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ON GROUPS WITH TWO ISOMORPHISM CLASSES OF DERIVED SUBGROUPS

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Abstract. The structure of groups which have at most two isomorphism classes of derived subgroups ($\mathfrak{D}_2$-groups) is investigated. A complete description of $\mathfrak{D}_2$-groups is obtained in the case where the derived subgroup is finite: the solution leads an interesting number theoretic problem. In addition, detailed information is obtained about soluble $\mathfrak{D}_2$-groups, especially those with finite rank, where algebraic number fields play an important role. Also, detailed structural information about insoluble $\mathfrak{D}_2$-groups is found, and the locally free $\mathfrak{D}_2$-groups are characterized.

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1. Introduction and results. By a *derived subgroup* in a group G is meant the derived (or commutator) subgroup of a subgroup of G. It is a natural question: how important the set of derived subgroups is within the lattice of all subgroups? Recently there has been interest in imposing restrictions on the number of derived subgroups in a group and investigating the resulting effect on the structure of the group. Let $\mathfrak{C}_n$ denote the class of groups in which there are at most n derived subgroups, and let $\mathfrak{C}$ denote the union of all the classes $\mathfrak{C}_n, n = 1, 2, \ldots$. The structure of $\mathfrak{C}_n$-groups for small n has been investigated in [**5**], while it is shown in [**3**] and [**5**] that a locally graded $\mathfrak{C}$-group has finite derived subgroup – also see [**2**] for related work.

In this paper we are concerned with groups for which the set of *isomorphism types* of derived subgroup is very small. If n is a positive integer, let

$$\mathfrak{D}_n$$

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denote the class of groups whose derived subgroups fall into at most n isomorphism classes. Clearly $\mathfrak{D}_1 = \mathfrak{C}_1$ is the class of abelian groups, but one would expect the class $\mathfrak{D}_n$ to be much larger than $\mathfrak{C}_n$ if $n > 1$.

Our attention here is focused on the class $\mathfrak{D}_2$. Notice that a group G belongs to $\mathfrak{D}_2$ if and only if $H' \simeq G'$ whenever H is a non-abelian subgroup of G. While this may seem a highly restricted class of groups, it contains groups of many diverse types: apart from abelian groups, $\mathfrak{D}_2$ contains free groups of countable rank, groups whose derived subgroups are cyclic of prime or infinite order, Tarski groups, (i.e. finitely generated infinite groups with every proper subgroup abelian) and a whole range of soluble groups. It turns out to be possible to describe in a precise way some large classes of $\mathfrak{D}_2$-groups. For example, $\mathfrak{D}_2$-groups with finite derived subgroup can be characterized modulo the centre by pairs (p, m) where p is a prime, $m > 1$ is an integer not divisible by p and the order of p mod m equals the order of p mod q for each prime divisor q of m: this is Theorem 2. We call such integer pairs *allowable pairs*. These pairs turn out to be mysterious objects and their study leads to an apparently difficult number theoretic problem: given a prime p, does there exist a prime q such that (p, q^2) is allowable?

A great deal of information about infinite soluble $\mathfrak{D}_2$-groups is obtained: for example, such groups are metabelian, and their derived subgroup is free or elementary abelian or torsion-free of finite rank (Theorem 3). In addition, infinite soluble $\mathfrak{D}_2$-groups of finite rank with trivial centre can be described in a similar manner to those with finite derived subgroup (Theorem 4). It turns out that they can be constructed to within finite index from an algebraic number field and a finitely generated subgroup of its multiplicative group which satisfy a condition analogous to the allowability condition for pairs of integers. These results appear in Section 4.

Among other types of $\mathfrak{D}_2$-group studied are groups whose derived subgroup is not perfect, groups satisfying the Tits alternative and groups whose derived subgroups satisfy the minimal condition on subgroups. Results about these groups are to be found in Sections 5 and 6. For example, it is proved that a locally free group G is a $\mathfrak{D}_2$-group if and only if G' is free with countable rank (Corollary 7).

In conclusion we point out that the study of $\mathfrak{D}_2$-groups involves groups of many different types, and that progress requires the application of a variety of techniques, both group theoretic and number theoretic. It seems probable that some of our methods may be extended to higher classes $\mathfrak{D}_n$, and work is already underway on the class $\mathfrak{D}_3$.

NOTATION. G', G'', $G^{(\alpha)}$: *terms of the derived series of a group* G.
$\gamma_n(G)$: *the nth term of the lower central series of* G.
$Z(G)$: *the centre of* G.
$Z_i(G)$: *the ith term of the upper central series of* G.
$[H,_n K]$: *the commutator subgroup* $[H, K, \dots, K]$ *with n subgroups* K.
$r_0(G)$: *the torsion-free rank of an abelian group.*
$r(G)$: *the Prüfer rank of a group.*

2. Elementary results. We begin by assembling some elementary facts about the class $\mathfrak{D}_2$.

LEMMA 1.
(i) *The class* $\mathfrak{D}_2$ *is subgroup closed.*
(ii) *Let* $G \in \mathfrak{D}_2$ *and assume that* G' *satisfies min, the minimal condition on subgroups. If* $N \lhd G$, *then* $G/N \in \mathfrak{D}_2$.

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(iii) *If $G \in \mathfrak{D}_2$, then G' is countable.*

Proof. The first statement is obvious. To prove the second, let H/N be any non-abelian subgroup of G/N. Then H is non-abelian and $G' \simeq H'$. Since G' satisfies min, it has the cohopfian property and so $G' = H'$. Hence, $(G/N)' = (H/N)'$ and $G/N \in \mathfrak{D}_2$. To prove (iii) we may assume that $G' \neq 1$. Then G has a finitely generated non-abelian subgroup H and $G' \simeq H'$. Hence, G' is countable. □

On the other hand, in the context of Lemma 1(iii) it is worth noting that if $G \in \mathfrak{D}_2$, the quotient group $G/Z(G)$ may be uncountable. Indeed it is easy to construct an extra-special p-group G for which $G/Z(G)$ is uncountable: of course G' has order p, so $G \in \mathfrak{D}_2$.

The class $\mathfrak{D}_2$ is not closed with respect to forming quotients, as the next result shows.

LEMMA 2. *A free group F belongs to $\mathfrak{D}_2$ if and only if it has countable rank.*

Proof. If F has uncountable rank and H is a free subgroup of rank 2, then F' and H' have different ranks. Hence, $F' \not\simeq H'$ and $F \notin \mathfrak{D}_2$. On the other hand, if F has countable rank and H is a non-abelian subgroup, then H' is free with countable rank and $H' \simeq F'$, so F belongs to $\mathfrak{D}_2$. □

The next result plays a fundamental role in the study of infinite $\mathfrak{D}_2$-groups.

PROPOSITION 1. *Let G be a perfect group in $\mathfrak{D}_2$. Then G has no proper subgroups of finite index.*

Proof. Assume that G/N is a non-trivial finite quotient of G. Suppose that A/N' is an abelian normal subgroup of G/N'; it will be shown that $[A, G] \leq N'$. Let $x \in G$ and assume that $[A, x] \neq 1$. Now $\langle x, A\rangle' = A'[A, x] \leq A$ and $\langle x, A\rangle'' \leq A' \leq N'$ since A/N' is abelian. Also, $\langle x, A\rangle$ is non-abelian, so we have $\langle x, A\rangle' \simeq G' = G$ and $\langle x, A\rangle'$ is perfect. Hence, $\langle x, A\rangle' = \langle x, A\rangle'' \leq N'$ and $[A, x] \leq N'$, which establishes our claim. In particular $N/N' \leq Z(G/N')$ and $(G/N')/Z(G/N')$ is finite.

The last step in the proof is to show that $(G/N')/Z(G/N')$ has odd order: it will then follow via the Odd Order Theorem that G/N' is soluble. Since G is perfect, the conclusion will be that $N = G$, a contradiction.

Suppose that $x, y \in G$ satisfy $x^2N', y^2N' \in Z(G/N')$. Then $[x^2, y] \in N'$ and hence $[x, y]^x \equiv [x, y]^{-1} \bmod N'$: similarly $[x, y]^y \equiv [x, y]^{-1} \bmod N'$. Consequently, $\langle [x, y]\rangle N' \lhd \langle x, y\rangle N'$. It follows that $(\langle x, y\rangle N')' \leq \langle [x, y]\rangle N'$. Also, $(\langle [x, y]\rangle N')' \leq N'$ and either $\langle x, y\rangle N'$ is abelian or else $(\langle x, y\rangle N')' \simeq G'$, which is perfect. Hence,

$$(\langle x, y\rangle N')' = (\langle x, y\rangle N')'' \leq (\langle [x, y]\rangle N')' \leq N'.$$

Thus, in any event $\langle x, y\rangle N'/N'$ is abelian. Therefore, $(xy)^2N' \in Z(G/N')$. Now define W/N' to be the set $\{xN' \mid x^2N' \in Z(G/N')\}$. The argument just presented shows that W/N' is an abelian subgroup of G/N', while clearly $W \lhd G$. By the first part of the proof $W/N' \leq Z(G/N')$, which shows that $(G/N')/Z(G/N')$ contains no elements of order 2, as required. □

COROLLARY 1. *Let G be a $\mathfrak{D}_2$-group and assume that G' has a proper subgroup of finite index. Then the derived series of G reaches the identity subgroup transfinitely, i.e. G is a hypoabelian group.*

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Proof. There is an ordinal $\alpha \geq 1$ such that $G^{(\alpha)} = G^{(\alpha+1)}$. Suppose that $G^{(\alpha)} \neq 1$. Then $G^{(\alpha+1)} \neq 1$ and $G^{(\alpha)}$ is not abelian so that $G' \simeq (G^{(\alpha)})' = G^{(\alpha+1)} = G^{(\alpha)}$. Therefore, G' is perfect; moreover, by Lemma 1 it belongs to $\mathfrak{D}_2$, which contradicts Proposition 1. □

In this context it is a reasonable to ask whether a residually finite $\mathfrak{D}_2$-group is always residually soluble.

Since our aim is to study non-abelian $\mathfrak{D}_2$-groups, it is natural to look first at nilpotent $\mathfrak{D}_2$-groups: these admit a very simple description, (cf. [**5**], Corollary 2).

THEOREM 1. *Let G be a non-abelian group. Then G is nilpotent and belongs to $\mathfrak{D}_2$ if and only if G' is cyclic of prime or infinite order and $G' \leq Z(G)$.*

Proof. Assume that $G \in \mathfrak{D}_2$ is nilpotent and put $Z_i = Z_i(G)$. Suppose that Z_2/Z_1 is not torsion-free so that it contains an element xZ_1 of some prime order p. There exists a $y \in G$ such that $[x, y] \neq 1$. Then $[x, y]^p = [x^p, y] = 1$, so $[x, y]$ has order p. Put $H = \langle x, y\rangle$; then $H' = \langle [x, y]\rangle$ has order p, and so does G'. Since G is nilpotent, G' must be contained in the centre of G.

Now assume that Z_2/Z_1 is torsion-free. The group G has a finitely generated non-abelian subgroup K and $G' \simeq K'$. Thus, we may assume that G is finitely generated, say by $g_1, g_2, \ldots, g_n$. Let $x \in Z_2 \backslash Z_1$. Now there is an i such that $[x, g_i]$ has infinite order: for otherwise there would exist $m > 0$ such that $[x^m, g_i] = [x, g_i]^m = 1$ for all i and $x^m \in Z_1$. With this i set $L = \langle x, g_i\rangle$. Then $L' = \langle [x, g_i]\rangle \simeq G'$ and G' is infinite cyclic: once again $G' \leq Z(G)$. The converse is clearly true. □

Note that a nilpotent $\mathfrak{D}_2$-group has class at most 2 so that locally nilpotent $\mathfrak{D}_2$-groups are nilpotent.

3. Groups with finite derived subgroup. In this section we classify $\mathfrak{D}_2$-groups with finite derived subgroup. The essential components of such groups are certain finite metabelian groups constructed from pairs of integers. We begin by describing these groups.

Let p be a prime and $m > 1$ an integer not divisible by p. Let F be a field of order p^n, where n is the order of p modulo m, which will be written as

$$n = |p|_m.$$

The multiplicative group F^* contains a unique (cyclic) subgroup $X = \langle x\rangle$ of order m. Also, $(F : \mathbb{Z}_p) = n = |p|_m = (\mathbb{Z}_p(x) : \mathbb{Z}_p)$ and hence $F = \mathbb{Z}_p(x)$. Regard $A = F^+$, the additive group of F, as an X-module via the field multiplication: then it is easy to show that A is a simple X-module and $C_A(y) = 0$ if $1 \neq y \in X$.

Next form the semi-direct product

$$G(p, m) = X \ltimes A.$$

Then $(G(p, m))' = A$ and $|G(p, m)| = mp^n$.

LEMMA 3. *The group $G(p, m)$ belongs to $\mathfrak{D}_2$ if and only if $|p|_m = |p|_d$ for every divisor $d > 1$ of m.*

It will be convenient to call the pair (p, m) *allowable* if the condition in Lemma 3 holds.

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Proof of Lemma 3. Let $G = G(p, m) = X \ltimes A$, as constructed above. Assume that (p, m) is an allowable pair and H is a non-abelian subgroup of G. Then H has the form $\langle x^r a_o, H \cap A\rangle$, where $H \cap A \neq 1$, $1 \leq r < m$ and $a_0 \in A$. Note that $H \cap A$ is a non-trivial $\langle x^r\rangle$-submodule of A and by Maschke's Theorem it is a direct sum of simple $\langle x^r\rangle$-modules. By allowability, each of the latter has dimension $|p|_d = |p|_m = n$, where $d = |x^r| = m/\gcd(m, r)$. Hence, $H \cap A = A$ and $A \leq H$. Therefore, $H = \langle x^r, A\rangle$ and $H' = [A, x^r] = A = G'$ since $x^r \neq 1$. Thus, $G \in \mathfrak{D}_2$.

Conversely, assume that $G \in \mathfrak{D}_2$ and let $d > 1$ divide m: put $r = \frac{m}{d}$. Suppose that B is a non-zero $\langle x^r\rangle$-submodule of A and set $H = \langle x^r, B\rangle$. Then H is non-abelian and hence $A \simeq H' = [B, x^r] \leq B$, which shows that $A = B$ and A is a simple $\langle x^r\rangle$-module. It follows that $F = \mathbb{Z}_p(x^r)$ and $n = (F : \mathbb{Z}_p) = |p|_{|x^r|} = |p|_d$ so that (p, m) is allowable. □

We can now state the main result on $\mathfrak{D}_2$-groups with finite derived subgroup. (A weaker form of this appears in [**5**, Theorem 8).

THEOREM 2. *Let G be a non-nilpotent group with G' finite. Then $G \in \mathfrak{D}_2$ if and only if the following conditions hold:*

(i) *$G = X \ltimes A$ where $A = G'$ is an elementary abelian p-group, $Z(G) = C_X(A)$ and $X/Z(G)$ is cyclic of order m;*

(ii) *(p, m) is an allowable pair and $G/Z(G) \simeq G(p, m)$.*

Proof. Assume that $G \in \mathfrak{D}_2$. First suppose that G is not soluble. Then G' is not abelian and, since it is finite, G' contains a minimal non-abelian subgroup H. By a classical theorem of Miller and Moreno ([**7**]), the subgroup H is soluble. However, $G' \simeq H'$, which yields the contradiction that G is soluble. Write $A = G'$. If A is non-abelian, $G' \simeq A' = G''$ and $G' = 1$, which shows that A is abelian and G is metabelian. Note that $A \not\leq Z(G)$ since G is not nilpotent.

Let $x \in G\backslash C_G(A)$ and set $H = \langle x, A\rangle$. Then $A = G' \simeq H' = [A, x]$, so $A = [A, x]$, which implies that $C_A(x) = 1$ because A is finite. Thus, x acts fixed point freely on A. Next, let $1 < B \leq A$ and assume that $B = B^x$. If $K = \langle x, B\rangle$, we have $A \simeq K' = [B, x] \leq B$, which shows that $A = B$ and A is a simple $\langle x\rangle$-module for all $x \in G\backslash C_G(A)$. Consequently, A is an elementary abelian p-group for some prime p.

Since A is a non-trivial, finite, simple G-module and G/A is abelian, G splits over A (see [**9**, Theorem 1]): Let us say $G = X \ltimes A$. If $ya \in Z(G)$ with $y \in X, a \in A$, then $[A, y] = 1$. Also, $[a, X] = 1$, which shows that $a = 1$ and $Z(G) = C_X(A) = Z$, say. Since A is a faithful simple X/Z-module, X/Z is cyclic with order m prime to p.

Put $\overline{G} = G/Z$, $\overline{A} = AZ/Z$ and $\overline{X} = X/Z$. Then $\overline{A}$ is a faithful simple $\overline{X}$-module and $\overline{X}$ is cyclic of order m, so we can identify $\overline{A}$ with the additive group of a field of order p^n, where $n = |p|_m$, and $\overline{X} = \langle\overline{x}\rangle$ with a subgroup of order m of the multiplicative group. Thus, $\overline{G} = \overline{X} \ltimes \overline{A} \simeq G(p, m)$. Since $\bar{G} \in \mathfrak{D}_2$ by Lemma 1, the pair (p, m) is allowable by Lemma 3.

Conversely, assume that G has the structure indicated in the theorem. Then $G/Z \simeq G(p, m) \in \mathfrak{D}_2$ by Lemma 3. Let H be a non-abelian subgroup of G. Then HZ/Z is non-abelian, since otherwise $H' \leq A \cap Z = 1$. Hence, $(HZ/Z)' \simeq (G/Z)'$, i.e. $H'Z = G'Z$ since G' is finite. Therefore, $G' = G' \cap (H'Z) = H'$ and $G \in \mathfrak{D}_2$. □

COROLLARY 2. *If G is a non-nilpotent, locally finite group in $\mathfrak{D}_2$, then G' is finite and the structure of G is given by Theorem 2.*

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For the group G has a finite non-abelian subgroup H and $G' \simeq H'$. Note, however, that the corollary does not hold if only G' is locally finite: indeed the wreath product $\mathbb{Z}_2 \operatorname{wr} \mathbb{Z}$ is easily seen to belong to $\mathfrak{D}_2$.

3.1. Digression on allowable pairs of integers. Since allowable pairs play a central role in the theory of $\mathfrak{D}_2$-groups with finite derived subgroup, we interrupt the narrative with a brief discussion of their properties.

LEMMA 4. *Let p be a prime and $m > 1$ an integer not divisible by p. Then (p, m) is allowable if and only if $|p|_m = |p|_q$ for every prime q dividing m.*

Proof. Let $d > 1$ be a divisor of m and let q be a prime dividing d. Then $|p|_q \mid |p|_d \mid |p|_m$ and sufficiency follows at once. The converse is obvious. □

COROLLARY 3. *If $m = q_1^{e_1} \cdots q_k^{e_k}$ is the primary decomposition of m, then (p, m) is allowable if and only if each $(p, q_i^{e_i})$ is allowable and $|p|_{q_1} = \cdots = |p|_{q_k}$.*

Thus, the problem of finding allowable pairs (p, m) is reduced to the case where $m = q^e$, with $q \neq p$ a prime. In this case allowability is expressed by a simple congruence.

LEMMA 5. *Let p, q be distinct primes and let e be a positive integer. Then (p, q^e) is allowable if and only if $p^{q-1} \equiv 1 \pmod{q^e}$.*

Proof. If $q = 2$, we have $|p|_q = 1$ and hence $(p, 2^e)$ is allowable if and only if $|p|_{2^e} = 1$, i.e. $p \equiv 1 \pmod{2^e}$. Now let $q > 2$. Then $\mathbb{Z}^*_{q^e} = A \times B$, where $A \simeq \mathbb{Z}_{q-1}$, $B \simeq \mathbb{Z}_{q^{e-1}}$. Write $[a]$ for $[a]_{q^e} \in \mathbb{Z}^*_{q^e}$. If $|p|_q = |p|_{q^e} = |[p]|$, then $|[p]|$ divides $q - 1$ and $p^{q-1} \equiv 1 \pmod{q^e}$. Conversely, suppose that $p^{q-1} \equiv 1 \pmod{q^e}$, so $[p]^{q-1} = [1]$ and $[p] \in A$. The assignment $[x] \mapsto [x]_q$ yields a surjective homomorphism from $\mathbb{Z}^*_{q^e}$ to $\mathbb{Z}^*_q \simeq \mathbb{Z}_{q-1}$ with kernel B. Therefore, $|[p]| = |[p]_q|$ and $|p|_{q^e} = |p|_q$, showing that (p, q^e) is allowable. □

For distinct primes p and q put $n = |p|_q$ and define

$$e(p, q)$$

to be the largest integer such that $p^n \equiv 1 \pmod{q^{e(p,q)}}$. Note that $1 \leq e(p, q) < p^n$, so $e(p, q)$ is finite. Plainly a pair (p, q^e) is allowable if and only if $e \leq e(p, q)$ exists. At this point we formulate a question: Given any prime p, does there exist a prime q such that $e(p, q) \geq 2$, or equivalently such that $p^{q-1} \equiv 1 \pmod{q^2}$? Such a prime q is called a *base-p Wieferich prime*, after the German number theorist Arthur Wieferich. Group theoretically we are asking if there is a prime q such that $G(p, q^2) \in \mathfrak{D}_2$.

This is a difficult number theoretic problem. A computer search reveals that the answer is positive for all primes $p < 100$ with the possible exception of 47. The case $p = 2$ is of special interest: $e(2, q) \geq 2$ if and only if $2^{q-1} \equiv 1 \pmod{q^2}$. Only two such primes q are known, 1093 and 3511. There is a curious connection between the Wieferich primes and the so-called first case of Fermat's Last Theorem, for which see **[4]**. (We are grateful to M. Mazur and S. Ullom for information about the Wieferich primes).

4. Soluble groups. In this section the structure of infinite soluble $\mathfrak{D}_2$-groups is analysed.

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THEOREM 3. *Let G be a non-nilpotent, soluble $\mathfrak{D}_2$-group and set $A = G'$. Then:*

(i) *A is abelian, so that G is metabelian.*[1]

(ii) *A is an elementary p-group for some p, a free abelian group or a torsion-free minimax group.*

(iii) *If A is torsion-free minimax and $x \in G \backslash C_G(A)$, then $C_A(x) = 1$.*

(iv) *If $1 < [B, \langle x\rangle] \leq B \leq A$ and $x \in G$, then $B \simeq A$.*

(v) *Nilpotent subgroups of G are abelian.*

Proof. (i) Suppose G has derived length $d > 2$. Then $G^{(d-2)}$ is non-abelian and $A = G' \simeq (G^{(d-2)})' = G^{(d-1)}$, which yields the contradiction that A is abelian.

(ii) Assume that A contains an element of prime order p. Since G is not nilpotent, there exists $x \in G\backslash C_G(A)$ and then $A \simeq \langle x, A\rangle' = [A, x]$, showing that $A \simeq A/C$ where $C = C_A(x)$. Hence, A/C has an element of order p, say aC. Thus, $[a, x]^p = [a^p, x] = 1$ and $[a, x]$ has order p since $a \notin C$. Hence, $A \simeq \langle x, a\rangle' = [a, x]^{\langle x\rangle}$ and it follows that A is an elementary p-group.

Now assume that A is torsion-free. There exist $a \in A$ and $x \in G$ such that $[a, x] \neq 1$, and hence $A \simeq \langle x, a\rangle' = [a, x]^{\langle x\rangle} \leq a^{\langle x\rangle}$. If $\{a^{x^i} \mid i \in \mathbb{Z}\}$ is linearly independent, $a^{\langle x\rangle}$, and hence A, is free abelian. Otherwise there is the largest $r > 0$ such that $\{a, a^x, \ldots, a^{x^{r-1}}\}$ is linearly independent. Then $F = \langle a, a^x, \ldots, a^{x^{r-1}}\rangle$ is free abelian of rank r and $a^{\langle x\rangle}/F$ is periodic. Hence, $a^{\langle x\rangle}$ has rank r, as must A. Finally, $H = \langle x, a\rangle$ is a finitely generated soluble group of finite rank, so it is a minimax group ([**8**]). Therefore, $A \simeq H'$ is also minimax.

(iii) If A is torsion-free minimax, the isomorphism $A \simeq A/C_A(x)$ implies that $C_A(x) = 1$.

(iv) Assume that $1 < [B, \langle x\rangle] \leq B \leq A$, so that $B = B^x$. In addition, $A \simeq \langle x, B\rangle' = [B, x]$. If A is either elementary p or free abelian, then so is B. Also, $r(A) = r([B, x]) \leq r(B) \leq r(A)$, whence $r(A) = r(B)$ and $B \simeq A$. If, on the other hand, A is torsion-free minimax, $C_A(x) = 1$ by (iii) and hence $[B, x] \simeq B$. Therefore, once again $B \simeq A$.

(v) Suppose that N is a non-abelian, nilpotent subgroup of G. Since $N \in \mathfrak{D}_2$, we have $A \simeq N' \simeq \mathbb{Z}$ or $\mathbb{Z}_p$ for some prime p by Theorem 1; therefore, $[N', N] = 1$. Since $1 \neq N' \leq A$ and $A \simeq \mathbb{Z}$ or $\mathbb{Z}_p$, it follows that $[A, N] = 1$. Consequently, $[N', G] \leq [N, G, N] \leq [A, N] = 1$ and $N' \leq Z(G)$. This gives the contradiction $A \leq Z(G)$. □

There is a simple converse to Theorem 3.

PROPOSITION 2. *Let G be a metabelian group and set $A = G'$. Assume that the following conditions hold:*

(i) *If $1 < [B, \langle x\rangle] \leq B \leq A$ for some $x \in G$, then $B \simeq A$.*

(ii) *Nilpotent subgroups of G are abelian.*

Then $G \in \mathfrak{D}_2$.

Proof. Let H be a non-abelian subgroup of G and put $B = H'$. Then H is not nilpotent, so there exists $x \in H$ such that $[B, x] \neq 1$. Also, $[B, \langle x\rangle] \leq B$, so $B \simeq A$ by (i) and $G \in \mathfrak{D}_2$. □

[1]In fact, a soluble $\mathfrak{D}_n$-group has derived length at most n for every n.

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COROLLARY 4. *Let G be a free soluble group. Then $G \in \mathfrak{D}_2$ if and only if G is free abelian or free metabelian of countable rank.*

Proof. Indeed, assume that $G \in \mathfrak{D}_2$ is non-abelian; then G is metabelian by Theorem 3 and clearly it must be free metabelian. By Lemma 1 G' is countable, which shows that G has countable rank. Conversely, if G is free metabelian of countable rank, it is clear that nilpotent subgroups are abelian. Suppose that $1 < [B, \langle x \rangle] \leq B \leq A = G'$; then the map $b \mapsto [b, x]$, $b \in B$, is injective, so $B \simeq [B, x]$, while it is easy to see that $[B, x]$ cannot be finitely generated. Hence, $B \simeq A$ and $G \in \mathfrak{D}_2$ by Proposition 2. □

REMARK. (a) The three possibilities for A envisaged in Theorem 3 all occur, as is shown by the wreath products $\mathbb{Z}_p \operatorname{wr} \mathbb{Z}$, $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ and the infinite dihedral group.

(b) When A is finitely generated and free abelian, condition (iv) of Theorem 3 is equivalent to A being rationally irreducible as an $\langle x \rangle$-module for all $x \in G \backslash C_G(A)$.

(c) The fixed point free action in (iii) of Theorem 3 need not hold when A is elementary or free abelian of infinite rank. To see this let $X = \langle x \rangle \times \langle y \rangle$ be free abelian of rank 2 and let $A_0 = RX \oplus R\langle y \rangle$, where $R = \mathbb{Z}_p$ or $\mathbb{Z}$. Regard A_0 as an X-module via the natural action of X on RX and of $\langle y \rangle$ on $R\langle y \rangle$, with x acting trivially on $R\langle y \rangle$. Define G to be the semi-direct product $X \ltimes A_0$. Then $A = G' = I_X \oplus I_{\langle y \rangle}$, where I_X, $I_{\langle y \rangle}$ are augmentation ideals. If H is a non-abelian subgroup of G, it is straightforward to see that $H' \simeq G'$ so that $G \in \mathfrak{D}_2$. However, $C_A(x) \neq 0$.

4.1. Groups of finite rank. Soluble $\mathfrak{D}_2$-groups with finite rank have additional structure over and above that described in Theorem 3. We can restrict ourselves to the case where the derived subgroup is torsion-free minimax in view of Theorems 2 and 3.

THEOREM 4. *Let G be a non-nilpotent, soluble $\mathfrak{D}_2$-group such that $A = G'$ is a torsion-free minimax group. Then the following hold.*

(i) *If $1 < B \leq A$ and $B = B^x$, where $x \in G \backslash C_G(A)$, then $|A : B|$ is finite: hence A is $\langle x \rangle$-rationally irreducible.*
(ii) *If $C = C_G(A)$, then G/C is finitely generated and A is a noetherian G/C-module.*
(iii) *There is an abelian subgroup U such that $U \cap A = 1$ and $|G : UA|$ is finite.*
(iv) *$G/Z(G)$ is a finitely generated metabelian minimax group in $\mathfrak{D}_2$.*

Proof. (i) By Theorem 3(iii) we have $[B, x] \neq 1$ and hence $B \simeq A$. Since $r_0(B) = r_0(A)$ and A is minimax, A/B is a Černikov group. Now a torsion-free abelian minimax group has a series whose infinite factors are cyclic or quasi-cyclic; moreover, the (multi-)set of infinite factors is an invariant of the group. Applying this fact to a series in A through B, we conclude that A/B is finite.

(ii) By (i) A is rationally irreducible as a G/C-module and a theorem of Baer [1] shows that G/C is finitely generated. Let $1 \neq a \in A$; then A/a^G is finite by (i), so A is a finitely generated G/C-module and hence it is a noetherian G/C-module.

(iii) Since G is not nilpotent, there exists $x \in G$ such that $[A, x] \neq 1$ and thus $C_A(x) = 1$ by Theorem 3(iii). Now apply [**6**, 6.1.4] to obtain a subgroup U as described. (In fact, we could take U to be $C_G(x)$.)

(iv) Set $G_0 = UA$, noting that $G_0 \lhd G$ and $Z_0 = Z(G_0) \lhd G$: in fact, $Z_0 = C_U(A)$. Also, $[Z_0, G] \leq Z_0 \cap A = 1$ and thus $Z_0 \leq Z = Z(G)$. Next $G/(G_0 \cap C)$ is a finitely generated abelian group and $G_0 \cap C = C_{G_0}(A) = Z_0 \times A$. Since $Z_0A/Z_0 \overset{G}{\simeq} A$, the G-module Z_0A/Z_0 is noetherian and hence G/Z_0 is finitely generated as is G/Z. Since G/Z has finite rank, it is a minimax group.

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The last step in the proof is to show that $G/Z \in \mathfrak{D}_2$ by using Proposition 2. If N/Z is nilpotent, then N is nilpotent and hence abelian by Theorem 3. Thus, nilpotent subgroups of G/Z are abelian. Next assume that $1 < [B/Z, \langle xZ\rangle] \leq B/Z \leq AZ/Z$ for some $x \in G$. Then $B = B^x$ and $B = B \cap (AZ) = B_0Z$, where $B_0 = B \cap A$. Since $B_0 \neq 1$ and $B_0^x = B_0$, we have $B_0 \simeq A$ by (i). Therefore, $B/Z = B_0Z/Z \simeq B_0 \simeq A \simeq AZ/Z$ and it follows that $G/Z \in \mathfrak{D}_2$. $\square$

4.2. Constructing soluble $\mathfrak{D}_2$-groups of finite rank. As is apparent from the proof of Theorem 4, the essential part of an infinite, non-nilpotent soluble $\mathfrak{D}_2$-group G with finite rank is a factor

$$\overline{G} = \overline{U} \ltimes A_0 :$$

here $A_0 = a^G$ for a fixed $a \neq 1$ in $A = G'$, $\overline{U} = U/C_U(A_0)$ is abelian, A is torsion-free minimax and $\overline{U}$-rationally irreducible, and $\overline{G}$ is finitely generated. There is a well-established connection between groups with this structure and algebraic number fields. Note that $F = A_0 \otimes \mathbb{Q}$ is a simple $\mathbb{Q}\overline{U}$-module and the assignment $r + I \mapsto (a \otimes 1)r$, $(r \in \mathbb{Q}\overline{U})$ yields a ring isomorphism $\mathbb{Q}\overline{U}/I \to F$, where $I = \mathrm{Ann}_{\mathbb{Q}\overline{U}}(a)$, a maximal ideal of $\mathbb{Q}\overline{U}$. Thus, F is an algebraic number field and we may identify A_0 and $\overline{U}$ with subgroups of F^+ and F^* respectively. Moreover, $A_0 = \mathrm{Rg}\langle\overline{U}\rangle$ and $F = \mathbb{Q}(\overline{U})$.

Conversely, suppose we start with an algebraic number field F and a non-trivial finitely generated subgroup X of F^* such that $F = \mathbb{Q}(X)$. Let C be the subring of F generated by X and regard C as an X-module in the natural way. Now form the group

$$G = G(F, X) = X \ltimes C.$$

Since $G = \langle X, 1_F\rangle$, this is a finitely generated metabelian group. Also, $F = \mathbb{Q}(X)$, so we have $r_0(C) = (F : \mathbb{Q})$ and G has finite rank; hence it is a minimax group. Note that if X is a subgroup of the group of units of F, then G will be polycyclic.

It is easy to see that any nilpotent subgroup of G is abelian and that $A := G' = [C, X]$. By Proposition 2 the group G belongs to $\mathfrak{D}_2$ if and only if $B \simeq A$ whenever $0 \neq B = Bx \leq A$ and $1 \neq x \in X$. Let us call the pair (F, X) *allowable* if this condition is valid, the analogy with allowable pairs of integers being evident. In conclusion $G(F, X) \in \mathfrak{D}_2$ *if and only if* (F, X) *is an allowable pair*. Note that if X is a group of units of F, then (F, X) is allowable if and only if $C = \mathrm{Rg}\langle X\rangle$ is $\langle x\rangle$-rationally irreducible for all $x \neq 1$ in X.

EXAMPLES. (i) If $F = \mathbb{Q}$ and $X = \langle -1\rangle$, then $C = \mathbb{Z}$ and $G(F, X)$ is the infinite dihedral group. Obviously, (F, X) is allowable and $G(F, X) \in \mathfrak{D}_2$.

(ii) Let $m > 1$ be an integer and let $F = \mathbb{Q}$, $X = \langle m\rangle$. Thus, $C = \left\{\frac{r}{m^s} \,\middle|\, r, s \in \mathbb{Z}\right\}$. Allowability is easy to check and thus $G(F, X) \in \mathfrak{D}_2$: this group is isomorphic with the Baumslag–Solitar group $\langle x, a \mid a^x = a^m\rangle$.

(iii) Let $F = \mathbb{Q}(\sqrt{2})$ and $X = \langle c\rangle$, where $c = 1 + \sqrt{2}$. Since $c^2 - 2c - 1 = 0$, c is an algebraic unit in F and $G(F, X)$ is polycyclic. Also, $C = \mathrm{Rg}\langle c\rangle$ is free abelian of rank 2 and $c^k \notin \mathbb{Q}$ for $k > 0$, so C is $\langle x^k\rangle$-rationally irreducible and (F, X) is allowable; hence, $G(F, X) \in \mathfrak{D}_2$. In this case $G(F, X)$ has the presentation $\langle t, u, v \mid [u, v] = 1,\ u^t = v,\ v^t = uv^2\rangle$.

(iv) Let $F = \mathbb{Q}(\sqrt{3})$ and $X = \langle c\rangle$, where $c = 1 + \sqrt{3}$. Then $c^2 - 2c - 2 = 0$ and $C = \mathrm{Rg}\langle c\rangle$ satisfies $C = 2C$. Hence, C is a free $\mathbb{Q}_2$-module of rank 2, where $\mathbb{Q}_2$ is the ring of dyadic rationals. Also, $A = G(F, X)'$ is a free module since $\mathbb{Q}_2$ is a principal ideal

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domain. Let $k > 0$. A routine calculation reveals that c^k has irreducible polynomial of the form $t^2 + 2rt + 2s$, $(r, s \in \mathbb{Z})$. If $B \neq 0$ is a $\langle c^k \rangle$-submodule of C, then $B = 2B$ and B is a free $\mathbb{Q}_2$-module of rank 2. Thus, $B \simeq A$, (F, X) is allowable and $G(F, X) \in \mathfrak{D}_2$.

5. Groups with non-perfect derived subgroup. We continue our study of $\mathfrak{D}_2$-groups by considering groups G such that G' is not perfect. Under the additional hypothesis that G'/G'' has finite abelian ranks, i.e. the p-rank is finite for $p = 0$ or a prime, it emerges that these groups are soluble, so they fall within the scope of the classification theorems of the previous two sections.

THEOREM 5. *Let G be a $\mathfrak{D}_2$-group such that G'/G'' is non-trivial and has finite abelian ranks. Then G is soluble and G' is either finite elementary abelian or a torsion-free abelian minimax group.*

Note that the hypothesis of finite rank cannot be omitted from the theorem since free groups of countable rank belong to $\mathfrak{D}_2$. During the proof of Theorem 5 we will use two auxiliary results about nilpotent groups which may be known. If n is a positive integer, let $e(n)$ denote the sum of the exponents in the primary decomposition of n.

LEMMA 6. *Let G be a nilpotent group, n a positive integer and $S = \gamma_n(G)$. If S'/S'' is finite and $e(|S'/S''|) \leq n$, then S is metabelian.*

Proof. There is a G-central series

$$S'/S'' > [S', G]S''/S'' > \cdots > [S',_m G]S''/S'' = 1$$

for some $m \geq 0$. The inequality $e(|S'/S''|) \leq n$ shows that $m \leq n$ and hence that $[S',_n G] \leq S''$. In addition

$$S'' \leq [S', S, S] \leq [S',_n G, S] \leq [S'', S],$$

which shows that $S'' = [S'', S]$ and thus $S'' = 1$ since S is nilpotent. □

LEMMA 7. *Let G be a nilpotent group, n a positive integer and $S = \gamma_n(G)$. If $r_0(S'/S'') \leq n$, then S'' is periodic.*

Proof. There is a G-central series

$$S'/S'' > [S', G]S''/S'' > \cdots > [S',_m G]S''\big/S'' = 1$$

for some $m \geq 0$, and there is a least $i \geq 0$ such that the factor $[S',_i G]S''/[S',_{i+1} G]S''$ is periodic. By forming successive tensor products with G/G', we deduce that all subsequent factors in the central series are periodic, and since $r_0(S'/S'') \leq n$, it follows that $i \leq n$. Hence, $[S',_n G]S''/S''$ is periodic. By an easy argument using the nilpotency of G, we deduce that $[S',_n G, S][S'', S]\big/[S'', S]$ is periodic. As in the proof of Lemma 6, we have $S'' \leq [S', S, S] \leq [S',_n G, S]$, which implies that $S''/[S'', S]$ is periodic. It follows that S'' is periodic. □

Proof of Theorem 5. By Theorem 3 it is enough to prove that G is soluble, so assume this is not true. Then G has a finitely generated insoluble subgroup H since by Theorem 3 soluble $\mathfrak{D}_2$-groups are metabelian. Then $G' \simeq H'$ and $G'/G'' \simeq H'/H''$. Now H/H'' is a finitely generated metabelian group, so it satisfies max-n, the maximal condition on normal subgroups, by a theorem of P. Hall (see [**6**, 4.2.2]). Therefore, the

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torsion-subgroup of H'/H'' has finite exponent. Of course, the same is true of G'/G'', and since G'/G'' has finite abelian ranks, we deduce that its torsion-subgroup is finite, say of order t. Let $n = r_0(G'/G'')$, which is also finite.

First of all consider the case where $n = 0$, i.e. G'/G'' has finite order t. Put $e = e(t) > 0$. Observe that $G' \simeq G^{(i)}$ and $G'/G'' \simeq G^{(i)}/G^{(i+1)}$ for all $i \geq 1$: thus $|G^{(i)}/G^{(i+1)}| = t$. Writing $C_i = C_G(G^{(i)}/G^{(i+1)})$, we have $|G/C_i| \leq t! = m$, say. Since $G^{(i)} \leq C_i$, the group G/C_i is soluble and its derived length is at most m, so $G^{(m)} \leq C_i$ for all i. Hence, $G^{(m)}/G^{(i)}$ is nilpotent for $i \geq m$, as is $G'/G^{(i-m+1)}$. Thus, $G'/G^{(j)}$ is nilpotent for all $j \geq 1$.

Now set $j = e + 2$ and write $K = G'$. Then $K/K^{(j)} = G'/G^{(j+1)}$ is nilpotent. Put $S = \gamma_e(K)$ and $\overline{K} = K/K^{(j)}$; thus $\overline{S} := \gamma_e(\overline{K}) = SK^{(j)}/K^{(j)}$. If S is not abelian, $S'/S'' \simeq K'/K''$ and hence $e(|\overline{S}'|/|\overline{S}''|) \leq e(|K'/K''|) = e(t) = e$, a conclusion that is still valid if S is abelian. We are now in a position to apply Lemma 6 to the group $\bar{K}$, so we may conclude that $\overline{S}$ is metabelian. From this it follows that $S'' \leq K^{(j)}$. Also, $K^{(e-1)} \leq S$, so $K^{(e+1)} \leq S'' \leq K^{(j)}$, whence $K^{(e+1)} = K^{(e+2)}$ and $K^{(e+1)} = 1$, i.e. $G^{(e+2)} = 1$, a contradiction.

Next we address the case $n > 0$. As before, write $C_i = C_G(G^{(i)}/G^{(i+1)})$, so that G/C_i is isomorphic with a soluble group of automorphisms of the abelian group $G^{(i)}/G^{(i+1)}$. The latter has finite torsion-free rank n. By a well-known theorem of Zassenhaus (see [**6**, 3.1.10]), the derived length of a soluble linear group of degree n cannot exceed $f(n)$ for some function f. It follows that G/C_i has derived length at most $s := f(n) + t! + 1$, where t is the order of the torsion-subgroup of G'/G''. Hence, $G^{(s)} \leq C_i$ for all i and $G^{(s)}/G^{(\ell)}$ is nilpotent for $\ell \geq s$. Therefore, $G'/G^{(\ell-s+1)}$ is nilpotent, as is $G'/G^{(j)}$ for $j \geq 1$.

Now set $j = n + 2$ and write $K = G'$. Then $K/K^{(j)} = G'/G^{(j+1)}$ is nilpotent. Put $S = \gamma_n(K)$ and $\overline{K} = K/K^{(j)}$; thus $\overline{S} := \gamma_n(\overline{K}) = SK^{(j)}/K^{(j)}$. If S is not abelian, $S'/S'' \simeq K'/K''$ and hence $r_0(\overline{S}'/\overline{S}'') \leq r_0(G'/G'') = n$, a conclusion that is still valid if S is abelian. We can now apply Lemma 7 to the group $\bar{K}$ and conclude that $\overline{S}''$ is periodic. As above $K^{(n+1)} \leq S''$, which implies that $K^{(n+1)}/K^{(j)}$ is periodic. Consequently, G'/G'' is periodic, a final contradiction. □

COROLLARY 5. *Let G be a periodic $\mathfrak{D}_2$-group. If G' is not perfect, then G is soluble.*

Proof. Assume this is false: clearly we may assume G is finitely generated. Then G/G' is finite and G' is finitely generated, which gives a contradiction by the Theorem 5. □

COROLLARY 6. *Let G be a periodic $\mathfrak{D}_2$-group. If G is locally graded, then it is soluble.*

Proof. As usual, we can assume that G is finitely generated, so G' is also finitely generated. Since G is locally graded, G' has a non-trivial finite quotient and hence $G' \neq G''$ by Proposition 1. The result now follows from Corollary 5. □

5.1. Elements of finite order in $\mathfrak{D}_2$-groups. Elements of finite order in a $\mathfrak{D}_2$-group are subject to surprisingly strong restrictions, at least if the group is insoluble and its derived subgroup is not perfect.

THEOREM 6. *Let G be an insoluble $\mathfrak{D}_2$-group such that G' is not perfect. Then the elements of G with finite order form a subgroup F of $Z(G)$ and G/F is in $\mathfrak{D}_2$.*

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Proof. Let a, b be elements of finite order in G and put $H = \langle a, b\rangle$. Suppose that H is not abelian so that $H' \simeq G'$. Now H/H' is finite, so H' is finitely generated and H' is not perfect; hence, H is soluble by Theorem 5, and so is G. Consequently, H must be abelian and $[a, b] = 1$, which demonstrates that elements of G with finite order form an abelian normal subgroup F. If $[F, g] \neq 1$ for some $g \in G$, then $G' \simeq \langle g, F\rangle' \leq F$ and G is soluble. Therefore, $F \leq Z(G)$.

Finally, let K/F be a non-abelian subgroup of G/F. Then $K' \simeq G'$, while $K' \cap F$ and $G' \cap F$ are the respective torsion-subgroups of K' and G'. Therefore, $K'/K' \cap F \simeq G'/G' \cap F$ and $(K/F)' \simeq (G/F)'$ so that $G/F \in \mathfrak{D}_2$. □

On the other hand, the elements of finite order in a soluble $\mathfrak{D}_2$-group need not form a subgroup as the infinite dihedral group shows.

6. Insoluble $\mathfrak{D}_2$-groups. Up to this point all the $\mathfrak{D}_2$-groups of whose structure we have some knowledge have turned out to be soluble. We now consider some classes of insoluble $\mathfrak{D}_2$-groups. Let

$$\mathfrak{I}$$

denote the class of groups that satisfy the *Tits alternative*, i.e. $G \in \mathfrak{I}$ if and only if either G is soluble-by-finite or it contains a free subgroup of rank 2.

THEOREM 7. *Let G be a $\mathfrak{I}$-group. Then $G \in \mathfrak{D}_2$ if and only if either G is a soluble $\mathfrak{D}_2$-group or else G' is free with countably infinite rank and L' is not finitely generated whenever L is a non-abelian subgroup of G.*

Proof. Assume that $G \in \mathfrak{D}_2$. Suppose that G has a soluble normal subgroup S with finite index in G. If $S \not\leq Z(G)$, there is a $g \in G$ such that $G' \simeq \langle g, S\rangle' \leq S$, and G is soluble. Next assume that $S \leq Z(G)$; then $G/Z(G)$ is finite, so G' is finite and again G is soluble by Theorem 2. Now suppose that no such S exists. Then G has a free subgroup F of rank 2 since $G \in \mathfrak{I}$. Thus, $G' \simeq F'$ and G' is free with countably infinite rank. If L is a non-abelian subgroup of G, we have $L' \simeq G' \simeq F'$ showing that L' is not finitely generated.

Conversely, assume the conditions hold and G is insoluble. Let L be a non-abelian subgroup of G. Then $L' \leq G'$, so L' is free of countable rank. Since by hypothesis L' is not finitely generated, $L' \simeq G'$ and $G \in \mathfrak{D}_2$. □

COROLLARY 7. *Let G be a locally free group. Then $G \in \mathfrak{D}_2$ if and only if G' is a free group of countable rank.*

Proof. On the basis that G is locally free, it is easy to see that $G \in \mathfrak{I}$. By Theorem 7 the condition on G is necessary. To prove sufficiency it is enough to show that if G' is free with countable rank and L is a non-abelian subgroup of G, then L' is not finitely generated. If this is false, $L' = K'$, where K is some finitely generated subgroup. But K is free and non-cyclic, so K' cannot be finitely generated. □

For example, the free product $\mathbb{Q}_2 * \mathbb{Z}$ is a locally free $\mathfrak{D}_2$-group which is not free. Note that this group is residually finite, and hence locally graded.

REMARKS. (i) If $G \in \mathfrak{D}_2 \cap \mathfrak{I}$, then G is locally graded. This follows directly from Theorem 7.

(ii) It is *not sufficient* for $G \in \mathfrak{D}_2$ that G' be free of countably infinite rank.

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To see this, let F be the free group with a countably infinite basis $\{a_i, b_i \mid i = 1, 2, \dots\}$. Define an automorphism t of F by $a_i^t = a_i b_i$, $b_i^t = a_i$, and write

$$G = \langle t \rangle \ltimes F.$$

Put $F_i = \langle a_i, b_i \rangle$ and observe that $[F_i, t] = F_i$ and $G' = [F, t] = F$. Let $H_i = \langle t, a_i, b_i \rangle$; then $H_i' \geq [F_i, t] = F_i$ and $H_i' = F_i$. Thus, $H_i' \not\simeq G'$ and $G \notin \mathfrak{D}_2$.

(iii) The original theorem of Tits ([10]) states that finitely generated linear groups belong to $\mathfrak{I}$, so Theorem 7 is applicable. In fact the theorem applies to any linear $\mathfrak{D}_2$-group. Indeed, let G be a linear group in $\mathfrak{D}_2$ and assume G is not soluble. Then G has a finitely generated insoluble subgroup H and $G' \simeq H'$. By Tits' theorem and Theorem 7 we see that $H \in \mathfrak{I}$ and H', and hence G' is free of countably infinite rank and so $G \in \mathfrak{I}$.

6.1. Groups whose derived subgroup satisfies the minimal condition. Up to this point none of the special types of $\mathfrak{D}_2$-group that we have studied has involved a Tarski group, yet Tarski groups certainly belong to $\mathfrak{D}_2$. Our final result shows that every insoluble $\mathfrak{D}_2$-group whose derived subgroup satisfies the minimal condition has a factor which is of Tarski type.

THEOREM 8. *Let G be an insoluble $\mathfrak{D}_2$-group such that G' satisfies the minimal condition. Then G has the following properties.*

(i) *G' is the unique smallest non-abelian subgroup of G.*
(ii) *Soluble subgroups of G are abelian.*
(iii) *G' is finitely generated and perfect.*
(iv) *The subgroup $M := G' \cap Z(G)$ is the unique maximum normal subgroup of G', and G'/M is an infinite simple group.*
(v) *G/M is a $\mathfrak{D}_2$-group.*
(vi) *If $N \lhd G$, then $N \leq Z(G)$ or $G' \leq N$.*

Proof. (i) Let H be a non-abelian subgroup of G. Then $H' \simeq G'$ and by min we have $H' = G'$ and thus $G' \leq H$.

(ii) If H were a non-abelian soluble subgroup, we would have $H' \simeq G'$ and G soluble.

(iii) Since G' is non-abelian, $G' \simeq G''$ and hence $G' = G''$, so that G' is perfect. If G' were not finitely generated, every finitely generated subgroup of G' would be abelian by (i) and G' would be abelian.

(iv) By (iii) G' has a maximal normal subgroup M. Let $x \in G$. If $[M, x] \neq 1$, then $G' \simeq \langle x, M \rangle' \leq M^{\langle x \rangle}$. Now $M^{\langle x \rangle}$ is locally nilpotent since $M \lhd G'$ and M is abelian. Hence, $M^{\langle x \rangle}$ is nilpotent by Theorem 1 and G is soluble. Therefore, [M, x] = 1 for all $x \in G$ and $M \leq G' \cap Z(G) \lhd G'$. Since $G' \cap Z(G) \neq G'$, the maximality of M shows that $M = G' \cap Z(G)$. Obviously, G'/M is simple: If it were finite, G' would be abelian-by-finite and hence soluble by the proof of Theorem 7.

(v) This follows from Lemma 1.

(vi) If N is non-abelian, then $G' \leq N$ by (i). Assume that N is abelian. If $x \in G$, then $\langle x, N \rangle$ is soluble and thus abelian by (ii). Hence, $[N, x] = 1$ and $N \leq Z(G)$. □

REMARKS. (a) The group $G'/M \simeq G'Z(G)/Z(G)$ is a finitely generated infinite simple group with all its proper subgroups abelian, so it is a Tarski group.

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(b) In the opposite direction note that properties (i) and (ii) imply that $G \in \mathfrak{D}_2$. For, if H is a non-abelian subgroup, H contains G' and hence $H' \geq G'' = G'$. Thus, $H' = G'$.

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