition 3.10].

PROPOSITION 3.3. *When* $(a, b) \in \mathcal{F}^- \cap \{b \neq 0\}$, $V^u_{\mathrm{loc}}(p_3)_3$ *is a horizontal disk of degree one in* $\mathcal{B}_3^-$.

Let $(a, b) \in \mathcal{F}^+ \cap \{b \neq 0\}$. For a forward admissible sequence of the form $I = i_0 i_1 \cdots i_n \overline{0} \in \mathfrak{S}^+_{\mathrm{fwd}}$, we define

$$V^s_I(a, b)^+ \equiv \mathcal{B}^+_{i_0} \cap f^{-1}(\mathcal{B}^+_{i_1} \cap \cdots \cap f^{-1}(\mathcal{B}^+_{i_n} \cap f^{-1}(V^s_{\mathrm{loc}}(p_1)_0)) \cdots),$$

and for a backward admissible sequence of the form $J = \overline{0} j_{-n} \cdots j_{-1} j_0 \in \mathfrak{S}^+_{\mathrm{bwd}}$, we define

$$V^u_J(a, b)^+ \equiv \mathcal{B}^+_{j_0} \cap f(\mathcal{B}^+_{j_{-1}} \cap \cdots \cap f(\mathcal{B}^+_{j_{-n}} \cap f(V^u_{\mathrm{loc}}(p_1)_0)) \cdots).$$

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Let $(a, b) \in \mathcal{F}^- \cap \{b \neq 0\}$. For a forward admissible sequence of the form $I = i_0 i_1 \cdots i_n \overline{0} \in \mathfrak{S}^-_{\mathrm{fwd}}$, we define

$$V^s_I(a, b)^- \equiv \mathcal{B}^-_{i_0} \cap f^{-1}(\mathcal{B}^-_{i_1} \cap \cdots \cap f^{-1}(\mathcal{B}^-_{i_n} \cap f^{-1}(V^s_{\mathrm{loc}}(p_1)_0)) \cdots),$$

and for a backward admissible sequence of the form $J = \overline{43} j_{-n} \cdots j_{-1} j_0 \in \mathfrak{S}^-_{\mathrm{bwd}}$, we define

$$V^u_J(a, b)^- \equiv \mathcal{B}^-_{j_0} \cap f(\mathcal{B}^-_{j_{-1}} \cap \cdots \cap f(\mathcal{B}^-_{j_{-n}} \cap f(V^u_{\mathrm{loc}}(p_3)_3)) \cdots).$$

Remark 3.4. The set K^u_J is all the points whose $\Sigma^\pm$-coding is J, but we defined $V^u_J(a, b)^\pm$ as a piece of a certain local unstable manifold. Hence, $V^u_J(a, b)^\pm \subset K^u_J$ always holds but $V^u_J(a, b)^\pm$ is not necessarily equal to K^u_J. For example, $K^u_{\overline{43}}$ consists of infinitely many mutually disjoint horizontal disks in $\mathcal{B}^-_3$ in the case of $b < 0$. However, $V^u_{\overline{43}}(a, b)^-$ is one of those disks. For the motivation of our definition, see Remark 4.10.

3.2. *Dynamics in* $\mathbb{R}^2$. Let us assume that $(a, b) \in \mathcal{E}^\pm_{\mathbb{R}}$. We denote the real slice of $\mathcal{B}^\pm_i$ by $\mathcal{B}^\pm_{i,\mathbb{R}} \equiv \mathcal{B}^\pm_i \cap \mathbb{R}^2$. A curve γ in $\mathcal{B}^\pm_{i,\mathbb{R}}$ is said to be *horizontal* (respectively *vertical*) if the projection $\pi_x : \gamma \to D_{x,\mathbb{R}}$ (respectively $\pi_y : \gamma \to D_{y,\mathbb{R}}$) is a bijection, where $D_{x,\mathbb{R}}$ and $D_{y,\mathbb{R}}$ are intervals so that $\mathcal{B}^\pm_{j,\mathbb{R}} = D_{x,\mathbb{R}} \times_{\mathrm{pr}} D_{y,\mathbb{R}}$.

Recall that p_i is either the unique saddle fixed point (for $i = 1, 3$) or the unique saddle periodic point of period two (for $i = 2, 4$) in the ith quadrant. Denote by $W^{u/s}_{\mathrm{loc}}(p_i)_j$ the connected component of $W^{u/s}(p_i) \cap \mathcal{B}^\pm_{j,\mathbb{R}}$ containing p_i. We have $W^{u/s}_{\mathrm{loc}}(p_i)_j = V^{u/s}_{\mathrm{loc}}(p_i)_j \cap \mathbb{R}^2$.

COROLLARY 3.5. *Let* $(a, b) \in \mathcal{E}^+_{\mathbb{R}}$. *Then, we have the following.*

(1) $W^s_{\mathrm{loc}}(p_1)_0$ *is a vertical curve in* $\mathcal{B}^+_{0,\mathbb{R}}$ *denoted by* $\gamma^+_{0,\mathrm{right}}$ *and* $W^s_{\mathrm{loc}}(p_1)_0 = K^s_{\overline{0}} \cap \mathbb{R}^2$.
(2) $W^s_{\mathrm{loc}}(p_4)_2$ *is a vertical curve in* $\mathcal{B}^+_{2,\mathbb{R}}$ *denoted by* $\gamma^+_{2,\mathrm{right}}$ *and* $W^s_{\mathrm{loc}}(p_4)_2 = K^s_{\overline{2'310}} \cap \mathbb{R}^2$.
(3) $W^s_{\mathrm{loc}}(p_2)_0$ *is a vertical curve in* $\mathcal{B}^+_{0,\mathbb{R}}$ *denoted by* $\gamma^+_{0,\mathrm{left}}$ *and* $W^s_{\mathrm{loc}}(p_2)_0 = K^s_{\overline{02'31}} \cap \mathbb{R}^2$.
(4) $W^s_{\mathrm{loc}}(p_4)_1$ *is a vertical curve in* $\mathcal{B}^+_{1,\mathbb{R}}$ *denoted by* $\gamma^+_{1,\mathrm{left}}$ *and* $W^s_{\mathrm{loc}}(p_4)_1 = K^s_{\overline{102'3}} \cap \mathbb{R}^2$.
(5) $W^s_{\mathrm{loc}}(p_2)_3$ *is a vertical curve in* $\mathcal{B}^+_{3,\mathbb{R}}$ *denoted by* $\gamma^+_{3,\mathrm{right}}$ *and* $W^s_{\mathrm{loc}}(p_2)_3 = K^s_{\overline{3102'}} \cap \mathbb{R}^2$.
(6) $f^{-1}_{\mathbb{R}}(W^s_{\mathrm{loc}}(p_4)_2) \cap \mathcal{B}^+_{2,\mathbb{R}}$ *is a vertical curve in* $\mathcal{B}^+_{2,\mathbb{R}}$ *denoted by* $\gamma^+_{2,\mathrm{left}}$ *and* $f^{-1}_{\mathbb{R}}(W^s_{\mathrm{loc}}(p_4)_2) \cap \mathcal{B}^+_{2,\mathbb{R}} = K^s_{2''\overline{2'310}} \cap \mathbb{R}^2$.

Proof. This immediately follows from Lemma 3.1 by taking the real parts. □

We also know that $V^u_{\mathrm{loc}}(p_1)_0$ is a horizontal disk of degree one in $\mathcal{B}^+_0$. Therefore, $W^u_{\mathrm{loc}}(p_1)_0$ is a horizontal curve in $\mathcal{B}^+_{0,\mathbb{R}}$.

When there are two mutually disjoint vertical curves in $\mathcal{B}^-_{i,\mathbb{R}}$, we can distinguish the left one and the right one.

COROLLARY 3.6. *Let* $(a, b) \in \mathcal{E}^-_{\mathbb{R}}$. *Then, we have the following.*

(1) $W^s_{\mathrm{loc}}(p_1)_0$ *is a vertical curve in* $\mathcal{B}^-_{0,\mathbb{R}}$ *denoted by* $\gamma^-_{0,\mathrm{right}}$ *and* $W^s_{\mathrm{loc}}(p_1)_0 = K^s_{\overline{0}} \cap \mathbb{R}^2$.
(2) $W^s_{\mathrm{loc}}(p_3)_3$ *is a vertical curve in* $\mathcal{B}^-_{3,\mathbb{R}}$ *denoted by* $\gamma^-_{3,\mathrm{left}}$ *and* $W^s_{\mathrm{loc}}(p_3)_3 = K^s_{\overline{3'4}} \cap \mathbb{R}^2$.
(3) $W^s_{\mathrm{loc}}(p_3)_4$ *is a vertical curve in* $\mathcal{B}^-_{4,\mathbb{R}}$ *denoted by* $\gamma^-_{4,\mathrm{right}}$ *and* $W^s_{\mathrm{loc}}(p_3)_4 = K^s_{\overline{43'}} \cap \mathbb{R}^2$.

(4) $f_{\mathbb{R}}^{-1}(W^s_{\mathrm{loc}}(p_3)_4) \cap \mathcal{B}^-_{2,\mathbb{R}}$ *consists of two mutually disjoint vertical curves in* $\mathcal{B}^-_{2,\mathbb{R}}$*; the left one is denoted by* $\gamma^-_{2,\mathrm{left}} \subset \mathcal{B}^-_{2',\mathbb{R}}$ *and the right one is denoted by* $\gamma^-_{2,\mathrm{right}} \subset \mathcal{B}^-_{2'',\mathbb{R}}$. *Moreover,* $\gamma^-_{2,\mathrm{left}} = K^s_{2'\overline{43'}} \cap \mathbb{R}^2$ *and* $\gamma^-_{2,\mathrm{right}} = K^s_{2''\overline{43'}} \cap \mathbb{R}^2$.

(5) $f_{\mathbb{R}}^{-1}(W^s_{\mathrm{loc}}(p_3)_4) \cap \mathcal{B}^-_{3,\mathbb{R}}$ *consists of two mutually disjoint vertical curves in* $\mathcal{B}^-_{3,\mathbb{R}}$*; the left one is denoted by* $\gamma^-_{3,\mathrm{left}} \subset \mathcal{B}^-_{3',\mathbb{R}}$ *and the right one is denoted by* $\gamma^-_{3,\mathrm{right}} \subset \mathcal{B}^-_{3'',\mathbb{R}}$. *Moreover,* $\gamma^-_{3,\mathrm{left}} = K^s_{3'\overline{43'}} \cap \mathbb{R}^2$ *and* $\gamma^-_{3,\mathrm{right}} = K^s_{3''\overline{43'}} \cap \mathbb{R}^2$.

(6) $f_{\mathbb{R}}^{-1}(\gamma^-_{2,\mathrm{left}}) \cap \mathcal{B}^-_{1,\mathbb{R}}$ *is a vertical curve in* $\mathcal{B}^-_{1,\mathbb{R}}$ *denoted by* $\gamma^-_{1,\mathrm{left}}$. *Moreover,* $\gamma^-_{1,\mathrm{left}} = K^s_{12'\overline{43'}} \cap \mathbb{R}^2$.

(7) $f_{\mathbb{R}}^{-1}(\gamma^-_{2,\mathrm{left}}) \cap \mathcal{B}^-_{0,\mathbb{R}}$ *is a vertical curve in* $\mathcal{B}^-_{0,\mathbb{R}}$ *denoted by* $\gamma^-_{0,\mathrm{left}}$. *Moreover,* $\gamma^-_{0,\mathrm{left}} = K^s_{02'\overline{43'}} \cap \mathbb{R}^2$.

Proof. This immediately follows from Lemma 3.2 by taking the real parts. □

Below, when the context is clear, we drop the superscript $\pm$ in $\gamma^{\pm}_{j,\mathrm{left}}$ and $\gamma^{\pm}_{j,\mathrm{right}}$, and write $\gamma_{j,\mathrm{left}}$ and $\gamma_{j,\mathrm{right}}$, respectively.

Since $p_3 \in \gamma_{3,\mathrm{left}}$, we have $\gamma_{3,\mathrm{left}} = W^s_{\mathrm{loc}}(p_3)_3$. We also know that $V^u_{\mathrm{loc}}(p_3)_3$ is a horizontal disk of degree one in $\mathcal{B}^-_3$ (see Proposition 3.3), and hence $W^u_{\mathrm{loc}}(p_3)_3$ is a horizontal curve in $\mathcal{B}^-_{3,\mathbb{R}}$.

Let $(a, b) \in \mathcal{E}^+_{\mathbb{R}}$. For a forward admissible sequence of the form $I = i_0 i_1 \cdots i_n \overline{0} \in \mathfrak{S}^+_{\mathrm{fwd}}$, we define

$$W^s_I(a, b)^+ \equiv \mathcal{B}^+_{i_0,\mathbb{R}} \cap f_{\mathbb{R}}^{-1}(\mathcal{B}^+_{i_1,\mathbb{R}} \cap \cdots \cap f_{\mathbb{R}}^{-1}(\mathcal{B}^+_{i_n,\mathbb{R}} \cap f_{\mathbb{R}}^{-1}(W^s_{\mathrm{loc}}(p_1)_0)) \cdots),$$

and for a backward admissible sequence of the form $J = \overline{0} j_{-n} \cdots j_{-1} j_0 \in \mathfrak{S}^+_{\mathrm{bwd}}$, we define

$$W^u_J(a, b)^+ \equiv \mathcal{B}^+_{j_0,\mathbb{R}} \cap f_{\mathbb{R}}(\mathcal{B}^+_{j_{-1},\mathbb{R}} \cap \cdots \cap f_{\mathbb{R}}(\mathcal{B}^+_{j_{-n},\mathbb{R}} \cap f_{\mathbb{R}}(W^u_{\mathrm{loc}}(p_1)_0)) \cdots).$$

Let $(a, b) \in \mathcal{E}^-_{\mathbb{R}}$. For a forward admissible sequence of the form $I = i_0 i_1 \cdots i_n \overline{0} \in \mathfrak{S}^-_{\mathrm{fwd}}$, we define

$$W^s_I(a, b)^- \equiv \mathcal{B}^-_{i_0,\mathbb{R}} \cap f_{\mathbb{R}}^{-1}(\mathcal{B}^-_{i_1,\mathbb{R}} \cap \cdots \cap f_{\mathbb{R}}^{-1}(\mathcal{B}^-_{i_n,\mathbb{R}} \cap f_{\mathbb{R}}^{-1}(W^s_{\mathrm{loc}}(p_1)_0)) \cdots),$$

and for a backward admissible sequence of the form $J = \overline{43} j_{-n} \cdots j_{-1} j_0 \in \mathfrak{S}^-_{\mathrm{bwd}}$, we define

$$W^u_J(a, b)^- \equiv \mathcal{B}^-_{j_0,\mathbb{R}} \cap f_{\mathbb{R}}(\mathcal{B}^-_{j_{-1},\mathbb{R}} \cap \cdots \cap f_{\mathbb{R}}(\mathcal{B}^-_{j_{-n},\mathbb{R}} \cap f_{\mathbb{R}}(W^u_{\mathrm{loc}}(p_3)_3)) \cdots).$$

As in [**AI**, Definition 4.4], $W^u_{\overline{434124}}(a, b)^-$ is decomposed into two parts: the 'inner part' $W^u_{\overline{434124}}(a, b)^-_{\mathrm{inner}}$ and the 'outer part' $W^u_{\overline{434124}}(a, b)^-_{\mathrm{outer}}$.

4. *Dynamics near the boundary* $\partial\mathcal{H}^{\pm}_{\mathbb{R}}$

4.1. *The* $\{L, R\}$*-coding.* In this subsection, we define the $\{L, R\}$-coding with respect to our box systems $\{\mathcal{B}^{\pm}_i\}_{i \in \Sigma^{\pm}}$. We note that our partition to define the $\{L, R\}$-coding is more involved than that given in [$\mathbf{BS_{\mathbb{R}}3}$].

When $\gamma \subset \mathcal{B}^{\pm}_{i,\mathbb{R}}$ is a vertical curve, the notion of *left/right component* of $\mathcal{B}^{\pm}_{i,\mathbb{R}} \setminus \gamma$ can be defined and denoted as $\mathrm{left}(\mathcal{B}^{\pm}_{i,\mathbb{R}} \setminus \gamma)$ and $\mathrm{right}(\mathcal{B}^{\pm}_{i,\mathbb{R}} \setminus \gamma)$.

Propositions 4.1 and 4.4 will be crucial in the rest of this paper.

PROPOSITION 4.1. *Suppose that $(a, b) \in \mathcal{E}^{+}_{\mathbb{R}}$. Let $z \in K_{\mathbb{R}}$ and let $(i_n)_{n\geq 0} \in \mathfrak{S}^{+}_{\mathrm{fwd}}$ be its forward Σ^{+}-coding.*

(1) *If $i_0 = 0$, then z belongs to the closure of* $\mathrm{right}(\mathcal{B}^{+}_{0,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_2)_0)$.
(2) *If $i_0 = 1$, then z belongs to the closure of* $\mathrm{right}(\mathcal{B}^{+}_{1,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_4)_1)$.
(3) *If $i_0 = 2$, then z belongs to the closure of* $\mathrm{left}(\mathcal{B}^{+}_{2,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_4)_2)$.
(4) *If $i_0 = 3$, then z belongs to the closure of* $\mathrm{left}(\mathcal{B}^{+}_{3,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_2)_3)$.

Proof. Let $i_0 = 0$ and assume that z belongs to $\mathrm{left}(\mathcal{B}^{+}_{0,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_2)_0)$. Since $(i_n)_{n\geq 0}$ is admissible, i_1 is either 2 or 3 (but not 1). It follows that $f_{\mathbb{R}}(z)$ belongs to $\mathrm{right}(\mathcal{B}^{+}_{2,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_4)_2)$ (compare [**AI**, Lemma 4.7]) and $i_1 = 2$. Similarly, since i_2 is either 2 or 3 (but not 0), it follows that $f^{2}_{\mathbb{R}}(z)$ belongs to $\mathrm{right}(\mathcal{B}^{+}_{3,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_2)_3)$ and $i_2 = 3$. A similar argument shows that $f^{3}_{\mathbb{R}}(z)$ belongs to $\mathrm{left}(\mathcal{B}^{+}_{1,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_4)_1)$ and $i_3 = 1$, and $f^{4}_{\mathbb{R}}(z)$ belongs to $\mathrm{left}(\mathcal{B}^{+}_{0,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_2)_0)$ and $i_4 = 0$. By repeating this, we see that $z \in K^{s}_{\overline{0231}} \cap \mathbb{R}^2$. Moreover, since $\mathrm{right}(\mathcal{B}^{+}_{2,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_4)_2) \subset \mathcal{B}^{+}_{2',\mathbb{R}}$, we obtain $z \in K^{s}_{\overline{02'31}} \cap \mathbb{R}^2$. It follows from Lemma 3.1(1) that $z \in W^{s}_{\mathrm{loc}}(p_2)_0$, which contradicts to the assumption that $z \in \mathrm{left}(\mathcal{B}^{+}_{0,\mathbb{R}} \setminus W^{s}_{\mathrm{loc}}(p_2)_0)$. Hence, claim (1) of this proposition is proved. The proofs for the other cases are similar. □

To define the $\{L, R\}$-partition, it is important to control the slopes of local stable manifolds. To obtain a simple criterion for estimates of the slopes, we employ the Poincaré metric and (vertical) Poincaré cone field (see [**I**, Definition 2.5]). Let $\mathcal{P} = D_x \times D_y$ be a polydisk, where $D_x \subset \mathbb{C}$ is a round disk of radius $\delta > 0$ and $D_y \subset \mathbb{C}$ is the round disk of radius $R > 0$. Take $0 < r < R$ and set $\mathcal{P}' = D_x \times D'_y$, where $D'_y \subset \mathbb{C}$ is the round disk of radius r. Denote by $|v_x|_P$ (respectively $|v_y|_P$) the Poincaré metric in D_x (respectively D_y).

Let D be a vertical disk of degree one in $\mathcal{P}$. It has been shown in [**I**, Corollary 2.10] that every tangent vector $v \in T_z D$ belongs to the vertical Poincaré cone at z:

$$C^{v}_{z} = \{v = (v_x, v_y) \in T_z\mathcal{P} : |v_y|_P \geq |v_x|_P\}.$$

As in the example following [**I**, Corollary 2.10], the condition $|v_y|_P \geq |v_x|_P$ is rewritten as

$$\frac{|v_y|_E}{|v_x|_E} \geq \frac{R^2 - |y|^2}{\delta^2 - |x|^2},$$

where $|v_x|_E$ (respectively $|v_y|_E$) denotes the Euclidean metric in $\mathbb{C}$. Moreover, if we restrict z to $\mathcal{P}'$, the estimate above yields

$$\frac{|v_y|_E}{|v_x|_E} \geq \frac{R^2 - r^2}{\delta^2}$$

for any $v \in T_{z'}D$ and any $z' \in \mathcal{P}'$. Finally, let us replace $\mathcal{P} = D_x \times D_y$ by the projective polydisk $D_x \times_{\mathrm{pr}} D_y$, where the u-direction is $(1, 0)$ and the v-direction is $(s, 1)$ with $s \neq 0$.

The computation

$$\begin{bmatrix} 1 & |s| \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{R^2-r^2}{\delta^2} \end{bmatrix} = \begin{bmatrix} \frac{\delta^2+|s|(R^2-r^2)}{\delta^2} \\ \frac{R^2-r^2}{\delta^2} \end{bmatrix}$$

shows that the slope of $v \in T_{z'}D$ in modulus is estimated from below by

$$\frac{R^2 - r^2}{\delta^2 + |s|(R^2 - r^2)} \tag{4.1}$$

for any $z' \in \mathcal{P}'$.

In the following lemma, a *horizontal edge* (respectively *vertical edge*) of a quadrilateral $\mathcal{B}^{\pm}_{i,\mathbb{R}}$ in $\mathbb{R}^2$ is a connected component of the real part of $D^{\pm}_{u,i} \times_{\mathrm{pr}} \partial D^{\pm}_{v,i}$ (respectively $\partial D^{\pm}_{u,i} \times_{\mathrm{pr}} D^{\pm}_{v,i}$). Given a closed interval $I = [\alpha, \beta]$ in the y-axis, we put $|I| = |\beta - \alpha|$ and let D_I be the disk in the complex y-plane centered at $(\alpha + \beta)/2$ with radius $|I|/2$. As before, given a polydisk $\mathcal{P}_i$, denote by $\mathcal{P}_{i,\mathbb{R}}$ its real part.

LEMMA 4.2. (Numerical Check 5) *Let $(a, b) \in \mathcal{E}^{+}_{\mathbb{R}}$.*

(1) *One can find two concentric intervals $I_2 = [y_2 - R_2, y_2 + R_2]$ and $I'_2 = [y_2 - r_2, y_2 + r_2]$ in the y-axis with $r_2 < R_2$ and a disk D_2 of radius $\delta_2 > 0$ so that:*
   (a) *a projective polydisk of the form $\mathcal{P}_2 = D_2 \times_{\mathrm{pr}} D_{I_2}$ contains p_2;*
   (b) *$f^2 : \mathcal{P}_2 \cap f^{-2}(\mathcal{P}_2) \to \mathcal{P}_2$ is a crossed mapping of degree one;*
   (c) *$\mathrm{pr}_y(\mathcal{B}^{+}_{0,\mathbb{R}} \cup \mathcal{B}^{+}_{3,\mathbb{R}}) \subset I'_2$;*
   (d) *$\mathcal{P}_{2,\mathbb{R}} \cap \mathcal{B}^{+}_{i,\mathbb{R}}$ does not intersect the vertical edges of $\mathcal{B}^{+}_{i,\mathbb{R}}$ for $i = 0, 3$;*
   (e) *the slopes of the horizontal edges of $\mathcal{B}^{+}_{0,\mathbb{R}}$ and $\mathcal{B}^{+}_{3,\mathbb{R}}$ in modulus are bounded by*

$$\frac{R_2^2 - r_2^2}{\delta_2^2 + |s_2|(R_2^2 - r_2^2)}$$

   from above, where $(s_2, 1)$ is the v-direction in defining $\mathcal{P}_2$.

(2) *One can find two concentric intervals $I_4 = [y_4 - R_4, y_4 + R_4]$ and $I'_4 = [y_4 - r_4, y_4 + r_4]$ in the y-axis with $r_4 < R_4$ and a disk D_4 of radius $\delta_4 > 0$ so that:*
   (a) *a projective polydisk of the form $\mathcal{P}_4 = D_4 \times_{\mathrm{pr}} D_{I_4}$ contains p_4;*
   (b) *$f^2 : \mathcal{P}_4 \cap f^{-2}(\mathcal{P}_4) \to \mathcal{P}_4$ is a crossed mapping of degree one;*
   (c) *$\mathrm{pr}_y(\mathcal{B}^{+}_{1,\mathbb{R}} \cup \mathcal{B}^{+}_{2,\mathbb{R}}) \subset I'_4$;*
   (d) *$\mathcal{P}_{4,\mathbb{R}} \cap \mathcal{B}^{+}_{i,\mathbb{R}}$ does not intersect the vertical edges of $\mathcal{B}^{+}_{i,\mathbb{R}}$ for $i = 1, 2$;*
   (e) *the slopes of the horizontal edges of $\mathcal{B}^{+}_{1,\mathbb{R}}$ and $\mathcal{B}^{+}_{2,\mathbb{R}}$ in modulus are bounded by*

$$\frac{R_4^2 - r_4^2}{\delta_4^2 + |s_4|(R_4^2 - r_4^2)}$$

   from above, where $(s_4, 1)$ is the v-direction in defining $\mathcal{P}_4$.

Proof. See Appendix A. □

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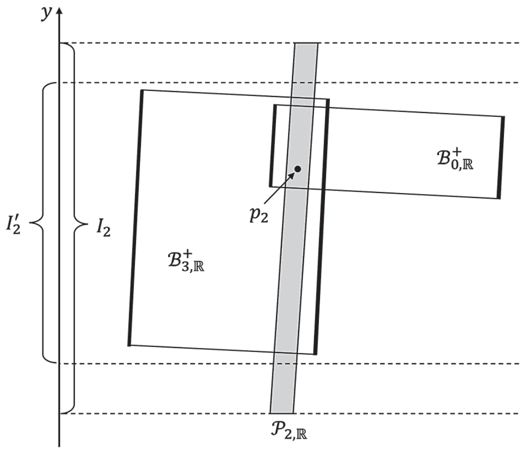

FIGURE 7. The polydisk $\mathcal{P}_2$.

See Figure 7 for Lemma 4.2.

Let P_L^+ be the union of left$(\mathcal{B}^+_{2,\mathbb{R}} \setminus W^s_{\mathrm{loc}}(p_4)_2)$ and left$(\mathcal{B}^+_{3,\mathbb{R}} \setminus W^s_{\mathrm{loc}}(p_2)_3)$. Let P_R^+ be the union of right$(\mathcal{B}^+_{0,\mathbb{R}} \setminus W^s_{\mathrm{loc}}(p_2)_0)$ and right$(\mathcal{B}^+_{1,\mathbb{R}} \setminus W^s_{\mathrm{loc}}(p_4)_1)$.

THEOREM 4.3. *For $(a, b) \in \mathcal{E}^+_{\mathbb{R}}$, the pair $\{P_L^+, P_R^+\}$ forms a partition for $K_{\mathbb{R}} \setminus (W^s(p_2) \cap W^s(p_4))$, that is: (1) $P_L^+ \cap P_R^+ \cap K_{\mathbb{R}} = \emptyset$ and (2) $P_L^+ \cup P_R^+ \supset K_{\mathbb{R}} \setminus (W^s(p_2) \cap W^s(p_4))$.*

Proof. Let C_2 be the real local stable manifold at p_2 in the real part $\mathcal{P}_{2,\mathbb{R}}$. By Lemma 4.2(1)(b), C_2 is a vertical curve connecting the upper boundary and the lower boundary of $\mathbb{R} \times I_2$ so that it divides $\mathbb{R} \times I_2$ into the left part and the right part. Note that $C_2 \cap \mathcal{P}_{2,\mathbb{R}} \neq \emptyset$ by Lemma 4.2(1)(a). This together with Lemma 4.2(1)(d) yields that C_2 intersects $\partial\mathcal{B}^+_{i,\mathbb{R}}$ only at its horizontal edges for $i = 0, 3$.

However, thanks to the estimate in equation (4.1), C_2 intersects each of the horizontal edges exactly once by Lemma 4.2(1)(c) and (e). It follows that C_2 divides $\mathcal{B}^+_{i,\mathbb{R}}$ $(i = 0, 3)$ into two connected components, one is in the left part of $\mathbb{R} \times I_2$ and the other is in the right part of $\mathbb{R} \times I_2$. By the uniqueness of the stable manifold $W^s(p_2)$, we see that $C_2 \cap \mathcal{B}^+_{0,\mathbb{R}} = W^s_{\mathrm{loc}}(p_2)_0$ and $C_2 \cap \mathcal{B}^+_{3,\mathbb{R}} = W^s_{\mathrm{loc}}(p_2)_3$. Therefore, it follows that left$(\mathcal{B}^+_{3,\mathbb{R}} \setminus W^s_{\mathrm{loc}}(p_2)_3) \cap$ right$(\mathcal{B}^+_{0,\mathbb{R}} \setminus W^s_{\mathrm{loc}}(p_2)_0) = \emptyset$. A similar argument with Lemma 4.2(2) shows that left$(\mathcal{B}^+_{2,\mathbb{R}} \setminus W^s_{\mathrm{loc}}(p_4)_2) \cap$ right$(\mathcal{B}^+_{1,\mathbb{R}} \setminus W^s_{\mathrm{loc}}(p_4)_1) = \emptyset$. This together with $\mathcal{B}^+_{0,\mathbb{R}} \cap \mathcal{B}^+_{2,\mathbb{R}} \cap K_{\mathbb{R}} = \mathcal{B}^+_{1,\mathbb{R}} \cap \mathcal{B}^+_{3,\mathbb{R}} \cap K_{\mathbb{R}} = \emptyset$ (see item (1) of Lemma 2.4) yields item (1) of Theorem 4.3.

Take $z \in K_{\mathbb{R}} \setminus (W^s(p_2) \cap W^s(p_4))$. Since $z \notin W^s(p_2) \cap W^s(p_4)$, it does not belong to the boundary of the pieces in Proposition 4.1). Hence, there exists a Σ^+-coding $(i_n)_{n\in\mathbb{Z}} \in \mathfrak{S}^+$ of z by Proposition 2.6. Then, Proposition 4.1 yields that z belongs to either P_L^+ or P_R^+. This proves item (2) of Theorem 4.3. □

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Therefore, this defines a unique $\{L, R\}$-coding for every point in $K_{\mathbb{R}} \setminus (W^s(p_2) \cap W^s(p_4))$.

PROPOSITION 4.4. *Suppose that $(a, b) \in \mathcal{E}_{\mathbb{R}}^-$. Let $z \in K_{\mathbb{R}}$ and let $(i_n)_{n\geq 0} \in \mathfrak{S}_{\mathrm{fwd}}^-$ be its forward Σ^--coding.*

(1) *If $i_0 = 0$, then z belongs to the closure of* $\mathrm{left}(\mathcal{B}_{0,\mathbb{R}}^- \setminus W_{\mathrm{loc}}^s(p_1)_0)$.
(2) *If $i_0 = 1$, then z belongs to the closure of* $\mathrm{right}(\mathcal{B}_{1,\mathbb{R}}^- \setminus \gamma_{1,\mathrm{left}})$.
(3) *If $i_0 = 2$, then z belongs to the closure of* $\mathrm{right}(\mathcal{B}_{2,\mathbb{R}}^- \setminus \gamma_{2,\mathrm{left}})$.
(4) *If $i_0 = 3$, then z belongs to the closure of* $\mathrm{right}(\mathcal{B}_{3,\mathbb{R}}^- \setminus \gamma_{3,\mathrm{left}}) \cap \mathrm{left}(\mathcal{B}_{3,\mathbb{R}}^- \setminus \gamma_{3,\mathrm{right}})$.
(5) *If $i_0 = 4$, then z belongs to the closure of* $\mathrm{left}(\mathcal{B}_{4,\mathbb{R}}^- \setminus W_{\mathrm{loc}}^s(p_3)_4)$.

Proof. The proof is similar to that of Proposition 4.1, and hence is omitted. □

LEMMA 4.5. (Numerical Check 6) *Let $(a, b) \in \mathcal{E}_{\mathbb{R}}^-$.*

(1) *One can find two concentric intervals $I_3 = [y_3 - R_3, y_3 + R_3]$ and $I_3' = [y_3 - r_3, y_3 + r_3]$ in the y-axis with $r_3 < R_3$ and a disk D_3 of radius $\delta_3 > 0$ so that:*
   (a) *a projective polydisk of the form $\mathcal{P}_3 = D_3 \times_{\mathrm{pr}} D_{I_3}$ contains p_3;*
   (b) *$f : \mathcal{P}_3 \cap f^{-1}(\mathcal{P}_3) \to \mathcal{P}_3$ is a crossed mapping of degree one;*
   (c) $\mathrm{pr}_y(\mathcal{B}_{2,\mathbb{R}}^- \cup \mathcal{B}_{3,\mathbb{R}}^- \cup \mathcal{B}_{4,\mathbb{R}}^-) \subset I_3'$;
   (d) *$\mathcal{P}_{3,\mathbb{R}} \cap \mathcal{B}_{i,\mathbb{R}}^-$ does not intersect the vertical edges of $\mathcal{B}_{i,\mathbb{R}}^-$ for $i = 2, 3, 4$;*
   (e) *the slopes of the horizontal edges of $\mathcal{B}_{2,\mathbb{R}}^-$, $\mathcal{B}_{3,\mathbb{R}}^-$ and $\mathcal{B}_{4,\mathbb{R}}^-$ in modulus are bounded by*
$$\frac{R_3^2 - r_3^2}{\delta_3^2 + |s_3|(R_3^2 - r_3^2)}$$
   from above, where $(s_3, 1)$ is the v-direction in defining $\mathcal{P}_3$.

(2) *One can find two concentric intervals $I_1 = [y_1 - R_1, y_1 + R_1]$ and $I_1' = [y_1 - r_1, y_1 + r_1]$ in the y-axis with $r_1 < R_1$ and a disk D_1 of radius $\delta_1 > 0$ so that:*
   (a) *a projective polydisk of the form $\mathcal{P}_1 = D_1 \times_{\mathrm{pr}} D_{I_1}$ does not contain p_3;*
   (b) *$f : \mathcal{P}_1 \cap f^{-1}(\mathcal{P}_3) \to \mathcal{P}_3$ is a crossed mapping of degree one;*
   (c) $\mathrm{pr}_y(\mathcal{B}_{1,\mathbb{R}}^- \cup \mathcal{B}_{3,\mathbb{R}}^-) \subset I_1'$;
   (d) *$\mathcal{P}_{1,\mathbb{R}} \cap \mathcal{B}_{i,\mathbb{R}}^-$ does not intersect the vertical edges of $\mathcal{B}_{i,\mathbb{R}}^-$ for $i = 1, 3$;*
   (e) *the slopes of the horizontal edges of $\mathcal{B}_{1,\mathbb{R}}^-$ and $\mathcal{B}_{3,\mathbb{R}}^-$ in modulus are bounded by*
$$\frac{R_1^2 - r_1^2}{\delta_1^2 + |s_1|(R_1^2 - r_1^2)}$$
   from above, where $(s_1, 1)$ is the v-direction in defining $\mathcal{P}_1$.

Proof. See Appendix A. □

As an immediate consequence of Lemma 2.11, we know that $f_{\mathbb{R}}^2(\mathcal{B}_{i',\mathbb{R}}^-)$ and $f_{\mathbb{R}}^2(\mathcal{B}_{i'',\mathbb{R}}^-)$ are connected components of $f_{\mathbb{R}}(f_{\mathbb{R}}(\mathcal{B}_{i,\mathbb{R}}^-) \cap \mathcal{B}_{4,\mathbb{R}}^-) \cap \mathcal{B}_{3,\mathbb{R}}^-$ for $i = 2, 3$. Let P_L^- be the union of $\mathrm{left}(f_{\mathbb{R}}^2(\mathcal{B}_{2',\mathbb{R}}^- \cup \mathcal{B}_{3',\mathbb{R}}^-) \setminus \gamma_{3,\mathrm{right}})$ and $\mathrm{left}(\mathcal{B}_{4,\mathbb{R}}^- \setminus W_{\mathrm{loc}}^s(p_3)_4)$. Let P_R^- be the union of $\mathcal{B}_{0,\mathbb{R}}^-$, $\mathrm{right}(\mathcal{B}_{1,\mathbb{R}}^- \setminus \gamma_{1,\mathrm{left}})$, $\mathrm{right}(\mathcal{B}_{2,\mathbb{R}}^- \setminus \gamma_{2,\mathrm{left}})$, and $\mathrm{right}(f_{\mathbb{R}}^2(\mathcal{B}_{2'',\mathbb{R}}^- \cup \mathcal{B}_{3'',\mathbb{R}}^-) \setminus \gamma_{3,\mathrm{left}})$.

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THEOREM 4.6. *For $(a, b) \in \mathcal{E}_{\mathbb{R}}^{-}$, the pair $\{P_L^-, P_R^-\}$ forms a partition for $K_{\mathbb{R}} \setminus W^s(p_3)$, that is (1) $P_L^- \cap P_R^- \cap K_{\mathbb{R}} = \emptyset$ and (2) $P_L^- \cup P_R^- \supset K_{\mathbb{R}} \setminus W^s(p_3)$.*

Proof. We proceed as in the proof of Theorem 4.3. First, it follows from Lemma 4.5 that $\mathrm{left}(\mathcal{B}_{3,\mathbb{R}}^- \setminus \gamma_{3,\mathrm{right}}) \cap \mathrm{right}(\mathcal{B}_{1,\mathbb{R}}^- \setminus \gamma_{1,\mathrm{left}}) = \mathrm{left}(\mathcal{B}_{4,\mathbb{R}}^- \setminus W^s_{\mathrm{loc}}(p_3)_4)) \cap \mathrm{right}(\mathcal{B}_{2,\mathbb{R}}^- \setminus \gamma_{2,\mathrm{left}}) = \mathrm{left}(\mathcal{B}_{4,\mathbb{R}}^- \setminus W^s_{\mathrm{loc}}(p_3)_4)) \cap \mathrm{right}(\mathcal{B}_{3,\mathbb{R}}^- \setminus \gamma_{3,\mathrm{left}}) = \emptyset$. This together with $\mathcal{B}_{0,\mathbb{R}}^- \cap \mathcal{B}_{3,\mathbb{R}}^- \cap K_{\mathbb{R}} = \mathcal{B}_{0,\mathbb{R}}^- \cap \mathcal{B}_{4,\mathbb{R}}^- \cap K_{\mathbb{R}} = \mathcal{B}_{1,\mathbb{R}}^- \cap \mathcal{B}_{4,\mathbb{R}}^- \cap K_{\mathbb{R}} = \mathcal{B}_{2,\mathbb{R}}^- \cap \mathcal{B}_{3,\mathbb{R}}^- \cap K_{\mathbb{R}} = \emptyset$ (see item (2) of Lemma 2.4) yields item (1) of Theorem 4.6.

The proof of item (2) of Theorem 4.6 is slightly different from that of item (2) of Theorem 4.3 since we replace $\mathcal{B}_{3,\mathbb{R}}^-$ by $f_{\mathbb{R}}^2(\mathcal{B}_{2',\mathbb{R}}^- \cup \mathcal{B}_{3',\mathbb{R}}^-)$ and $f_{\mathbb{R}}^2(\mathcal{B}_{2'',\mathbb{R}}^- \cup \mathcal{B}_{3'',\mathbb{R}}^-)$. Notice that we use $\mathcal{B}_{i',\mathbb{R}}^-$ and $\mathcal{B}_{i'',\mathbb{R}}^-$ $(i = 2, 3)$ to define the partition $\{P_L^-, P_R^-\}$, but we do not use $\mathcal{B}_{2',\mathbb{R}}^+$ and $\mathcal{B}_{2'',\mathbb{R}}^+$ to define the partition $\{P_L^+, P_R^+\}$. Take any $z \in K_{\mathbb{R}} \setminus W^s(p_3)$. Then, it follows from Propositions 2.6 and 2.9(2) that z has a unique Σ^--coding $(i_n)_{n\in\mathbb{Z}} \in \mathfrak{S}^-$. If $i_0 \neq 3$, then Proposition 4.4(1), (2), (3), and (5) yield that z belongs to either P_L^- or P_R^-. So, suppose that $i_0 = 3$. Then, the forward Σ^--coding of $f_{\mathbb{R}}^{-2}(z)$ is either $243\cdots$ or $343\cdots$. Hence, $f_{\mathbb{R}}^{-2}(z) \in (\mathcal{B}_{2,\mathbb{R}}^- \cup \mathcal{B}_{3,\mathbb{R}}^-) \cap f_{\mathbb{R}}^{-1}(\mathcal{B}_{4,\mathbb{R}}^- \cap f_{\mathbb{R}}^{-1}(\mathcal{B}_{3,\mathbb{R}}^-))$. It then follows that

$$
\begin{aligned}
K_{\mathbb{R}} \cap \mathcal{B}_{3,\mathbb{R}}^- &\subset (f_{\mathbb{R}}(f_{\mathbb{R}}(\mathcal{B}_{2,\mathbb{R}}^-) \cap \mathcal{B}_{4,\mathbb{R}}^-) \cap \mathcal{B}_{3,\mathbb{R}}^-) \cup (f_{\mathbb{R}}(f_{\mathbb{R}}(\mathcal{B}_{3,\mathbb{R}}^-) \cap \mathcal{B}_{4,\mathbb{R}}^-) \cap \mathcal{B}_{3,\mathbb{R}}^-) \\
&= f_{\mathbb{R}}^2(\mathcal{B}_{2',\mathbb{R}}^- \cup \mathcal{B}_{3',\mathbb{R}}^-) \cup f_{\mathbb{R}}^2(\mathcal{B}_{2'',\mathbb{R}}^- \cup \mathcal{B}_{3'',\mathbb{R}}^-).
\end{aligned}
\tag{4.2}
$$

Thanks to claim (4) of Proposition 4.4, we observe that $z \in (K_{\mathbb{R}} \cap \mathcal{B}_{3,\mathbb{R}}^-) \setminus W^s(p_3)$ is a point in $\mathrm{right}(\mathcal{B}_{3,\mathbb{R}}^- \setminus \gamma_{3,\mathrm{left}}) \cap \mathrm{left}(\mathcal{B}_{3,\mathbb{R}}^- \setminus \gamma_{3,\mathrm{right}})$. This together with the inclusion in equation (4.2) yields that z belongs to either P_L^- or P_R^-, and hence we have item (2) of Theorem 4.6. □

Therefore, this defines a unique $\{L, R\}$-coding for every point in $K_{\mathbb{R}} \setminus W^s(p_3)$.

4.2. *Proof of Theorem 1.1.* Below, $\mathrm{card}(A)$ denotes the cardinality of A counted without multiplicity. The proof of Theorem 1.1 is based on the following two results from [**AI**].

THEOREM 4.7. [**AI**, Theorem 5.1] *When $(a, b) \in \mathcal{E}_{\mathbb{R}}^{+}$, we have $h_{\mathrm{top}}(f_{\mathbb{R}}) = \log 2$ if and only if $\mathrm{card}(W^s_{31\bar{0}}(a, b)^+ \cap W^u_{\bar{0}23}(a, b)^+) \geq 1$. When $(a, b) \in \mathcal{E}_{\mathbb{R}}^{-}$, we have $h_{\mathrm{top}}(f_{\mathbb{R}}) = \log 2$ if and only if $\mathrm{card}(W^s_{41\bar{0}}(a, b)^- \cap W^u_{\overline{434124}}(a, b)^-_{\mathrm{inner}}) \geq 1$.*

THEOREM 4.8. [**AI**, Theorem 5.14] *When $(a, b) \in \mathcal{E}_{\mathbb{R}}^{+}$, $f_{\mathbb{R}}$ is a hyperbolic horseshoe on $\mathbb{R}^2$ if and only if $\mathrm{card}(W^s_{31\bar{0}}(a, b)^+ \cap W^u_{\bar{0}23}(a, b)^+) = 2$. When $(a, b) \in \mathcal{E}_{\mathbb{R}}^{-}$, $f_{\mathbb{R}}$ is a hyperbolic horseshoe on $\mathbb{R}^2$ if and only if $\mathrm{card}(W^s_{41\bar{0}}(a, b)^- \cap W^u_{\overline{434124}}(a, b)^-_{\mathrm{inner}}) = 2$.*

Let us set

$$S(a, b)^+ \equiv W^s_{31\bar{0}}(a, b)^+ \cap W^u_{\bar{0}23}(a, b)^+$$

and

$$S(a, b)^- \equiv W^s_{41\bar{0}}(a, b)^- \cap W^u_{\overline{434124}}(a, b)^-.$$

Note that we dropped 'inner' from $W^u_{\overline{434124}}(a,b)^-_{\mathrm{inner}}$ in the definition above because the inner part and the outer part of $W^u_{\overline{434124}}(a,b)^-$ cannot be distinguished by the Σ^--coding. As an immediate consequence of Theorems 4.7 and 4.8, we have the following corollary.

COROLLARY 4.9. *We have the following.*

(1) *Let* $(a,b)\in\mathcal{E}^+_{\mathbb{R}}$. *Then, we have* $\mathrm{card}(S(a,b)^+)\le 2$. *Moreover:*
  (a) $f_{\mathbb{R}}$ *is a hyperbolic horseshoe on* $\mathbb{R}^2$ *if and only if* $\mathrm{card}(S(a,b)^+)=2$*;*
  (b) $f_{\mathbb{R}}$ *has a quadratic tangency but* $h_{\mathrm{top}}(f_{\mathbb{R}})=\log 2$ *if and only if* $\mathrm{card}(S(a,b)^+)=1$*;*
  (c) $h_{\mathrm{top}}(f_{\mathbb{R}})<\log 2$ *if and only if* $\mathrm{card}(S(a,b)^+)=0$.

(2) *Let* $(a,b)\in\mathcal{E}^-_{\mathbb{R}}$. *Then, we have* $\mathrm{card}(S(a,b)^-)\le 4$. *Moreover:*
  (a) $f_{\mathbb{R}}$ *is a hyperbolic horseshoe on* $\mathbb{R}^2$ *if and only if* $\mathrm{card}(S(a,b)^-)=4$*;*
  (b) $f_{\mathbb{R}}$ *has a quadratic tangency but* $h_{\mathrm{top}}(f_{\mathbb{R}})=\log 2$ *if and only if* $\mathrm{card}(S(a,b)^-)=3$*;*
  (c) $h_{\mathrm{top}}(f_{\mathbb{R}})<\log 2$ *if and only if* $\mathrm{card}(S(a,b)^-)\le 2$.

Proof. The bound $\mathrm{card}(S(a,b)^+)\le 2$ in case (1) follows from the fact that $V^s_{31\overline{0}}(a,b)^+$ is a vertical disk of degree one in $\mathcal{B}^+_3$ and $V^u_{\overline{0}23}(a,b)^+$ is a horizontal disk of degree two in $\mathcal{B}^+_3$. The bound $\mathrm{card}(S(a,b)^-)\le 4$ in case (2) follows from the fact that $V^s_{41\overline{0}}(a,b)^-$ is a vertical disk of degree one in $\mathcal{B}^-_4$ and $V^u_{\overline{434124}}(a,b)^-$ consists of two horizontal disks of degree two in $\mathcal{B}^-_4$.

Consider first the classification in case (2). If $W^s_{41\overline{0}}(a,b)^-$ intersects $W^u_{\overline{434124}}(a,b)^-_{\mathrm{inner}}$, then it follows that $W^s_{41\overline{0}}(a,b)^-$ intersects $W^u_{\overline{434124}}(a,b)^-_{\mathrm{outer}}$ exactly twice (see [**AI**, Figure 25 as well as Definition 4.4]). Hence,

$$\mathrm{card}(W^s_{41\overline{0}}(a,b)^-\cap W^u_{\overline{434124}}(a,b)^-_{\mathrm{inner}})=\max\{0,\mathrm{card}(S(a,b)^-)-2\}.$$

Then, claim (a) follows from Theorem 4.8, claim (c) follows from Theorem 4.7 and that (a,b) belongs to the closure of $\mathcal{H}^{\pm}_{\mathbb{R}}$ if and only if $h_{\mathrm{top}}(f_{a,b})=\log 2$ (see [**AI**, Main Corollary]), and claim (b) follows from claims (a), (c), and that $\mathcal{H}^{\pm}_{\mathbb{R}}$ is open. Proof for the classification in case (1) is similar and hence omitted. □

Thanks to this corollary, our task to finish the proof of Theorem 1.1 is to transfer the $\Sigma^{\pm}$-coding to the $\{L,R\}$-coding. Notice that the paper [**BS$_{\mathbb{R}}$3**] lacks a proof of its theorem 1 which corresponds to our Theorem 1.1.

Proof of Theorem 1.1(1). For a point $z\in K_{\mathbb{R}}\setminus(W^s(p_2)\cap W^s(p_4))$, we consider the following conditions:

(a) the Σ^+-coding of z exists and is $\overline{0}2\cdot 31\overline{0}$;
(b) $f^{-2}(z)\in W^u_{\mathrm{loc}}(p_1)_0$;
(c) $f^2(z)\in W^s_{\mathrm{loc}}(p_1)_0$;
(d) the $\{L,R\}$-coding of z is $\overline{R}L\cdot L\overline{R}$.

It is easy to see that $z\in S^+(a,b)$ if and only if z satisfies conditions (a), (b), and (c). Assume that condition (a) holds. Then, the forward Σ^+-coding of $f^2(z)$ is $\overline{0}\in\mathfrak{S}^+_{\mathrm{fwd}}$. Since

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$f : \mathcal{B}_0^+ \cap f^{-1}(\mathcal{B}_0^+) \to \mathcal{B}_0^+$ is a crossed mapping of degree one, this yields condition (c) as in Corollary 3.5(1). Since the backward Σ^+-coding of $f^{-2}(z)$ including the 0th digit is $\overline{0} \in \mathfrak{S}_{\mathrm{bwd}}^+$, a similar argument yields condition (b). Therefore, it is sufficient to prove that condition (a) holds if and only if condition (d) holds. Note that $\overline{R}L \cdot L\overline{R} = \overline{R}L \cdot LR\overline{R}$.

(a) implies (d): Since $z \notin W^s(p_2) \cup W^s(p_4)$, Proposition 4.1 and the definition of the partition $\{P_L^+, P_R^+\}$ imply the following.

- If $i_0 = 0$, then $z \in \mathrm{right}(\mathcal{B}_{0,\mathbb{R}}^+ \setminus W_{\mathrm{loc}}^s(p_2)_0) \subset P_R^+$.
- If $i_0 = 1$, then $z \in \mathrm{right}(\mathcal{B}_{1,\mathbb{R}}^+ \setminus W_{\mathrm{loc}}^s(p_4)_1) \subset P_R^+$.
- If $i_0 = 2$, then $z \in \mathrm{left}(\mathcal{B}_{2,\mathbb{R}}^+ \setminus W_{\mathrm{loc}}^s(p_4)_2) \subset P_L^+$.
- If $i_0 = 3$, then $z \in \mathrm{left}(\mathcal{B}_{3,\mathbb{R}}^+ \setminus W_{\mathrm{loc}}^s(p_2)_3) \subset P_L^+$.

Therefore, we replace the 0s and 1s in $\overline{0}2 \cdot 31\overline{0}$ by R, and the 2s and 3s in $\overline{0}2 \cdot 31\overline{0}$ by L to obtain $\overline{R}L \cdot LR\overline{R}$.

(d) implies (a): We note that z has a unique Σ^+-coding thanks to Proposition 2.6 and claim (1) of Proposition 2.9.

First consider the forward $\{L, R\}$-coding $LR\overline{R}$. By the definition of the $\{L, R\}$-partition, the 0th digit of the Σ^+-coding is either 2 or 3. There is no arrow from 2 to the R-side in the transition graph, but there is an arrow $3 \to 1$ from 3 to the R-side. Hence, the 0th digit of the Σ^+-coding is 3 and the 1st digit is 1. Then, the only arrow from 1 to the R-side is $1 \to 0$. By repeating this argument, we obtain $31\overline{0}$ as the forward Σ^+-coding.

Next, consider the backward $\{L, R\}$-coding $\overline{R}LL$ (including the 0th digit). As discussed above, the 0th digit of the Σ^+-coding is 3. The only arrow in the transition graph from the L-side to 3 is $2 \to 3$, and hence the -1th digit is 2. Then, there are two arrows $0 \to 2$ and $1 \to 2$ from the R-side to 2. There is no arrow from the R-side to 1, but there is an arrow $1 \to 2$ from the R-side. Hence, the -2th digit of the Σ^+-coding is 1. By repeating this argument, we obtain $\overline{0}23$ as the backward Σ^+-coding including the 0th digit.

By combining the forward and the backward Σ^+-codings above, we obtain condition (a). $\square$

Proof of Theorem 1.1(2). For a point $z \in K_{\mathbb{R}} \setminus W^s(p_3)$, we consider the following conditions:

(a) the Σ^--coding of z exists and is $\overline{43}412 \cdot 41\overline{0}$;
(b) $f^{-4}(z) \in W_{\mathrm{loc}}^u(p_3)_3$;
(c) $f^2(z) \in W_{\mathrm{loc}}^s(p_1)_0$;
(d) the $\{L, R\}$-coding of z is $\overline{L}RR \cdot L\overline{R}$.

It is easy to see that $z \in S^-(a, b)$ if and only if z satisfies conditions (a), (b), and (c). Assume that condition (a) holds. Then, the forward Σ^--coding of $f^2(z)$ is $\overline{0} \in \mathfrak{S}_{\mathrm{fwd}}^-$. Since $f : \mathcal{B}_0^- \cap f^{-1}(\mathcal{B}_0^-) \to \mathcal{B}_0^-$ is a crossed mapping of degree one, this yields condition (c) as in Corollary 3.6(1). Therefore, it is sufficient to prove that conditions (a) and (b) hold if and only if condition (d) holds. Note that $\overline{L}RR \cdot L\overline{R} = \overline{LL}LRR \cdot LR\overline{R}$.

(a) and (b) imply (d): Since $z \notin W^s(p_3)$, Proposition 4.4 and the definition of the partition $\{P_L^-, P_R^-\}$ imply the following.

- If $i_0 = 0$, then $z \in \mathrm{left}(\mathcal{B}_{0,\mathbb{R}}^- \setminus W_{\mathrm{loc}}^s(p_1)_0) \subset P_R^-$.
- If $i_0 = 1$, then $z \in \mathrm{right}(\mathcal{B}_{1,\mathbb{R}}^- \setminus \gamma_{1,\mathrm{left}}) \subset P_R^-$.

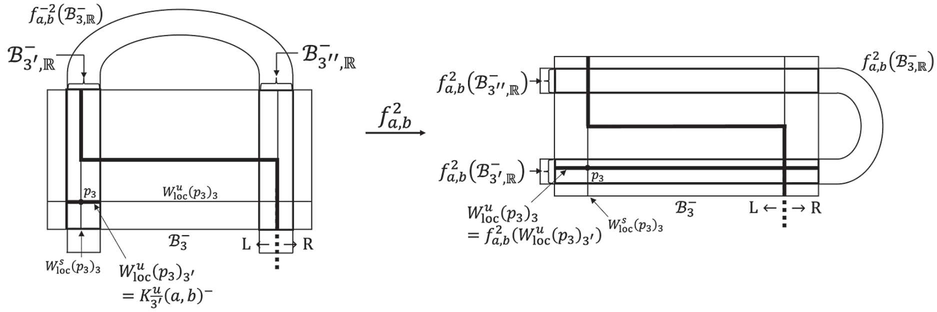

FIGURE 8. The real box $\mathcal{B}_{3,\mathbb{R}}^-$.

- If $i_0 = 2$, then $z \in \mathrm{right}(\mathcal{B}_{2,\mathbb{R}}^- \setminus \gamma_{2,\mathrm{left}}) \subset P_R^-$.
- If $i_0 = 3$, then $z \in \mathrm{right}(\mathcal{B}_{3,\mathbb{R}}^- \setminus \gamma_{3,\mathrm{left}}) \cap \mathrm{left}(\mathcal{B}_{3,\mathbb{R}}^- \setminus \gamma_{3,\mathrm{right}})$.
- If $i_0 = 4$, then $z \in \mathrm{left}(\mathcal{B}_{4,\mathbb{R}}^- \setminus W^s_{\mathrm{loc}}(p_3)_4) \subset P_L^-$.

Therefore, we replace the 0s, 1s, and 2s in $\overline{43}412 \cdot 41\overline{0}$ by R, and the 4s in $\overline{43}412 \cdot 41\overline{0}$ by L to obtain $\overline{L{*}}LRR \cdot LR\overline{R}$.

The problem here is that $\mathcal{B}_{3,\mathbb{R}}^-$ is divided into both a part of P_L^- and a part of P_R^-, and hence we need to determine the letters in the $-(2n+4)$th digits ($n \geq 0$) which are denoted by $*$ above. Here, we claim that

$$f_{\mathbb{R}}^{-(2n+4)}(z) \in W^u_{\mathrm{loc}}(p_3)_3 \tag{4.3}$$

holds for all $n \geq 0$. Indeed, the case $n = 0$ is identical to assumption (b) and we see that $f_{\mathbb{R}}^{-(2(n+1)+4)}(z) = f_{\mathbb{R}}^{-2}(f_{\mathbb{R}}^{-(2n+4)}(z)) \in f_{\mathbb{R}}^{-2}(W^u_{\mathrm{loc}}(p_3)_3) \subset W^u_{\mathrm{loc}}(p_3)_3$, and hence equation (4.3) is proved. The claim in equation (4.3) together with case (4) of Proposition 4.4 implies that $f_{\mathbb{R}}^{-(2n+4)}(z) \in P_L^-$ (see Figure 8). Therefore, we conclude $* = L$ and condition (d) follows.

(d) implies (a): We note that z has a unique Σ^--coding thanks to Proposition 2.6 and case (2) of Proposition 2.9. Then, the rest of the argument is similar to the proof of '(d) implies (a)' in the case $b > 0$, and hence is omitted.

(d) implies (b): A discussion similar to the previous paragraph yields that the only transition with $LRRL$ is 4124. Therefore, the backward Σ^--coding of z including the 0th digit is $\overline{43}4124 \in \mathfrak{S}^+_{\mathrm{bwd}}$. Moreover, by case (3) of Proposition 4.4 and the definition of the partition piece P_L^-, we see that $f_{\mathbb{R}}^{-(2n+4)}(z) \in f_{\mathbb{R}}^2(\mathcal{B}_{3',\mathbb{R}}^-)$. This yields $f_{\mathbb{R}}^{-(2n+6)}(z) \in \mathcal{B}_{3',\mathbb{R}}^-$ for all $n \geq 0$. Since $f^2 : \mathcal{B}_{3'}^- \to \mathcal{B}_3^-$ is a crossed mapping of degree one, we have $f_{\mathbb{R}}^{-(2n+6)}(z) \in W^u_{\mathrm{loc}}(p_3)_{p_3'}$ (compare item (2) of Corollary 3.6). Hence, $f_{\mathbb{R}}^{-4}(z) \in f_{\mathbb{R}}^2(W^u_{\mathrm{loc}}(p_3)_{3'}) = W^u_{\mathrm{loc}}(p_3)_3$ and claim (b) is proved. □

Remark 4.10

(a) When $b < 0$, the crossed mapping $f^2 : \mathcal{B}_3^- \cap f^{-1}(\mathcal{B}_4^- \cap f^{-1}(\mathcal{B}_3^-)) \to \mathcal{B}_3^-$ has degree two. Then the set $K^u_{\overline{43}}$ consists of infinitely many horizontal disks in $\mathcal{B}_3^-$, and $K^u_{\overline{43}4124} \cap K^s_{41\overline{0}}$ consists of infinitely many points. Hence, we cannot classify the dynamics of the Hénon maps only by the number of points with the Σ^--coding $\overline{43}412 \cdot 41\overline{0}$. In Corollary 4.9(2), we classify the dynamics by $S(a, b)^-$ instead. Here,

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$S(a,b)^-$ is the number of intersection points of $W^u_{\overline{434124}}(a,b)^-$ and $W^s_{41\overline{0}}(a,b)^-$, and $W^u_{\overline{434124}}(a,b)^-$ is defined as a certain preimage of the local unstable manifold $W^u_{\overline{43}}(a,b)^-$, one of the disks which $K^u_{\overline{43}}$ consists of.

(b) Our goal was to classify the dynamics in terms of $\{L, R\}$-coding. Hence, our $\{L, R\}$-partition must distinguish $W^u_{\overline{43}}(a,b)^-$ from the other horizontal disks in $K^u_{\overline{43}}$ (more precisely, claim (d) must imply claim (a) in the proof of Theorem 1.1(2)). This is the reason why we refined the box $\mathcal{B}^-_3$ into two parts $\mathcal{B}^-_{3'}$ and $\mathcal{B}^-_{3''}$ so that $f^2 : \mathcal{B}^-_{3'} \to \mathcal{B}^-_{3'}$ has degree one and hence we have $K^u_{\overline{43'}} = W^u_{\overline{43}}(a,b)^-$.

(c) However, since $f : \mathcal{B}^\pm_0 \cap f^{-1}(\mathcal{B}^\pm_0) \to \mathcal{B}^\pm_0$ is degree one, $K^{u/s}_{\overline{0}}$ are disks of degree one without any refinement.

5. *Dynamics at the boundary* $\partial\mathcal{H}^\pm_{\mathbb{R}}$

The purpose of this section is to prove Theorem 1.2.

For simplicity, we write $\mathrm{left}(\mathcal{B}^+_{j,\mathbb{R}}) \equiv \mathrm{left}(\mathcal{B}^+_{j,\mathbb{R}} \setminus \gamma^+_{j,\mathrm{left}})$ $(j = 0, 1, 2)$ and $\mathrm{right}(\mathcal{B}^+_{j,\mathbb{R}}) \equiv \mathrm{right}(\mathcal{B}^+_{j,\mathbb{R}} \setminus \gamma^+_{j,\mathrm{right}})$ $(j = 2, 3)$ below. Let $\mathfrak{M}^+_1$ be the family of these sets. Similarly, we write $\mathrm{left}(\mathcal{B}^-_{j,\mathbb{R}}) \equiv \mathrm{left}(\mathcal{B}^-_{j,\mathbb{R}} \setminus \gamma^-_{j,\mathrm{left}})$ $(j = 0, 1, 2, 3)$ and $\mathrm{right}(\mathcal{B}^-_{j,\mathbb{R}}) \equiv \mathrm{right}(\mathcal{B}^-_{j,\mathbb{R}} \setminus \gamma^-_{j,\mathrm{right}})$ $(j = 2, 3, 4)$ below. Let $\mathfrak{M}^-_1$ be the family of these sets. We call the elements of $\mathfrak{M}^\pm_1$ *margins* of $\mathcal{B}^\pm_{j,\mathbb{R}}$.

5.1. *Refined dynamics.*

We first refine local stable/unstable manifolds for our purpose. The proofs of the following lemmas are analogous to Lemmas 3.1 and 3.2, and hence are omitted.

LEMMA 5.1. *Let $(a,b) \in \mathcal{E}^+_{\mathbb{R}}$. Then, we have the following.*

(1) *$V^s_{\overline{0}}(a,b)^+$ is a vertical disk of degree one in $\mathcal{B}^+_0$ and $V^s_{\overline{0}}(a,b)^+ = K^s_{\overline{0}}$.*
(2) *$V^s_{1\overline{0}}(a,b)^+$ is a vertical disk of degree one in $\mathcal{B}^+_1$ and $V^s_{1\overline{0}}(a,b)^+ = K^s_{1\overline{0}}$.*
(3) *$V^s_{31\overline{0}}(a,b)^+$ is a vertical disk of degree one in $\mathcal{B}^+_3$ and $V^s_{31\overline{0}}(a,b)^+ = K^s_{31\overline{0}}$.*
(4) *$V^u_{\overline{0}}(a,b)^+$ is a horizontal disk of degree one in $\mathcal{B}^+_0$ and $V^u_{\overline{0}}(a,b)^+ = K^u_{\overline{0}}$.*
(5) *$V^u_{\overline{0}3}(a,b)^+$ is a horizontal disk of degree two in $\mathcal{B}^+_3$ and $V^u_{\overline{0}3}(a,b)^+ = K^u_{\overline{0}3}$.*
(6) *$V^u_{\overline{0}2}(a,b)^+$ is a horizontal disk of degree one in $\mathcal{B}^+_2$ and $V^u_{\overline{0}2}(a,b)^+ = K^u_{\overline{0}2}$.*
(7) *$V^u_{\overline{0}31}(a,b)^+$ is a horizontal disk of degree two in $\mathcal{B}^+_1$ and $V^u_{\overline{0}31}(a,b)^+ = K^u_{\overline{0}31}$.*
(8) *$V^u_{\overline{0}23}(a,b)^+$ is a horizontal disk of degree two in $\mathcal{B}^+_3$ and $V^u_{\overline{0}23}(a,b)^+ = K^u_{\overline{0}23}$.*
(9) *$V^u_{\overline{0}22}(a,b)^+$ is a horizontal disk of degree one in $\mathcal{B}^+_2$ and $V^u_{\overline{0}22}(a,b)^+ = K^u_{\overline{0}22}$.*
(10) *$V^u_{\overline{0}310}(a,b)^+$ is a horizontal disk of degree two in $\mathcal{B}^+_0$ and $V^u_{\overline{0}310}(a,b)^+ = K^u_{\overline{0}310}$.*

LEMMA 5.2. *Let $(a,b) \in \mathcal{E}^-_{\mathbb{R}}$. Then, we have the following.*

(1) *$V^s_{\overline{0}}(a,b)^-$ is a vertical disk of degree one in $\mathcal{B}^-_0$ and $V^s_{\overline{0}}(a,b)^- = K^s_{\overline{0}}$.*
(2) *$V^s_{1\overline{0}}(a,b)^-$ is a vertical disk of degree one in $\mathcal{B}^-_1$ and $V^s_{1\overline{0}}(a,b)^- = K^s_{1\overline{0}}$.*
(3) *$V^s_{41\overline{0}}(a,b)^-$ is a vertical disk of degree one in $\mathcal{B}^-_4$ and $V^s_{41\overline{0}}(a,b)^- = K^s_{41\overline{0}}$.*
(4) *$V^u_{\overline{43}}(a,b)^-$ is a horizontal disk of degree one in $\mathcal{B}^-_3$ and $V^u_{\overline{43}}(a,b)^- \subset K^u_{\overline{43}}$.*
(5) *$V^u_{\overline{43}4}(a,b)^-$ is a horizontal disk of degree two in $\mathcal{B}^-_4$ and $V^u_{\overline{43}4}(a,b)^- \subset K^u_{\overline{43}4}$.*

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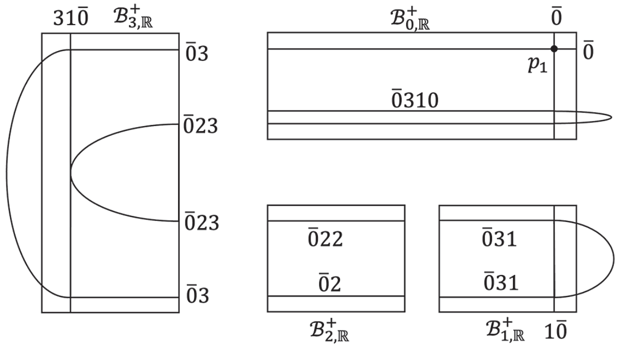

FIGURE 9. The Σ^+-coding of pieces of stable/unstable manifolds for $b > 0$.

(6) $V^u_{\overline{4343}}(a, b)^-$ *consists of two mutually disjoint disks of degree one in* $\mathcal{B}^-_3$. *Moreover,* $V^u_{\overline{4343}}(a, b)^- \subset K^u_{\overline{43}}$.

(7) $V^u_{\overline{4341}}(a, b)^-$ *is a horizontal disk of degree two in* $\mathcal{B}^-_1$ *and* $V^u_{\overline{4341}}(a, b)^- \subset K^u_{\overline{4341}}$.

(8) $V^u_{\overline{43410}}(a, b)^-$ *is a horizontal disk of degree two in* $\mathcal{B}^-_0$ *and* $V^u_{\overline{43410}}(a, b)^- \subset K^u_{\overline{43410}}$.

(9) $V^u_{\overline{43412}}(a, b)^-$ *consists of two mutually disjoint disks of degree one in* $\mathcal{B}^-_2$. *Moreover,* $V^u_{\overline{43412}}(a, b)^- \subset K^u_{\overline{43412}}$.

(10) $V^u_{\overline{434100}}(a, b)^-$ *is a horizontal disk of degree one in* $\mathcal{B}^-_0$ *and* $V^u_{\overline{434100}}(a, b)^- \subset K^u_{\overline{434100}}$.

(11) $V^u_{\overline{434102}}(a, b)^-$ *consists of two mutually disjoint disks of degree one in* $\mathcal{B}^-_2$. *Moreover,* $V^u_{\overline{434102}}(a, b)^- \subset K^u_{\overline{434102}}$.

(12) $V^u_{\overline{434124}}(a, b)^-$ *consists of two mutually disjoint disks of degree two in* $\mathcal{B}^-_4$. *Moreover,* $V^u_{\overline{434124}}(a, b)^- \subset K^u_{\overline{434124}}$.

The corollary below follows from Lemma 5.1 (see Figure 9).

COROLLARY 5.3. *Let* $(a, b) \in \partial\mathcal{H}^+_{\mathbb{R}}$. *Then, we have the following.*

(1) $W^s_{\bar{0}}(a, b)^+$ *is a vertical curve in* $\mathcal{B}^+_{0,\mathbb{R}}$ *and* $W^s_{\bar{0}}(a, b)^+ = K^s_{\bar{0}} \cap \mathbb{R}^2$.

(2) $W^s_{1\bar{0}}(a, b)^+$ *is a vertical curve in* $\mathcal{B}^+_{1,\mathbb{R}}$ *and* $W^s_{1\bar{0}}(a, b)^+ = K^s_{1\bar{0}} \cap \mathbb{R}^2$.

(3) $W^s_{31\bar{0}}(a, b)^+$ *is a vertical curve in* $\mathcal{B}^+_{3,\mathbb{R}}$ *and* $W^s_{31\bar{0}}(a, b)^+ = K^s_{31\bar{0}} \cap \mathbb{R}^2$.

(4) $W^u_{\bar{0}}(a, b)^+$ *is a horizontal curve in* $\mathcal{B}^+_{0,\mathbb{R}}$ *and* $W^u_{\bar{0}}(a, b)^+ = K^u_{\bar{0}} \cap \mathbb{R}^2$.

(5) *The intersection of* $W^u_{\bar{0}3}(a, b)^+$ *and* $\mathcal{B}^+_{3,\mathbb{R}} \setminus \overline{\mathrm{left}(\mathcal{B}^+_{3,\mathbb{R}} \setminus W^s_{31\bar{0}}(a, b)^+)}$ *consists of two horizontal curves in* $\mathcal{B}^+_{3,\mathbb{R}} \setminus \overline{\mathrm{left}(\mathcal{B}^+_{3,\mathbb{R}} \setminus W^s_{31\bar{0}}(a, b)^+)}$ *and* $W^u_{\bar{0}3}(a, b)^+ = K^u_{\bar{0}3} \cap \mathbb{R}^2$.

(6) $W^u_{\bar{0}2}(a, b)^+$ *is a horizontal curve in* $\mathcal{B}^+_{2,\mathbb{R}}$ *and* $W^u_{\bar{0}2}(a, b)^+ = K^u_{\bar{0}2} \cap \mathbb{R}^2$.

(7) *The intersection of* $W^u_{\bar{0}31}(a, b)^+$ *and* $\mathcal{B}^+_{1,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^+_{1,\mathbb{R}} \setminus W^s_{1\bar{0}}(a, b)^+)}$ *consists of two horizontal curves in* $\mathcal{B}^+_{1,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^+_{1,\mathbb{R}} \setminus W^s_{1\bar{0}}(a, b)^+)}$ *and* $W^u_{\bar{0}31}(a, b)^+ = K^u_{\bar{0}31} \cap \mathbb{R}^2$.

(8) $W^u_{\bar{0}23}(a, b)^+$ *is a horizontal curve in* $\mathcal{B}^+_{3,\mathbb{R}}$ *from its right boundary to itself. Moreover,* $W^u_{\bar{0}23}(a, b)^+ = K^u_{\bar{0}23} \cap \mathbb{R}^2$ *and this curve is tangent to* $W^s_{31\bar{0}}(a, b)^+$ *at just one point.*

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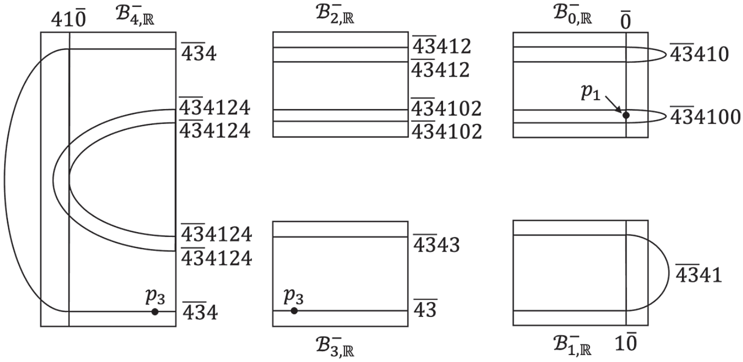

FIGURE 10. The Σ^--coding of pieces of stable/unstable manifolds for $b < 0$.

(9) *$W^u_{\bar{0}22}(a,b)^+$ is a horizontal curve in $\mathcal{B}^+_{2,\mathbb{R}}$ and $W^u_{\bar{0}22}(a,b)^+ = K^u_{\bar{0}22} \cap \mathbb{R}^2$.*

(10) *The intersection of $W^u_{\bar{0}310}(a,b)^+$ and $\mathcal{B}^+_{0,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^+_{0,\mathbb{R}} \setminus W^s_{\bar{0}}(a,b)^+)}$ consists of two horizontal curves in $\mathcal{B}^+_{0,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^+_{0,\mathbb{R}} \setminus W^s_{\bar{0}}(a,b)^+)}$ and $W^u_{\bar{0}310}(a,b)^+ = K^u_{\bar{0}310} \cap \mathbb{R}^2$.*

Proof. The claims (1)–(4), (6), and (9) immediately follow from Lemma 5.1. Claim (8) follows from this lemma and the assumption that $W^u_{\bar{0}23}(a,b)^+$ is tangent to $W^s_{31\bar{0}}(a,b)^+$ at just one point. Since $W^u_{\bar{0}3}(a,b)^+$ is further out than $W^u_{\bar{0}23}(a,b)^+$, $W^u_{\bar{0}23}(a,b)^+$ intersects with $W^s_{31\bar{0}}(a,b)^+$ at distinct two points. This proves the claim (5). Claim (7) follows from claim (5), and claim (10) follows from claim (7). □

Similarly, we have the corollary below from Lemma 5.2 (see Figure 10). The proof is analogous to that of Corollary 5.3, and hence is omitted.

COROLLARY 5.4. *Let $(a,b) \in \partial\mathcal{H}^-_{\mathbb{R}}$. Then, we have the following.*

(1) *$W^s_{\bar{0}}(a,b)^-$ is a vertical curve in $\mathcal{B}^-_{0,\mathbb{R}}$ and $W^s_{\bar{0}}(a,b)^- = K^s_{\bar{0}} \cap \mathbb{R}^2$.*

(2) *$W^s_{1\bar{0}}(a,b)^-$ is a vertical curve in $\mathcal{B}^-_{1,\mathbb{R}}$ and $W^s_{1\bar{0}}(a,b)^- = K^s_{1\bar{0}} \cap \mathbb{R}^2$.*

(3) *$W^s_{41\bar{0}}(a,b)^-$ is a vertical curve in $\mathcal{B}^-_{4,\mathbb{R}}$ and $W^s_{41\bar{0}}(a,b)^- = K^s_{41\bar{0}} \cap \mathbb{R}^2$.*

(4) *$W^u_{\overline{43}}(a,b)^-$ is a horizontal curve in $\mathcal{B}^-_{3,\mathbb{R}}$ and $W^u_{\overline{43}}(a,b)^- \subset K^u_{\overline{43}} \cap \mathbb{R}^2$.*

(5) *The intersection of $W^u_{\overline{43}4}(a,b)^-$ and $\mathcal{B}^-_{4,\mathbb{R}} \setminus \mathrm{left}(\mathcal{B}^-_{4,\mathbb{R}} \setminus W^s_{41\bar{0}}(a,b)^-)$ consists of two horizontal curves in $\mathcal{B}^-_{4,\mathbb{R}} \setminus \mathrm{left}(\mathcal{B}^-_{4,\mathbb{R}} \setminus W^s_{41\bar{0}}(a,b)^-)$ and $W^u_{\overline{43}4}(a,b)^- \subset K^u_{\overline{43}4} \cap \mathbb{R}^2$.*

(6) *$W^u_{\overline{43}43}(a,b)^-$ consists of two mutually disjoint horizontal curves in $\mathcal{B}^-_{3,\mathbb{R}}$. Moreover, $W^u_{\overline{43}43}(a,b)^- \subset K^u_{\overline{43}} \cap \mathbb{R}^2$.*

(7) *The intersection of $W^u_{\overline{43}41}(a,b)^-$ and $\mathcal{B}^-_{1,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^-_{1,\mathbb{R}} \setminus W^s_{1\bar{0}}(a,b)^-)}$ consists of two horizontal curves in $\mathcal{B}^-_{1,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^-_{1,\mathbb{R}} \setminus W^s_{1\bar{0}}(a,b)^-)}$ and $W^u_{\overline{43}41}(a,b)^- \subset K^u_{\overline{43}41} \cap \mathbb{R}^2$.*

(8) *The intersection of $W^u_{\overline{43}410}(a,b)^-$ and $\mathcal{B}^-_{0,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^-_{0,\mathbb{R}} \setminus W^s_{\bar{0}}(a,b)^-)}$ consists of two horizontal curves in $\mathcal{B}^-_{0,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^-_{0,\mathbb{R}} \setminus W^s_{\bar{0}}(a,b)^-)}$ and $W^u_{\overline{43}410}(a,b)^- \subset K^u_{\overline{43}410} \cap \mathbb{R}^2$.*

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(9) $W^u_{\overline{43}412}(a, b)^-$ *consists of two mutually disjoint horizontal curves in* $\mathcal{B}^-_{2,\mathbb{R}}$. *Moreover,* $W^u_{\overline{43}412}(a, b)^- \subset K^u_{\overline{43}412} \cap \mathbb{R}^2$.

(10) *The intersection of* $W^u_{\overline{43}4100}(a, b)^-$ *and* $\mathcal{B}^-_{0,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^-_{0,\mathbb{R}} \setminus W^s_{\bar{0}}(a, b)^-)}$ *consists of two horizontal curves in* $\mathcal{B}^-_{0,\mathbb{R}} \setminus \overline{\mathrm{right}(\mathcal{B}^-_{0,\mathbb{R}} \setminus W^s_{\bar{0}}(a, b)^-)}$ *and* $W^u_{\overline{43}4100}(a, b)^- \subset K^u_{\overline{43}4100} \cap \mathbb{R}^2$.

(11) $W^u_{\overline{43}4102}(a, b)^-$ *consists of two mutually disjoint horizontal curves in* $\mathcal{B}^-_{2,\mathbb{R}}$. *Moreover,* $W^u_{\overline{43}4102}(a, b)^- \subset K^u_{\overline{43}4102} \cap \mathbb{R}^2$.

(12) $W^u_{\overline{43}4124}(a, b)^-$ *consists of two mutually disjoint horizontal curves in* $\mathcal{B}^-_{4,\mathbb{R}}$ *from the right boundary to itself and* $W^u_{\overline{43}4124}(a, b)^- \subset K^u_{\overline{43}4124} \cap \mathbb{R}^2$. *Moreover, the inner one of these two curves is* $W^u_{\overline{43}4124}(a, b)^-_{\mathrm{inner}}$ *and this curve is tangent to* $W^u_{41\bar{0}}(a, b)^-$ *at just one point.*

The following definitions of pieces of stable/unstable manifolds are necessary to construct the $\{\alpha, \beta\}$-partition for the proof of Theorem 1.2. First, we let $(a, b) \in \partial\mathcal{H}^+_{\mathbb{R}}$.

- We write $\gamma^+_{0,\mathrm{right}}$ as $W^s_{\bar{0}}(a, b)^+$.
- We write $\gamma^+_{1,\mathrm{right}}$ as $W^s_{1\bar{0}}(a, b)^+$.
- We write $\gamma^+_{3,\mathrm{left}}$ as $W^s_{31\bar{0}}(a, b)^+$.
- We write $\gamma^+_{0,\mathrm{upper}}$ as $W^u_{\bar{0}}(a, b)^+$.
- We write $\gamma^+_{3,\mathrm{upper}}$ (respectively $\gamma^+_{3,\mathrm{lower}}$) as the upper (respectively lower) connected component of $W^u_{\bar{0}3}(a, b)^+ \cap (\mathcal{B}^+_{3,\mathbb{R}} \setminus \mathrm{left}(\mathcal{B}^+_{3,\mathbb{R}} \setminus \gamma^+_{3,\mathrm{left}}))$.
- We write $\gamma^+_{2,\mathrm{upper}}$ (respectively $\gamma^+_{2,\mathrm{lower}}$) as the upper (respectively lower) connected component of $W^u_{\bar{0}2}(a, b)^+ \cap \mathcal{B}^+_{2,\mathbb{R}}$.
- We write $\gamma^+_{1,\mathrm{upper}}$ (respectively $\gamma^+_{1,\mathrm{lower}}$) as the upper (respectively lower) connected component of $W^u_{\bar{0}31}(a, b)^+ \cap (\mathcal{B}^+_{1,\mathbb{R}} \setminus \mathrm{right}(\mathcal{B}^+_{1,\mathbb{R}} \setminus \gamma^+_{1,\mathrm{right}}))$.
- We write $\gamma^+_{3,\mathrm{inner}}$ as $W^u_{\bar{0}23}(a, b)^+$.
- We write $\gamma^+_{2,\mathrm{upper}}$ as $W^u_{\bar{0}22}(a, b)^+$.
- We write $\gamma^+_{0,\mathrm{lower}}$ as the lower connected component of $W^u_{\bar{0}310}(a, b)^+ \cap (\mathcal{B}^+_{0,\mathbb{R}} \setminus \mathrm{right}(\mathcal{B}^+_{0,\mathbb{R}} \setminus \gamma^+_{0,\mathrm{right}}))$.

See Figure 11 for more details. Next, we let $(a, b) \in \partial\mathcal{H}^-_{\mathbb{R}}$.

- We write $\gamma^-_{0,\mathrm{right}}$ as $W^s_{\bar{0}}(a, b)^-$.
- We write $\gamma^-_{1,\mathrm{right}}$ as $W^s_{1\bar{0}}(a, b)^-$.
- We write $\gamma^-_{4,\mathrm{left}}$ as $W^s_{41\bar{0}}(a, b)^-$.
- We write $\gamma^-_{3,\mathrm{lower}}$ as $W^u_{\overline{43}}(a, b)^-$.
- We write $\gamma^-_{4,\mathrm{upper}}$ (respectively $\gamma^-_{4,\mathrm{lower}}$) as the upper (respectively lower) connected component of $W^u_{\overline{43}4}(a, b)^- \cap (\mathcal{B}^-_{4,\mathbb{R}} \setminus \mathrm{left}(\mathcal{B}^-_{4,\mathbb{R}} \setminus \gamma^-_{4,\mathrm{left}}))$.
- We write $\gamma^-_{3,\mathrm{upper}}$ (respectively $\gamma^-_{3,\mathrm{lower}}$) as the upper (respectively lower) connected component of $W^u_{\overline{43}43}(a, b)^-$.
- We write $\gamma^-_{1,\mathrm{upper}}$ (respectively $\gamma^-_{1,\mathrm{lower}}$) as the upper (respectively lower) connected component of $W^u_{\overline{43}41}(a, b)^- \cap (\mathcal{B}^-_{1,\mathbb{R}} \setminus \mathrm{right}(\mathcal{B}^-_{1,\mathbb{R}} \setminus \gamma^-_{1,\mathrm{right}}))$.

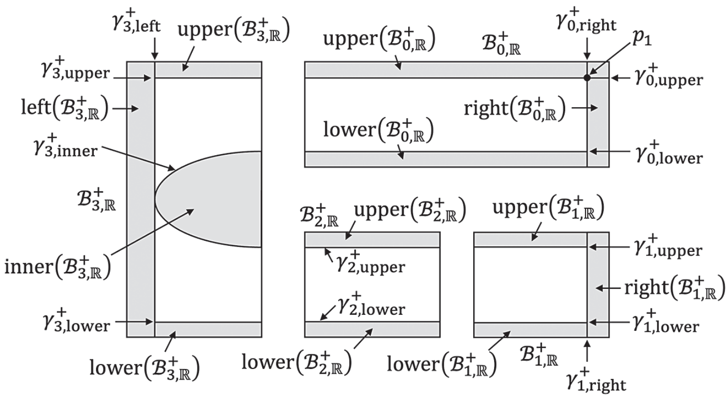

FIGURE 11. Margins in $\mathfrak{M}_2^+$.

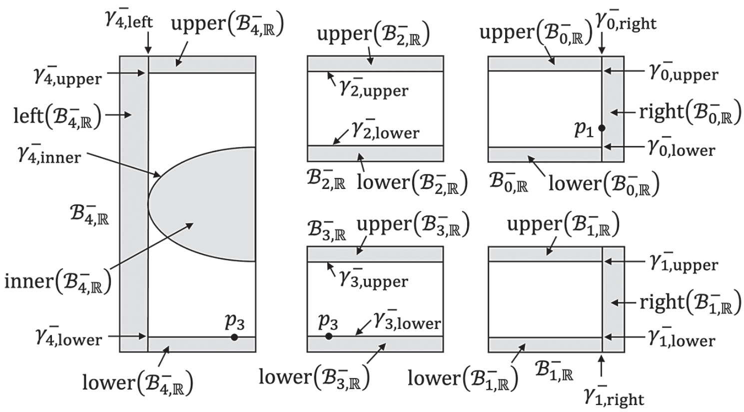

FIGURE 12. Margins in $\mathfrak{M}_2^-$.

- We write $\gamma^-_{0,\mathrm{lower}}$ as the lower connected component of $W^u_{43410}(a,b)^- \cap (\mathcal{B}^-_{0,\mathbb{R}} \setminus \mathrm{right}(\mathcal{B}^+_{0,\mathbb{R}} \setminus \gamma^-_{0,\mathrm{right}}))$.
- We write $\gamma^-_{2,\mathrm{lower}}$ as the lower connected component of $W^u_{43412}(a,b)^- \cap \mathcal{B}^-_{2,\mathbb{R}}$.
- We write $\gamma^-_{0,\mathrm{upper}}$ as the upper connected component of $W^u_{434100}(a,b)^- \cap (\mathcal{B}^-_{0,\mathbb{R}} \setminus \mathrm{right}(\mathcal{B}^-_{0,\mathbb{R}} \setminus \gamma^-_{0,\mathrm{right}}))$.
- We write $\gamma^-_{2,\mathrm{upper}}$ as the upper connected component of $W^u_{434102}(a,b)^- \cap \mathcal{B}^-_{2,\mathbb{R}}$.
- We write $\gamma^-_{4,\mathrm{inner}}$ as $W^u_{434124}(a,b)^-_{\mathrm{inner}}$.

See Figure 12 for more details.

5.2. *The* $\{\alpha,\beta\}$*-coding.* The purpose of this subsection is to define the $\{\alpha,\beta\}$-coding. We basically follow the construction of B^a and B^b in [**BS$_\mathbb{R}$3**, §6]. Hereafter, we always assume that $(a,b) \in \partial\mathcal{H}^\pm_\mathbb{R}$.

For $\gamma^+_{j,\mathrm{left}}$, $\gamma^+_{j,\mathrm{right}}$, $\gamma^+_{j,\mathrm{upper}}$, $\gamma^+_{j,\mathrm{lower}}$, $\gamma^+_{j,\mathrm{inner}}$ given in the previous subsection, we set $\mathrm{left}(\mathcal{B}^+_{j,\mathbb{R}})$, $\mathrm{right}(\mathcal{B}^+_{j,\mathbb{R}})$, $\mathrm{upper}(\mathcal{B}^+_{j,\mathbb{R}})$, $\mathrm{lower}(\mathcal{B}^+_{j,\mathbb{R}})$, $\mathrm{inner}(\mathcal{B}^+_{j,\mathbb{R}}) \subset \mathcal{B}^+_{j,\mathbb{R}}$ as follows (see Figure 11).

- We write $\mathrm{left}(\mathcal{B}^+_{3,\mathbb{R}}) \equiv \mathrm{left}(\mathcal{B}^+_{3,\mathbb{R}} \setminus \gamma^+_{3,\mathrm{left}})$.
- We write $\mathrm{right}(\mathcal{B}^+_{j,\mathbb{R}}) \equiv \mathrm{right}(\mathcal{B}^+_{j,\mathbb{R}} \setminus \gamma^+_{j,\mathrm{right}})$ $(j = 0, 1)$.
- We write $\mathrm{upper}(\mathcal{B}^+_{j,\mathbb{R}})$ as the part further up than $\gamma^+_{j,\mathrm{upper}}$ in either $\mathcal{B}^+_{j,\mathbb{R}} \setminus \mathrm{right}(\mathcal{B}^+_{j,\mathbb{R}})$ $(j = 0, 1)$, $\mathcal{B}^+_{j,\mathbb{R}}(j = 2)$, or $\mathcal{B}^+_{j,\mathbb{R}} \setminus \mathrm{left}(\mathcal{B}^+_{j,\mathbb{R}})(j = 3)$.
- We write $\mathrm{lower}(\mathcal{B}^+_{j,\mathbb{R}})$ as the part lower than $\gamma^+_{j,\mathrm{lower}}$ in either $\mathcal{B}^+_{j,\mathbb{R}} \setminus \mathrm{right}(\mathcal{B}^+_{j,\mathbb{R}})$ $(j = 0, 1)$, $\mathcal{B}^+_{j,\mathbb{R}}(j = 2)$, or $\mathcal{B}^+_{j,\mathbb{R}} \setminus \mathrm{left}(\mathcal{B}^+_{j,\mathbb{R}})(j = 3)$.
- We write $\mathrm{inner}(\mathcal{B}^+_{3,\mathbb{R}})$ as the part further in than $\gamma^+_{3,\mathrm{inner}}$ in $\mathcal{B}^+_{3,\mathbb{R}}$.

Let $\mathfrak{M}^+_2$ be the family of sets defined as above.

For $\gamma^-_{j,\mathrm{left}}$, $\gamma^-_{j,\mathrm{right}}$, $\gamma^-_{j,\mathrm{upper}}$, $\gamma^-_{j,\mathrm{lower}}$, $\gamma^-_{j,\mathrm{inner}}$ given in the previous subsection, we set $\mathrm{left}(\mathcal{B}^-_{j,\mathbb{R}})$, $\mathrm{right}(\mathcal{B}^-_{j,\mathbb{R}})$, $\mathrm{upper}(\mathcal{B}^-_{j,\mathbb{R}})$, $\mathrm{lower}(\mathcal{B}^-_{j,\mathbb{R}})$, $\mathrm{inner}(\mathcal{B}^-_{j,\mathbb{R}}) \subset \mathcal{B}^-_{j,\mathbb{R}}$ as follows (see Figure 12).

- We write $\mathrm{left}(\mathcal{B}^-_{4,\mathbb{R}}) \equiv \mathrm{left}(\mathcal{B}^-_{3,\mathbb{R}} \setminus \gamma^-_{4,\mathrm{left}})$.
- We write $\mathrm{right}(\mathcal{B}^-_{j,\mathbb{R}}) \equiv \mathrm{right}(\mathcal{B}^-_{j,\mathbb{R}} \setminus \gamma^-_{j,\mathrm{right}})$ $(j = 0, 1)$.
- We write $\mathrm{upper}(\mathcal{B}^-_{j,\mathbb{R}})$ as the part further up than $\gamma^-_{j,\mathrm{upper}}$ in either $\mathcal{B}^-_{j,\mathbb{R}} \setminus \mathrm{right}(\mathcal{B}^-_{j,\mathbb{R}})(j = 0, 1)$, $\mathcal{B}^-_{j,\mathbb{R}}(j = 2, 3)$, or $\mathcal{B}^-_{j,\mathbb{R}} \setminus \mathrm{left}(\mathcal{B}^-_{j,\mathbb{R}})(j = 4)$.
- We write $\mathrm{lower}(\mathcal{B}^+_{j,\mathbb{R}})$ as the part lower than $\gamma^-_{j,\mathrm{lower}}$ in either $\mathcal{B}^-_{j,\mathbb{R}} \setminus \mathrm{right}(\mathcal{B}^-_{j,\mathbb{R}})$ $(j = 0, 1)$, $\mathcal{B}^-_{j,\mathbb{R}}(j = 2, 3)$, or $\mathcal{B}^-_{j,\mathbb{R}} \setminus \mathrm{left}(\mathcal{B}^-_{j,\mathbb{R}})(j = 4)$.
- We write $\mathrm{inner}(\mathcal{B}^-_{4,\mathbb{R}})$ as the part further in than $\gamma^-_{4,\mathrm{inner}}$ in $\mathcal{B}^-_{4,\mathbb{R}}$.

Let $\mathfrak{M}^-_2$ be the family of sets defined as above. We call the elements of $\mathfrak{M}^\pm_2$ the *margins* of $\mathcal{B}^\pm_{j,\mathbb{R}}$ just like $\mathfrak{M}^\pm_1$.

We then construct the $\{\alpha, \beta\}$-partition as described in Figures 13 and 14. Define $\mathcal{B}^+_{j,\alpha,\mathbb{R}}$ for $j = 0$ and $\mathcal{B}^+_{j,\beta,\mathbb{R}}$ for $j = 1, 2$ as $\mathcal{B}^+_{j,\mathbb{R}} \setminus (\mathrm{left}(\mathcal{B}^+_{j,\mathbb{R}}) \cup \mathrm{right}(\mathcal{B}^+_{j,\mathbb{R}}) \cup \mathrm{upper}(\mathcal{B}^+_{j,\mathbb{R}}) \cup \mathrm{lower}(\mathcal{B}^+_{j,\mathbb{R}}))$. The set $\mathcal{B}^+_{3,\mathbb{R}} \setminus (\mathrm{left}(\mathcal{B}^+_{3,\mathbb{R}}) \cup \mathrm{right}(\mathcal{B}^+_{3,\mathbb{R}}) \cup \mathrm{upper}(\mathcal{B}^+_{3,\mathbb{R}}) \cup \mathrm{lower}(\mathcal{B}^+_{3,\mathbb{R}}))$ with one point $\star^+$ being removed has two connected components. We let $\mathcal{B}^+_{3,\alpha,\mathbb{R}}$ be the upper one with $\star^+$ being added, and let $\mathcal{B}^+_{3,\beta,\mathbb{R}}$ be the lower one with $\star^+$ being added.

Definition 5.5. We define $\mathcal{B}^+_{\alpha,\mathbb{R}} \equiv \mathcal{B}^+_{0,\alpha,\mathbb{R}} \cup \mathcal{B}^+_{3,\alpha,\mathbb{R}}$ and $\mathcal{B}^+_{\beta,\mathbb{R}} \equiv \mathcal{B}^+_{1,\beta,\mathbb{R}} \cup \mathcal{B}^+_{2,\beta,\mathbb{R}} \cup \mathcal{B}^+_{3,\beta,\mathbb{R}}$.

Similarly, define $\mathcal{B}^-_{j,\alpha,\mathbb{R}}$ for $j = 0, 1$ and $\mathcal{B}^-_{j,\beta,\mathbb{R}}$ for $j = 2, 3$ as $\mathcal{B}^-_{j,\mathbb{R}} \setminus (\mathrm{left}(\mathcal{B}^-_{j,\mathbb{R}}) \cup \mathrm{right}(\mathcal{B}^-_{j,\mathbb{R}}) \cup \mathrm{upper}(\mathcal{B}^-_{j,\mathbb{R}}) \cup \mathrm{lower}(\mathcal{B}^-_{j,\mathbb{R}}))$. The set $\mathcal{B}^-_{4,\mathbb{R}} \setminus (\mathrm{left}(\mathcal{B}^-_{4,\mathbb{R}}) \cup \mathrm{right}(\mathcal{B}^-_{4,\mathbb{R}}) \cup \mathrm{upper}(\mathcal{B}^-_{4,\mathbb{R}}) \cup \mathrm{lower}(\mathcal{B}^-_{4,\mathbb{R}}))$ with one point $\star^-$ being removed has two connected components. We let $\mathcal{B}^-_{4,\alpha,\mathbb{R}}$ be the upper one with $\star^-$ being added, and let $\mathcal{B}^-_{4,\beta,\mathbb{R}}$ be the lower one with $\star^-$ being added.

Definition 5.6. We define $\mathcal{B}^-_{\alpha,\mathbb{R}} \equiv \mathcal{B}^-_{0,\alpha,\mathbb{R}} \cup \mathcal{B}^-_{2,\alpha,\mathbb{R}} \cup \mathcal{B}^-_{4,\alpha,\mathbb{R}}$ and $\mathcal{B}^-_{\beta,\mathbb{R}} \equiv \mathcal{B}^-_{1,\beta,\mathbb{R}} \cup \mathcal{B}^-_{3,\beta,\mathbb{R}} \cup \mathcal{B}^-_{4,\beta,\mathbb{R}}$.

The transitions between the margins can be described in the following two lemmas.

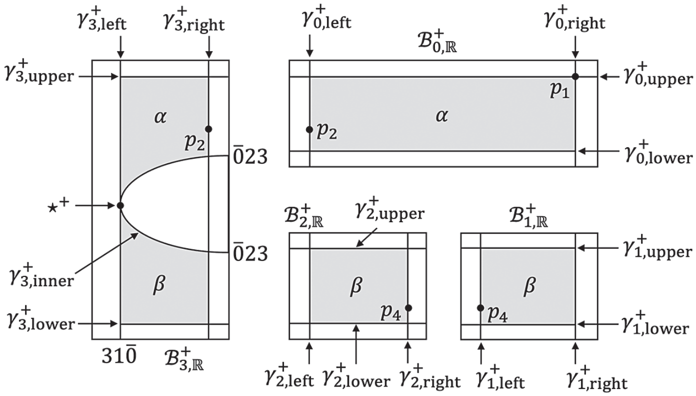

FIGURE 13. The $\{\alpha, \beta\}$-partition ($b > 0$).

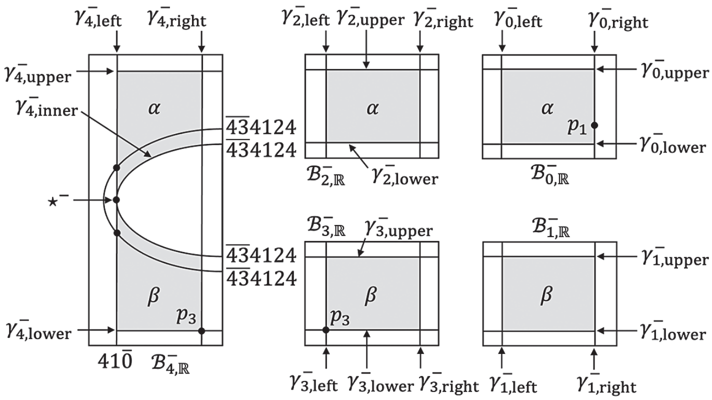

FIGURE 14. The $\{\alpha, \beta\}$-partition ($b < 0$).

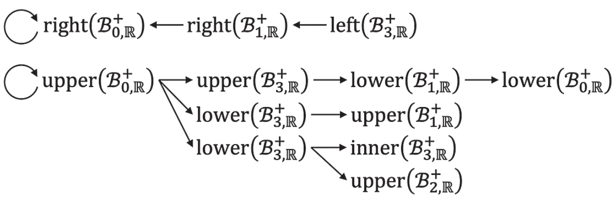

FIGURE 15. The transitions between margins in $\mathfrak{M}_2^+$.

LEMMA 5.7. *The transitions between margins in* $\mathfrak{M}_2^+$ *are given by Figure 15. In this figure, for example,* $\mathrm{left}(\mathcal{B}^+_{3,\mathbb{R}}) \to \mathrm{right}(\mathcal{B}^+_{1,\mathbb{R}})$ *means* $f(\mathrm{left}(\mathcal{B}^+_{3,\mathbb{R}})) \cap \mathcal{B}^+_{1,\mathbb{R}} \subset \mathrm{right}(\mathcal{B}^+_{1,\mathbb{R}})$.

LEMMA 5.8. *The transitions between margins in* $\mathfrak{M}_2^-$ *are given by Figure 16. In this figure,* $\mathrm{left}(\mathcal{B}^-_{4,\mathbb{R}}) \to \mathrm{right}(\mathcal{B}^-_{1,\mathbb{R}})$ *means* $f(\mathrm{left}(\mathcal{B}^-_{4,\mathbb{R}})) \cap \mathcal{B}^-_{1,\mathbb{R}} \subset \mathrm{right}(\mathcal{B}^-_{1,\mathbb{R}})$.

From these lemmas, we have the following proposition.

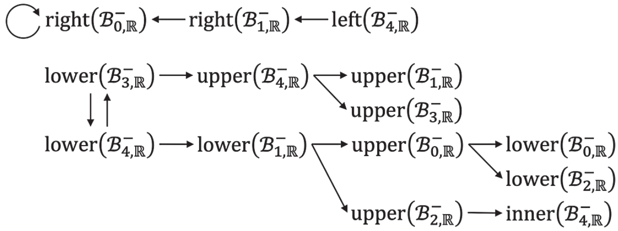

FIGURE 16. The transitions between margins in $\mathfrak{M}_2^-$.

PROPOSITION 5.9. *For* $(a, b) \in \partial\mathcal{H}^+_{\mathbb{R}}$, *we have* $K_{\mathbb{R}} \subset \mathcal{B}^+_{\alpha,\mathbb{R}} \cup \mathcal{B}^+_{\beta,\mathbb{R}}$.

Proof. Suppose $p \in K_{\mathbb{R}}$. It suffices to show that

$$\text{the 0th coordinate of the } \Sigma^+\text{-coding of } p \text{ is } 1 \Longrightarrow p \notin M$$

for any $M \in \mathfrak{M}_2^+$ (compare Proposition 4.1). Here, we show this claim for $M = \mathrm{right}(\mathcal{B}^+_{1,\mathbb{R}})$ for example. We can also show this for another M similarly.

Take a point $p \in K_{\mathbb{R}}$. Assume that the zeroth coordinate of the Σ^+-coding of p is 1 and $p \in \mathrm{right}(\mathcal{B}^+_{1,\mathbb{R}})$. By the transitions between symbols, the first coordinate of the Σ^+-coding of p is either 0 or 2. Since $p \in \mathrm{right}(\mathcal{B}^+_{1,\mathbb{R}})$, the first coordinate must be 0. Then, we have $f(p) \in \mathrm{right}(\mathcal{B}^+_{0,\mathbb{R}})$ by Lemma 5.7. By chasing the orbit of p similarly, we can find that $f^{n_0}(p)$ belongs to $\mathrm{right}(\mathcal{B}^+_{0,\mathbb{R}})$ and $f^n(p)$ also belongs to $\mathrm{right}(\mathcal{B}^+_{0,\mathbb{R}})$ for any $n > n_0$. Therefore, $f^{n_0}(p)$ lies on $W^s_{\mathrm{loc}}(p_1)^+_0$. This contradicts to the fact that $f^{n_0}(p) \in \mathrm{right}(\mathcal{B}^+_{0,\mathbb{R}})$. □

PROPOSITION 5.10. *For* $(a, b) \in \partial\mathcal{H}^-_{\mathbb{R}}$, *we have* $K_{\mathbb{R}} \subset \mathcal{B}^-_{\alpha,\mathbb{R}} \cup \mathcal{B}^-_{\beta,\mathbb{R}}$.

Proof. The proof is similar to that of the previous proposition, and hence is omitted. □

5.3. *Proof of Theorem 1.2.* In this subsection, we continue to assume that $(a, b) \in \partial\mathcal{H}^{\pm}_{\mathbb{R}}$. Below, we write

$$K_{\mathrm{fwd}} \equiv \{p \in \mathbb{C}^2 : (f^n(p))_{n\geq 0} \text{ is bounded}\}$$

and

$$K_{\mathrm{bwd}} \equiv \{p \in \mathbb{C}^2 : (f^n(p))_{n\leq 0} \text{ is bounded}\}$$

as the forward/backward filled Julia set of $f_{a,b}$, respectively. We also write $J_{\mathrm{fwd}} \equiv \partial K_{\mathrm{fwd}}$ and $J_{\mathrm{bwd}} \equiv \partial K_{\mathrm{bwd}}$ as the forward/backward Julia set, and let $J \equiv J_{\mathrm{fwd}} \cap J_{\mathrm{bwd}}$ be the Julia set. By [**AI**, Proposition 3.1], the filled Julia set $K_{\mathrm{fwd}} \cap K_{\mathrm{bwd}}$ coincides with $K \equiv K_{a,b}$. Moreover, the filled Julia set is included in the real plane $\mathbb{R}^2$ and coincides with the Julia set J by [**BLS**, Theorem 10.1(4), (7)] since $h_{\mathrm{top}}(f) = \log 2$ holds. Hence, we have $J = K = K_{\mathbb{R}}$.

We write $\mathcal{B}^+_{\mathbb{R}}$ as the region in $\mathbb{R}^2$ enclosed by the curves $W^u_{\mathrm{loc}}(p_1)_0$, $\gamma^+_{1,\mathrm{lower}}$, $\gamma^+_{2,\mathrm{lower}}$, $\gamma^+_{3,\mathrm{left}}$, $\gamma^+_{3,\mathrm{upper}}$, $\gamma^+_{3,\mathrm{lower}}$, and $f^{-1}(W^s_{\mathrm{loc}}(p_1)_0)$. Moreover, let $\mathcal{B}^+_{\tilde{\alpha},\mathbb{R}} = f^{-1}(\mathcal{B}^+_{\alpha,\mathbb{R}})$ and

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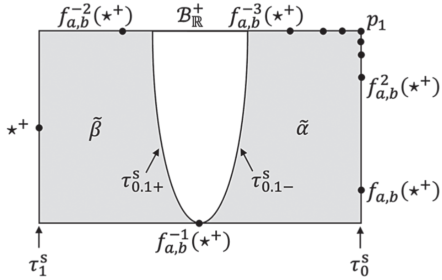

FIGURE 17. The $\{\tilde{\alpha}, \tilde{\beta}\}$-partition.

$\mathcal{B}^+_{\tilde{\beta},\mathbb{R}} = f^{-1}(\mathcal{B}^+_{\beta,\mathbb{R}})$. Also, we write $\mathcal{B}^-_{\mathbb{R}}$ as the region in $\mathbb{R}^2$ enclosed by the curves $W^u_{\mathrm{loc}}(p_3)_3$, $\gamma^-_{0,\mathrm{upper}}$, $\gamma^-_{1,\mathrm{lower}}$, $\gamma^-_{2,\mathrm{upper}}$, $\gamma^-_{4,\mathrm{left}}$, $\gamma^-_{4,\mathrm{upper}}$, $\gamma^-_{4,\mathrm{lower}}$, and $f^{-1}(W^s_{\mathrm{loc}}(p_1)_0)$. Moreover, let $\mathcal{B}^-_{\tilde{\alpha},\mathbb{R}} = f^{-1}(\mathcal{B}^-_{\alpha,\mathbb{R}})$ and $\mathcal{B}^-_{\tilde{\beta},\mathbb{R}} = f^{-1}(\mathcal{B}^-_{\beta,\mathbb{R}})$. Then, the $\{\tilde{\alpha}, \tilde{\beta}\}$-partition is described in Figure 17.

For any point $p \in J$, the $\{\tilde{\alpha}, \tilde{\beta}\}$-coding of $f^{-1}(p)$ coincides with the symbolic sequence obtained by replacing α and β in the $\{\alpha, \beta\}$-coding of p for $\tilde{\alpha}$ and $\tilde{\beta}$, respectively.

From now, we assume that $(a, b) \in \partial\mathcal{H}^+_{\mathbb{R}}$. The argument for the case $(a, b) \in \partial\mathcal{H}^-_{\mathbb{R}}$ is similar, and hence will be omitted.

For $p \in K$, let $s^+(p)$ be the set of all $\{\alpha, \beta\}$-codings of p, that is,

$$s^+(p) \equiv \{(s_j)_{j\in\mathbb{Z}} \in \{\alpha, \beta\}^{\mathbb{Z}} : (f^j(p) \in \mathcal{B}^+_{\alpha,\mathbb{R}} \text{ and } s_j = \alpha) \text{ or } (f^j(p) \in \mathcal{B}^+_{\beta,\mathbb{R}} \text{ and } s_j = \beta)\}.$$

In the case that $p = f^n(\star^+)$ for some $n \in \mathbb{Z}$, we have $s^+(p) = \{\sigma^n(\overline{\alpha}\beta \cdot \alpha\beta\overline{\alpha}), \sigma^n(\overline{\alpha}\beta \cdot \beta\beta\overline{\alpha})\}$. However, if $p \in J' \equiv J \setminus \{f^n(\star^+) : n \in \mathbb{Z}\}$, then $s^+(p)$ consists of one point. Therefore, $s^+ : p \mapsto s^+(p)$ is a map from J to $\{\alpha, \beta\}^{\mathbb{Z}}/\sim$, where $\sim$ is the equivalence relation generated by $\sigma^n(\overline{\alpha}\beta \cdot \alpha\beta\overline{\alpha}) \sim \sigma^n(\overline{\alpha}\beta \cdot \beta\beta\overline{\alpha})$, $n \in \mathbb{Z}$.

PROPOSITION 5.11. *The map $s^+ : J \to \{\alpha, \beta\}^{\mathbb{Z}}$ is continuous.*

Proof. The map s^+ is obviously continuous on J'. We show that it is continuous on $f^n(\star^+)$. The neighborhood basis for $s^+(f^n(\star^+))$ according to the quotient topology is given by $\{*\alpha^N\beta\cdot ?\beta\alpha^N*\}_{N\geq 0}$, where ? is either α or β, and $*$ means any one-sided infinite sequence. Take any N. The map f^j is continuous for each $j \in \mathbb{Z}$ such that $-N \leq j \leq N$, $j \neq 0$. Hence, we can choose a neighborhood U_j of $f^{n+j}(\star^+)$ such that $U_j \subset \mathcal{B}^+_\alpha$ if $-N \leq j \leq N$, $j \neq 0, \pm 1$ and $U_j \subset \mathcal{B}^+_\beta$ if $j = \pm 1$. Then, $U = \bigcap_{j\in\mathbb{Z}} f^{-j}(U_j)$ is a neighborhood of $f^n(\star^+)$ and satisfies $p \in U \cap J \Rightarrow s^+(p) \in \{\sigma^n(*\alpha^N\beta\cdot ?\beta\alpha^N*)\}$. $\square$

To prove Theorem 1.2 in the case $b > 0$, it suffices to show that the continuous map s^+ is bijective. In the rest this subsection, we will show its surjectivity and injectivity.

For each binomial number $0 \leq \theta \leq 1$, we define a curve τ^s_θ in $\mathcal{B}^+_{\mathbb{R}}$ included in $\overline{W^s(p_1)}$ as follows. For $\theta = 0, 1$, let τ^s_0 be the right-hand boundary of $\mathcal{B}^+_{\mathbb{R}}$ and let τ^s_1 be its left-hand boundary.

Next, for a finite binomial $0 < \theta < 1$, we define τ^s_θ inductively as follows. Assume that τ^s_θ is already defined for a finite binomial $0 < \theta \leq 1$. Then, we let $f^{-1}(\tau^s_\theta) \cap \mathcal{B}^+_{\alpha,\mathbb{R}} = \tau^s_{\theta/2}$

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and $f^{-1}(\tau^s_\theta) \cap \mathcal{B}^+_{\beta,\mathbb{R}} = \tau^s_{1-\theta/2}$ for $\theta' = \theta/2, 1-\theta/2$. However, $\tau^s_{1/2} = f^{-1}(\tau^s_1)$ for $\theta' = \frac{1}{2}$ instead. The curve τ^s_θ is represented as two connected curves $\tau^s_{\theta'\pm}$ from the upper hand boundary of $\mathcal{B}^+_{\mathbb{R}}$ to its lower hand boundary, where $\tau^s_{\theta'+}$ is left of $\tau^s_{\theta'-}$. By definition, τ^s_θ is included in $W^s(p_1)$ for any finite binomial $0 \leq \theta \leq 1$. Also, for any point in $W^s(p_1) \cap \mathcal{B}^+_{\mathbb{R}}$, there is a unique finite binomial θ such that the curve τ^s_θ passes that point.

Finally, any infinite binomial $0 \leq \theta \leq 1$ can be represented as the limit of some sequence $(\theta_j)_j$ of finite binomials. Then, we have $\tau^s_{\theta_j} \subset f^{-1}(\mathcal{B}^+_i)$ for some preimage $f^{-1}(\mathcal{B}^+_i)$ and large j. For each j, the connected component V_j of $W^s(p_1) \cap f^{-1}(\mathcal{B}^+_i)$ including $\tau^s_{\theta_j}$ can be described as a graph of some holomorphic function since this component is a vertical disk of $f^{-1}(\mathcal{B}^+_i)$. Since the sequence of these functions uniformly converges, its limit function is also holomorphic. Hence, the limit set V of V_j is a graph of this holomorphic function, which implies V is a vertical disk of $f^{-1}(\mathcal{B}^+_i)$. We let τ^s_θ be the limit set of $\tau^s_{\theta_j}$. Then, τ^s_θ is a curve from the upper hand boundary of $\mathcal{B}^+_{\mathbb{R}}$ to its lower hand boundary, and it is included in V.

Remark 5.12. A finite binomial number θ has two infinite binomial representations (e.g. $0.1 = 0.0111\cdots = 0.1000\cdots$). Each representation corresponds to each of the two curves $\tau^s_{\theta\pm}$ of degree one, respectively (e.g. $\tau^s_{0.0111\cdots} = \tau^s_{0.1-}$, $\tau^s_{0.1000\cdots} = \tau^s_{0.1+}$). From now, we distinguish two infinite binomial representations of a finite binomial number if needed.

For a point $p \in J'$, we choose $0 \leq \theta \leq 1$ so that τ^s_θ passes p. Let $0.d_1d_2d_3\cdots$ be the binomial representation of θ. By the definition of τ^s_θ, $d_j = 0$ if and only if $f^j(p) \in \mathcal{B}^+_{\alpha,\mathbb{R}}$ and $d_j = 1$ if and only if $f^j(p) \in \mathcal{B}^+_{\beta,\mathbb{R}}$ for each $j \geq 1$. Hence, we have the following proposition.

PROPOSITION 5.13. *The map $s^+ : J \to \{\alpha, \beta\}^{\mathbb{Z}}/\sim$ is surjective.*

Proof. Let $\mathcal{S}' = \{\alpha, \beta\}^{\mathbb{Z}} \setminus \bigcup_{j\in\mathbb{Z}} \sigma^j\{\overline{\alpha}\beta \cdot \alpha\beta\overline{\alpha}, \overline{\alpha}\beta \cdot \beta\beta\overline{\alpha}\}$. It suffices to show that there exists a point $p \in J'$ for each $s \in \mathcal{S}'$ with $s^+(p) = s$. Take any $s = (s_j)_j \in \mathcal{S}'$. For each j, let $d_j = 0$ if $s_j = \alpha$ and $d_j = 1$ if $s_j = \beta$, and consider a binomial number $\theta^j = d_j.d_{j+1}d_{j+2}d_{j+3}\cdots$. If we choose a point $p_n \in \tau^s_{\theta^{-n}} \cap J$ for each $n \in \mathbb{Z}$, then the jth coordinate of the $\{\alpha, \beta\}$-coding of $f^n(p_n)$ is s_j for $j \geq -n$. Since J is compact, $f^n(p_n) \in \tau^s_{\theta^0}$ has a cluster point $p \in \tau^s_{\theta^0}$. Here, if p is on the orbit of $\star^+$, then we have $s \in \mathcal{S}'$. Therefore, p belongs to J' and satisfies $s^+(p) = s$. $\square$

We can also construct a curve τ^u_ϑ included in $\overline{W^u(p_1)}$ for each binomial number $0 \leq \vartheta \leq 1$ by a similar argument according to $\mathcal{B}^+_{\alpha,\mathbb{R}}, \mathcal{B}^+_{\beta,\mathbb{R}}$ instead of $\mathcal{B}^+_{\tilde{\alpha},\mathbb{R}}, \mathcal{B}^+_{\tilde{\beta},\mathbb{R}}$. For a point $p \in J'$, we choose $0 < \vartheta < 1$ so that τ^u_ϑ passes p, and let $0.d_0d_{-1}d_{-2}\cdots$ be the infinite binomial representation of ϑ. Then, $d_j = 0$ if and only if $f^j(p) \in \mathcal{B}^+_{\alpha,\mathbb{R}}$ and $d_j = 1$ if and only if $f^j(p) \in \mathcal{B}^+_{\beta,\mathbb{R}}$ for each $j \leq 0$. Therefore, we have the following proposition.

PROPOSITION 5.14. *The map $s^+ : J \to \{\alpha, \beta\}^{\mathbb{Z}}/\sim$ is injective.*

Proof. It suffices to show that there exists at most one point $p \in J'$ for any $s = (s_j)_j \in \mathcal{S}'$ with $s^+(p) = s$. Let $d_j = 0$ if $s_j = \alpha$ and $d_j = 1$ if $s_j = \beta$ for each $j \in \mathbb{Z}$, and consider

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two binomial numbers $\theta = 0.d_1d_2d_3\cdots$ and $\vartheta = 0.d_0d_{-1}d_{-2}\cdots$. Then, θ corresponds to a curve τ^s_θ (or one of $\tau^s_{\theta\pm}$), and ϑ corresponds to a curve τ^u_ϑ (or one of $\tau^u_{\vartheta\pm}$). We can obtain a point p with $s^+(p) = s$ as an intersection point of τ^s_θ and τ^u_ϑ.

The curve $\tau^s_\theta \cap U^+$ is included in either $\mathcal{B}^+_0 \cup \mathcal{B}^+_1$, $\mathcal{B}^+_0 \cup \mathcal{B}^+_2$, or $\mathcal{B}^+_3$, and $\tau^u_\vartheta \cap U^+$ is included in either $\mathcal{B}^+_0 \cup \mathcal{B}^+_3$ or $\mathcal{B}^+_1 \cup \mathcal{B}^+_2 \cup \mathcal{B}^+_3$. Hence, $\tau^s_\theta \cap \tau^u_\vartheta$ is included in one of four boxes $\mathcal{B}^+_i$ $(i = 0, 1, 2, 3)$. We can obtain $\tau^s_\theta \cap \mathcal{B}^+_i$ as the intersection of a vertical disk of degree one in $\mathcal{B}^+_i$ and $\mathbb{R}^2$, or as one of two branches of the intersection of a vertical disk of degree two in $\mathcal{B}^+_i$ and $\mathbb{R}^2$. Also, we can obtain $\tau^u_\vartheta \cap \mathcal{B}^+_i$ as the intersection of a horizontal disk of degree one in $\mathcal{B}^+_i$ and $\mathbb{R}^2$, or as one of two branches of the intersection of a horizontal disk of degree two in $\mathcal{B}^+_i$ and $\mathbb{R}^2$. Therefore, τ^s_θ and τ^u_ϑ intersect at at most one point. □

Proof of Theorem 1.2. By Propositions 5.11, 5.13, and 5.14, we obtain Theorem 1.2. □

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A. *Appendix. Computer-assisted proofs*

In this appendix, we explain the proofs of Proposition A.1 and Lemmas 2.7, 2.8, 2.10, 2.11, 4.2, and 4.5. These proofs require computer-assistance, and are performed by Arai [**A2**]. Note that these lemmas concern only the real parameters $(a, b) \in \mathcal{E}^\pm_{\mathbb{R}}$, and hence the computational cost is not so high.

PROPOSITION A.1. *The locus $\mathcal{E}^\pm_{\mathbb{R}}$ forms a neighborhood of $\partial\mathcal{H}^\pm_{\mathbb{R}}$.*

Proof. When (a, b) belongs to the left boundary of $\mathcal{E}^\pm_{\mathbb{R}}$, we show that there exists a periodic point of period 7 in $\mathbb{C}^2 \setminus \mathbb{R}^2$ by using the interval Krawczyk method (see [**AI**, Theorem 2.12(i)]). This implies $h_{\mathrm{top}}(f_{a,b}) < \log 2$ by [**BLS**, Theorem 10.1]. When (a, b) belongs to the right boundary of $\mathcal{E}^\pm_{\mathbb{R}}$, we show that $f_{a,b}$ is hyperbolic by using a technique in [**A1**] (see [**AI**, Theorem 2.12(ii)]). These results together with [**AI**, Main Theorem] yields that the locus $\mathcal{E}^\pm_{\mathbb{R}}$ forms a neighborhood of $\partial\mathcal{H}^\pm_{\mathbb{R}}$. □

Proof of Lemmas 2.7 and 2.8. For Lemma 2.7, we first choose appropriate topological disks $\widetilde{D}^+_{u,i} \subset D^+_{u,i}$ $(i = 0, 1, 2)$ and set $\mathcal{P}^+_{0,3} \equiv \widetilde{D}^+_{u,0} \times_{\mathrm{pr}} D^+_{v,0} \subset \mathcal{B}^+_0$, $\mathcal{P}^+_{1,2} \equiv \widetilde{D}^+_{u,1} \times_{\mathrm{pr}} D^+_{v,1} \subset \mathcal{B}^+_1$ and $\mathcal{P}^+_{2,3} \equiv \widetilde{D}^+_{u,2} \times_{\mathrm{pr}} D^+_{v,2} \subset \mathcal{B}^+_2$ so that conditions (1) and (2) in the lemma are satisfied.

For Lemma 2.8, we choose appropriate topological disks $\widetilde{D}^-_{u,i} \subset D^-_{u,i}$ $(i = 0, 1, 2, 3)$ and set $\mathcal{P}^-_{0,2} \equiv \widetilde{D}^-_{u,0} \times_{\mathrm{pr}} D^-_{v,0} \subset \mathcal{B}^-_0$, $\mathcal{P}^-_{1,3} \equiv \widetilde{D}^-_{u,1} \times_{\mathrm{pr}} D^-_{v,1} \subset \mathcal{B}^-_0$, $\mathcal{P}^-_{2,4} \equiv \widetilde{D}^-_{u,2} \times_{\mathrm{pr}} D^-_{v,2} \subset \mathcal{B}^-_0$ and $\mathcal{P}^-_{3,4} \equiv \widetilde{D}^-_{u,3} \times_{\mathrm{pr}} D^-_{v,3} \subset \mathcal{B}^-_0$ so that conditions (1) and (2) in the lemma are satisfied. □

Proof of Lemmas 2.10 and 2.11. Let $\pi_{u,i} : \mathcal{B}_i^{\pm} \to D_{u,i}^{\pm}$ be the projection given by $\pi_{u,i}(u, v) = u$ and let $\iota_{v_0,i} : D_{u,i} \to \mathcal{B}_i^{\pm}$ be given by $\iota_{v_0,i}(u) = (u, v_0)$ for $v_0 \in D_{v,i}^{\pm}$. The proof is similar to [**AI**, Lemma 3.13] and goes as follows.

Let $(a, b) \in \mathcal{E}_{\mathbb{R}}^{+}$. For Lemma 2.10, we show the following claim by computer assistance. For every fixed $v_0 \in D_{v,2}^{+}$, we have

$$\frac{d}{du}\{\pi_{u,2}^{-} \circ f^4 \circ \iota_{v_0,2}(u)\} \neq 0$$

for $u \in D_{u,2}^{+}$ with $\iota_{v_0,2}(u) \in \mathcal{B}_2^{+} \cap f^{-1}(\mathcal{B}_3^{+} \cap f^{-1}(\mathcal{B}_1^{+} \cap f^{-1}(\mathcal{B}_0^{+} \cap f^{-1}(\mathcal{B}_2^{+}))))$.

Let $(a, b) \in \mathcal{E}_{\mathbb{R}}^{-}$. For Lemma 2.11, we show the following claims by computer assistance.

(1) For every fixed $v_0 \in D_{v,2}^{-}$, we have

$$\frac{d}{du}\{\pi_{u,3}^{-} \circ f^2 \circ \iota_{v_0,2}(u)\} \neq 0$$

for $u \in D_{u,2}^{-}$ with $\iota_{v_0,2}(u) \in \mathcal{B}_3^{-} \cap f^{-1}(\mathcal{B}_4^{-} \cap f^{-1}(\mathcal{B}_2^{-}))$.

(2) For every fixed $v_0 \in D_{v,3}^{-}$, we have

$$\frac{d}{du}\{\pi_{u,3}^{-} \circ f^2 \circ \iota_{v_0,3}(u)\} \neq 0$$

for $u \in D_{u,3}^{-}$ with $\iota_{v_0,3}(u) \in \mathcal{B}_3^{-} \cap f^{-1}(\mathcal{B}_4^{-} \cap f^{-1}(\mathcal{B}_3^{-}))$. □

Proof of Lemmas 4.2 and 4.5. We define the projective polydisk $\mathcal{P}_i = D_i \times_{\mathrm{pr}} D_{I_i}$ in claim (b) as follows. For $i = 2, 3, 4$, we first compute the stable direction at the saddle point p_i and define π_u to be the projection to this direction. We also define π_v as the projection pr_y orthogonal to the y-axis. We let D_i be a disk centered at $\pi_u(p_i)$ and determine its radius $\delta_i > 0$ is determined so that claims (a) and (c) hold. For $i = 1$, we approximately compute the tangential direction of $W^s_{\mathrm{loc}}(p_3)_4$ at some $p \in W^s_{\mathrm{loc}}(p_3)_4 \cap \mathcal{B}^{-}_{2,\mathbb{R}}$ and pull it back by f to define the direction of the projection π_u for $\mathcal{P}_1$. We also define π_v as the projection pr_y orthogonal to the y-axis. We let D_1 be a disk centered at $\pi_u(f^{-1}(p))$ and determine its radius $\delta_1 > 0$ is determined so that claims (a) and (c) hold. □

References

[A1] Z. Arai. On hyperbolic plateaus of the Hénon map. *Exp. Math.* **16**(2) (2007), 181–188.

[A2] Z. Arai. All programs for computer-assisted proofs and their instructions are available at: http://www.is.c.titech.ac.jp/zin/hironaka-ishii/.

[AI] Z. Arai and Y. Ishii. On parameter loci of the Hénon family. *Comm. Math. Phys.* **361**(2) (2018), 343–414.

[BD] E. Bedford and R. Dujardin. Topological and geometric hyperbolicity criteria for polynomial automorphisms of $\mathbb{C}^2$. *Ergod. Th. & Dynam. Sys.* **42**(7) (2022), 2151–2171.

[BLS] E. Bedford, M. Lyubich and J. Smillie. Polynomial diffeomorphisms of $\mathbf{C}^2$: IV. The measure of maximal entropy and laminar currents. *Invent. Math.* **112**(1) (1993), 77–125.

[BS$_{\mathbf{C}}$8] E. Bedford and J. Smillie. Polynomial diffeomorphisms of $\mathbf{C}^2$. VIII. Quasi-expansion. *Amer. J. Math.* **124**(2) (2002), 221–271.

[BS$_{\mathbb{R}}$1] E. Bedford and J. Smillie. Real polynomial diffeomorphisms with maximal entropy: Tangencies. *Ann. of Math. (2)* **160**(1) (2004), 1–26.

[BS$_{\mathbb{R}}$2] E. Bedford and J. Smillie. Real polynomial diffeomorphisms with maximal entropy. II. Small Jacobian. *Ergod. Th. & Dynam. Sys.* **26**(5) (2006), 1259–1283.

[BS$_{\mathbb{R}}$3] E. Bedford and J. Smillie. A symbolic characterization of the horseshoe locus in the Hénon family. *Ergod. Th. & Dynam. Sys.* **37**(5) (2017), 1389–1412.

[I] Y. Ishii. Hyperbolic polynomial diffeomorphisms of $\mathbb{C}^2$. I: A non-planar map. *Adv. Math.* **218**(2) (2008), 417–464.

[IS] Y. Ishii and J. Smillie. Homotopy shadowing. *Amer. J. Math.* **132**(4) (2010), 987–1029.