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EIGENVALUES IN TRAILING EDGE FLOWS

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Abstract

A wake similarity solution for symmetric uniform shear flows merging at the trailing edge of a flat plate has associated with it an eigenfunction problem which was overlooked by Hakkinen and O'Neil (1967). An asymptotic formula for large eigenvalues is obtained and compared with another such formula related to both the Goldstein (1930) inner wake solution and Tillett's (1968) similarity solution for a jet emerging from a two-dimensional channel.

1. Introduction

The problem of incompressible high Reynolds number flow past a finite flat plate in a uniform stream is of fundamental importance in fluid dynamics. The essentials of the trailing edge flow structure have been described by Stewartson (1969) and Messiter (1970). Let the plate of length L be aligned with the unperturbed mainstream velocity U_∞. Choose axes $0x^*y^*$ with 0 at the trailing edge and $0x^*$ along the wake centre line. (Asterisks label physical quantities.) The Reynolds number Re is defined by

$$\varepsilon^8 = \mathrm{Re}^{-1} = \frac{\nu}{U_\infty L}, \tag{1.1}$$

where ν is the kinematic viscosity. Stewartson and Messiter describe a triple deck region of extent $\varepsilon^3 L$ in the streamwise direction in which the transition from the Blasius profile to the Goldstein (1930) wake profile is effected. To achieve consistency, it is necessary to introduce three decks having scales $\varepsilon^3 L$, $\varepsilon^4 L$ and $\varepsilon^5 L$ in the y^*-direction. The upper deck is a region of potential flow. The middle or main deck is inviscid in character; in this deck, the Blasius profile is convected downstream with perturbations appearing as correction terms. The lower deck, or sublayer, is needed to match with the Goldstein inner wake downstream and to cope upstream with a velocity of slip over the plate associated with the main deck perturbations. It appears likely that a favourable pressure gradient operates in the upstream sublayer. Then the flow well within the lower deck very near the plate is a shear flow for which the vorticity increases from Ω^*_B to Ω^* as $\varepsilon^{-3}x^*$ increases from $-\infty$ to 0_-. Then $\Omega^* = \lambda_1\Omega^*_B$, where $\lambda_1 > 1$ and $\Omega^*_B = \varepsilon^{-4}\lambda L^{-1}U_\infty$ is the vorticity in the Blasius shear; $\lambda = 0.33206$. The limiting value of the uniform shear as $\varepsilon^{-3}x^* \to 0_-$ provides the forcing flow for a region $0(\varepsilon^6 L)$ in which the full Navier-Stokes equations are strictly required to describe the flow. Hakkinen and O'Neil (1967) found an approximate solution of the full equations in the form of matched asymptotic expansions valid for $1 \gg x^{*2} + y^{*2} \gg \varepsilon^{12}L^2$ and describing the flow of three regions (i) a wake region, (ii) an inviscid region, and (iii) an upstream lower order boundary later. Capell (1972) found similar expansions after linearizing the equations with respect to the uniform shear, in the manner of Oseén. The latter results are qualitatively and quantitatively consistent with those of Stewartson (1968) who solved this approximate problem exactly. They agree qualitatively only in part with the results of Hakkinen and O'Neil who overlooked the eigenfunction problems associated with the similarity solutions for regions (i) and (ii) above. Our aim here is to consider these eigenfunction problems. The inviscid case is easy to solve but the wake problem is more difficult. The first eigensolution is surmised to be associated, as usual, with arbitrariness in the choice of origin. An asymptotic formula for large inner eigenvalues is developed using the method of multiple scales. The method is similar to that described by Stewartson (1957) and Brown (1968).

The eigenvalue problems are compared with those arising in the Goldstein wake. The two problems are very much alike but the pressure fields are different. The Goldstein wake boundary-layer problem is identical with that of Tillett (1968) who discussed a jet flow emerging from a two-dimensional channel. The formula for large eigenvalues in this case has been found by Price (1968).

2. Asymptotic solution for the Navier-Stokes region

When the undisturbed flow is a uniform shear with vorticity of constant magnitude Ω^*, dimensionless variables may be defined by

$$\left\{\begin{array}{l} \psi^* = \nu\Psi = \varepsilon^8 U_\infty L\Psi,\; p^* - p^*_\infty = \rho\nu\Omega^* P, \\ x^* = X\sqrt{\dfrac{\nu}{\Omega^*}},\quad y^* = Y\sqrt{\dfrac{\nu}{\Omega^*}},\quad r^* = R\sqrt{\dfrac{\nu}{\Omega^*}}, \end{array}\right. \tag{2.1}$$

where ψ^* is the stream function, p^* the pressure, and $R^2 = X^2 + Y^2$. When the correct vorticity $\Omega^* = \lambda_1\Omega^*_B$ is used, (2.1) implies a stretching of the coordinates, $r^* = \varepsilon^6 RL(\lambda_1\lambda)^{-\frac{1}{2}}$, the angular coordinate θ remaining unstretched. The exact problem for the Navier-Stokes region may now be stated non-dimensionally:

$$\frac{\partial(\Psi, \nabla^2\Psi)}{\partial(X, Y)} = -\nabla^4\Psi, \tag{2.2}$$

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where $\nabla^2 = \partial^2/\partial X^2 + \partial^2/\partial Y^2$, with boundary conditions

(2.3) $$\Psi = \Psi_Y = 0 \text{ at } Y = 0, \text{ for } X < 0,$$

(2.4) $$\Psi = \Psi_{YY} = 0 \text{ at } Y = 0, \text{ for } X > 0,$$

(2.5) $$\Psi \to \tfrac{1}{2}Y^2 \text{ as } Y \to \infty \text{ or } X \to -\infty.$$

The transformation (2.1) is that used by Hakkinen and O'Neil (1967) except that they used Ω^*_B instead of the correct vorticity field Ω^*. Otherwise, this is precisely the problem solved asymptotically by them for $R \gg 1$, $-\pi \leqq \theta \leqq \pi$. They constructed stream function expansions for each of three regions: (i) a wake region, where inertia and viscous terms are of equal importance; (ii) an outer region where inertia effects dominate and the flow is essentially inviscid; (iii) an upstream lower order boundary layer to correct for a velocity of slip over the plate arising from lower order terms of the inviscid outer expansion. The corresponding expansions introduced by Hakkinen and O'Neil are respectively:

(2.6) $$\Psi^w = X^{2/3}f_0(\eta) + f_1(\eta) + X^{-2/3}f_2(\eta) + X^{-4/3}f_3(\eta) + \cdots,$$

(2.7) $$\Psi^o = R^2G_0(\theta) + R^{4/3}G_1(\theta) + R^{2/3}G_2(\theta) + G_3(\theta) + R^{-2/3}G_4(\theta) + \cdots$$
$$+ (\ln R)H_3(\theta) + (R^{-2}\ln R)H_6(\theta) + \cdots,$$

(2.8) $$\Psi^u = X^{2/3}h_0(\zeta) + h_1(\zeta) + X^{-2/3}h_2(\zeta) + X^{-4/3}h_3(\zeta) + \cdots,$$

where $\eta = Y/X^{1/3}$ with $X > 0$, and $\zeta = Y/X^{1/3}$ with $X < 0$. Then Ψ^w and Ψ^o may be matched consistently as $\eta \to \infty$ and $\theta \to 0$ while Ψ^o and Ψ^u may be matched as $\theta \to \pi$ and $\zeta \to -\infty$. However, the expansions differ qualitatively from those found by Capell (1972) who considered an Oseén type linearization of (2.2) with respect to the uniform shear. He constructed expansions similar to (2.6), (2.7) and (2.8) but included eigensolutions arising from inner and outer eigenfunction problems. Our aim is to investigate the eigenfunction problems associated with the inner and outer similarity solutions $X^{2/3}f_0(\eta)$ and $R^2G_0(\theta)$.

3. The eigenfunction problems

In the inviscid region, the basic flow satisfying $\nabla^2\Psi = 1$ for $R \gg 1$ is the similarity solution $R^2G_0(\theta)$, θ being the similarity variable. This solution is made to satisfy conditions at $\theta = 0, \pi$ but on no other boundaries. (The solution turns out to be simply the uniform shear $\frac{1}{2}R^2\sin^2\theta$.) The elliptic nature of the problem leads to an eigenfunctio nproblem. The eigensolutions are harmonic functions which are bounded as $R \to \infty$ and vanish on $\theta = 0, \pi$. The outer eigensolutions are, in fact, $R^{-k}\sin k\theta$, where $k = 1, 2, 3, \cdots$, as in the linearized problem.

In the wake, (2.2) is reduced to the boundary-layer equation

(3.1) $$\Psi_X\Psi_{YYY} - \Psi_Y\Psi_{YYX} + \Psi_{YYYY} = 0,$$

with boundary conditions

(3.2) $$\Psi = \Psi_{YY} = 0 \text{ at } Y = 0,$$

(3.3) $$\Psi \to \tfrac{1}{2}Y^2 \text{ as } Y \to \infty.$$

No boundary conditions are imposed at $X = X_0 > 0$, so that eigensolutions are expected, since (3.1) is parabolic. The boundary-layer solution is $\Psi_0{}^w = X^{2/3}f_0(\eta)$, where

(3.4) $$f_0'''' + \frac{2}{3}f_0f_0''' = 0,$$

with two boundary conditions at $\eta = 0$,

(3.5) $$f_0(0) = f_0''(0) = 0,$$

since f_0 must be an odd function. The vanishing of higher order even derivatives of f_0 at $\eta = 0$ is guaranteed by (3.4) and (3.5). Two more boundary conditions are found by matching (2.6) and (2.7). The asymptotic form of f_0 is obtained from (3.4). Matching shows that in this asymptotic form, namely

(3.6) $$f_0 \sim a_{00}\eta^2 + a_{01}\eta + a_{02} + 0(\eta^{-6}e^{-\eta^3/9}),$$

we must have $a_{00} = \frac{1}{2}$ and $a_{01} = 0$. Integration of (3.4) yields

(3.7) $$f_0''' + \frac{2}{3}f_0f_0'' - \frac{1}{3}f_0'^2 = C_0,$$

where the arbitrary constant C_0 is related to a pressure field $P_X = C_0X^{-1/3}$. The unknown constants C_0 and a_{02} can be found after $f_0'(0)$ and $f_0'''(0)$ have been determined numerically through their implicit dependence on a_{00} and a_{01}. In the numerical solution, Rott and Hakkinen (1965) obtained $C_0 = 0.4089$ and $a_0 = 3C_0/2 = 0.6133$. It is the change in boundary condition at 0, through its incorporation in the solution through matching, that leads to the non-zero pressure gradient in the problem for f_0.

Following Libby and Fox (1963), we may set up the eigenfunction problem by considering a small arbitrary symmetric disturbance in the near field and linearizing the boundary-layer equations about the basic wake similarity solution $\Psi_0{}^w$ The perturbed similarity solution is written

(3.8) $$\tilde{\Psi} = X^{2/3}f_0(\eta) + \delta f(X, \eta) + o(\delta),$$

where δ is a small parameter characterizing the small disturbance in a finite neighbourhood of 0 (i.e. R finite), for example, a shortening of the plate or a small symmetric bump on the plate. Such a disturbance will not substantially alter the

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wake similarity flow yet its influence must be communicated downstream across $X = X_0$ as a perturbation as described by (3.8). As the coefficient of δ when (3.8) is substituted in (3.1) we find

$$3f_{\eta\eta\eta\eta} + 2f_0 f_{\eta\eta\eta} + 2f_0' f_{\eta\eta} - 3Xf_0' f_{X\eta\eta} + 3Xf_0''' f_X = 0. \tag{3.9}$$

Let $f(X,\eta) = V(X)E(\eta)$. Then (3.9) is separable. We find $V = bX^{-n}$; further

$$3E'''' + 2f_0 E''' + 2f_0' f E'' + 3n(f_0' E'' - f_0''' E) = 0; \tag{3.10}$$

$$E(0) = E''(0) = 0, \text{ since } E \text{ is odd}; \tag{3.11}$$

$$E'(\eta) \to 0 \text{ exponentially as } \eta \to \infty. \tag{3.12}$$

Higher order even derivatives of E should vanish at $\eta = 0$ and this is guaranteed by (3.10) and (3.11). The outer velocity field is matched by that of the boundary-layer solution $X^{2/3} f_0(\eta)$ and the velocity decay in (3.12) guarantees undisturbed flow outside the wake. The eigenfunction E is the odd solution of (3.10) which asymptotes to a constant as $\eta \to \infty$. This constant remains undetermined in the expansion procedure. The exponential decay imposed on the vorticity term $E''(\eta)$ is discussed in the next section.

4. Limiting behaviour of the wake vorticity

We show that the boundary-layer vorticity for fixed X has a limiting behaviour as $Y \to \infty$ that exhibits an exponentially small error term, the exponent of which contains the similarity variable η. This suggests that the double limit as $Y \to \infty$, $X \to \infty$ is commutative which is turn leads to the assumption concerning the wake similarity vorticity as $\eta \to \infty$. For sufficiently large values of Y, the boundary-layer equations reduce to the linearized form

$$YW_X = W_{YY}, \tag{4.1}$$

where $W = 1 - \Psi_{YY}$. The boundary conditions at $Y = 0$ are not needed here. Following Brown and Stewartson (1965), we seek solutions of the form

$$W = W(X,Y) = w(X,Y)\exp[-\{Y - K(X)\}^\alpha / F(X)], \tag{4.2}$$

where w is algebraic in Y and $\alpha > 0$. When (4.2) is substituted in (4.1), a suitable balance of terms can be achieved only when $\alpha = 3$. It is then found that (i) $F = 9(X + \delta)$, where δ is a constant, (ii) $K = 0$, (iii) separable solutions for w may be found, namely $w_\sigma = B_\sigma Y^{-(\sigma+1)}(X+\delta)^{2\sigma/3}$. Thus, for finite values of X, there are solutions of (4.1) whose behaviour as $Y \to \infty$ is given by

$$W_\sigma = B_\sigma Y^{-(\sigma+1)}(X+\delta)^{2\sigma/3}\exp\left[-\frac{1}{9}Y^3/(X+\delta)\right]. \tag{4.3}$$

The form of the similarity variable $\eta = Y/X^{1/3}$ is comparable with that of

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the exponent in (4.3), the presence of the (finite) constant δ corresponding to an arbitrary choice of 0 along the X axis. The form of W for $Y \gg 1$ appears to be

$$W = \exp\left[-\frac{1}{9}Y^3/(X+\delta)\right] \int_{-\infty}^{\infty} B(\sigma)Y^{-(\sigma+1)}(X+\delta)^{2\sigma/3}d\sigma \tag{4.4}$$

where $B(\sigma)$ replaces B_σ. The freedom in $B(\sigma)$ allows description of the initial profile. The exponential factor persists as $X \to \infty$ indicating an exponential decay in the wake similarity eigenvorticity.

The argument may be extended to the upstream flow by letting $X \to -\infty$, since the exponent in (4.4) then contains the variable ζ. However, unless we write $W = 0$ for $X < -\delta$, there is exponential growth of vorticity. This is consistent with the fact that there is no eigenfunction problem for the upstream boundary-layer flow. (Hakkinen and O'Neil found it necessary to exclude such exponential growth from their upstream boundary-layer solution.) There is no question of algebraic decay upstream as $X \to -\infty$, since the dominant sublayer solution is precisely the uniform shear.

5. Method of solution for the inner eigenvalue problem

It is easy to find the eigenfunction $2f_0 - \eta f_0'$ corresponding to the arbitrariness in the choice of 0. Apart from the related eigenvalue $n = 1/3$, the eigenvalues for the problem (3.10) to (3.12) are not easily determined. An asymptotic formula for large eigenvalues is obtained by considering the problem for the eigenvorticity using the following argument.

Since $E'' \to 0$ exponentially as $\eta \to \infty$, suppose $E'' \sim \exp[-g(\eta)]$. A suitable balance of terms in (3.10) yields $g' = 2/3f_0$:

$$E'' \sim \exp\left[-\frac{2}{3}\int_0^\eta f_0(\bar{\eta})d\bar{\eta}\right]; \tag{5.1}$$

$$E'' = Z(\eta)\exp\left[-k\int_0^\eta f_0(\bar{\eta})d\bar{\eta}\right], \tag{5.2}$$

say, where $k < 2/3$ to ensure $Z \to 0$ exponentially as $\eta \to \infty$. There is close agreement between (5.1) and (4.4) when $f_0(\bar{\eta})$ is approximated by $\frac{1}{2}\bar{\eta}^2$. An approximate form of equation for Z can be found which is adequate both for $\eta \gg 1$ and $\eta = 0(1)$. In anticipation of rapidly oscillatory solutions of (3.10) for $n \gg 1$ and $\eta = 0(1)$, we ignore the terms $3nf_0'''E$ which is much smaller than $3nf_0'E''$ for such solutions. The same term is ignored for $\eta \gg 1$, since it contains the exponentially small factor f_0'''. These approximations are in qualitative agreement with the equation for the wake eigenvorticity in the linearized problem. The resulting approximate equation for Z may be simplified by the choice $k = 1/3 < 2/3$:

$$Z'' + \left(Nf_0' - \frac{1}{9} f_0^2\right) Z = 0, \tag{5.3}$$

where $N = n + 1/3$. The boundary conditions are

$$\begin{cases} Z(0) = 0, \text{ from (3.11)} \\ Z \sim \exp\left[-\frac{1}{3}\int^{\eta} f_0(\bar{\eta})d\bar{\eta}\right] \text{ as } \eta \to \infty. \end{cases} \tag{5.4}$$

If Z could be found, integration of (5.2) together with the use of (3.11) and (3.12) would yield the eigenfunction $E(\eta)$ at least approximately.

Solutions of (5.3), with $N \gg 1$, are discussed for $\eta \gg 1$ and $\eta = 0(1)$. For $\eta \gg 1$ there is a turning point problem. The solutions are exponential on one side of the turning point and oscillatory on the other. A limit-process expansion, valid close to the turning point, is matched with these, leading to the correct continuation of the solution through the turning point. Another matching between this oscillatory continuation and the solution for $\eta = 0(1)$ imposes a condition on N that yields an asymptotic formula for large eigenvalues.

6. Solution for $\eta \gg 1$

The coefficient of Z in (5.3) has a positive (minimum) value at $\eta = 0$, increases with η, but eventually decreases through zero as η grows large. The turning point occurs where $\eta \sim (36N)^{1/3}$, since $f_0 \sim \frac{1}{2}\eta^2 + a_{02} + 0(\eta^{-6}e^{-\eta^3/9})$. The asymptotic form of (5.3) is

$$\frac{d^2Z}{d\xi^2} - \varepsilon^{-2}Q(\xi;\varepsilon)Z = 0, \tag{6.1}$$

where $\xi = \eta/(36N)^{1/3}$ with $\varepsilon^{-1} = 6N$, and

$$Q = \xi(\xi^3 - 1) + \frac{48^{1/3}}{9} a_{02}\xi^2\varepsilon^{2/3} + \frac{6^{2/3}}{9} a_{02}^2\varepsilon^{4/3}. \tag{6.2}$$

The turning point is now at $\xi = 1 + \delta$, where $\delta = 0(\varepsilon^{2/3})$. The discussion is presented in three steps.

STEP 1. Following Cole (1968), page 107, define a fast variable

$$\bar{\xi} = \varepsilon^{-1}\int_{1+\delta}^{\xi} \sqrt{Q(t,\varepsilon)}\, dt. \tag{6.3}$$

for $\xi > 1 + \delta$. Then (6.1) becomes

$$\frac{d^2Z}{d\bar{\xi}^2} + \frac{1}{2}\varepsilon\frac{Q'}{Q^{3/2}}\frac{dZ}{d\bar{\xi}} - Z = 0, \tag{6.4}$$

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where $Q' = \partial Q/\partial \xi$. Consider a two-variable expansion with $\bar{\xi}$ the fast variable and ξ the slow variable:

$$Z(\bar{\xi}; \varepsilon) = T_0(\bar{\xi}, \xi) + \varepsilon T_1(\bar{\xi}, \xi) + o(\varepsilon T_1). \tag{6.5}$$

Here T_0, T_1 are permitted to depend on ε provided that $\varepsilon T_1/T_0 = o(1)$. The leading terms in (6.4) yield

$$T_{0\bar{\xi}\bar{\xi}} - T_0 = 0, \tag{6.6}$$

$$T_{1\bar{\xi}\bar{\xi}} - T_1 = -\frac{2}{\sqrt{Q}} T_{0\xi\bar{\xi}} - \frac{1}{2}\frac{Q'}{Q^{3/2}} T_{0\bar{\xi}}. \tag{6.7}$$

Now $\bar{\xi}$ is real and positive, since $\xi > 1 + \delta$. For a decaying solution at infinity, choose

$$T_0 = A(\xi) e^{-\bar{\xi}} \tag{6.8}$$

as the appropriate solution of (6.6). A particular integral of (6.7) is then seen to contain $\bar{\xi} e^{-\bar{\xi}}$ which implies $\varepsilon T_1 = 0(T_0)$. This is avoided by choosing $A(\xi) = aQ^{-1/4}$, where a is a constant, for then (6.7) is homogeneous. Consequently

$$T_0 = aQ^{-1/4} e^{-\bar{\xi}}. \tag{6.9}$$

STEP 2. For $\xi < 1 + \delta$, define the fast variable by

$$\bar{\xi} = \varepsilon^{-1} \int_{\xi}^{1+\delta} \sqrt{-Q(t;\varepsilon)}\, dt, \tag{6.10}$$

so that $\bar{\xi}$ is again real and positive in the region of interest. Repetition of the argument in step 1 leads to

$$T_0 = (-Q)^{-1/4} (b \cos \bar{\xi} + c \sin \bar{\xi}). \tag{6.11}$$

STEP 3. The correct choice of the constants b and c will ensure that (6.11) is the correct continuation of (6.9). This choice is made after matching (6.9) and (6.11) with a limit-process expansion near the turning point. Accordingly, write $\xi - (1 + \delta) = \tau$. The leading terms of (6.1) near $\tau = 0$ yield

$$\frac{d^2 Z}{d\tau^2} - \varepsilon^{-2} \tau Z = 0, \tag{6.12}$$

since $Q'(1 + \delta) \neq 0$ implies that $\xi = 1 + \delta$ is a simple turning point. In terms of $\bar{\tau} = \tau \varepsilon^{-2/3}$, (6.12) becomes

$$\frac{d^2 Z}{d\bar{\tau}^2} - \bar{\tau} Z = 0, \tag{6.13}$$

$$Z = \alpha Ai(\bar{\tau}) + \beta Bi(\bar{\tau}),$$

where α and β are constants that may depend on ε. Antosiewicz (1964), page 448,

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gives the asymptotic forms of the Airy functions $Ai(\tau)$ and $Bi(\tau)$ as $|\tau| \to \infty$:

$$Ai(\bar{\tau}) \sim \frac{1}{2\sqrt{\pi}} \varepsilon^{1/6}\tau^{-1/4} \exp\left(-\frac{2}{3}\varepsilon^{-1}\tau^{3/2}\right),$$

$$Bi(\bar{\tau}) \sim \frac{1}{\sqrt{\pi}} \varepsilon^{1/6}\tau^{-1/4} \exp\left(\frac{2}{3}\varepsilon^{-1}\tau^{3/2}\right).$$

Matching (6.14) with (6.9) as $\bar{\tau} \to \infty$, $\tau \to 0$ or $\xi \to (1+\delta)_+$ shows that $\beta = 0$ and $\alpha\varepsilon^{1/6} = 2a\sqrt{\pi}$. Again for $|z| \to \infty$, with $|\arg z| < 2\pi/3$,

$$Ai(-z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4} \cos\left(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\right). \tag{6.15}$$

Here z is real and positive with $\bar{\tau} = -z$. Now, as $\bar{\tau} \to -\infty$, $\alpha Ai(\bar{\tau})$ matches (6.11) as $\xi \to (1+\delta)_-$, or equivalently $\bar{\xi} \to 0_+$, provided b and c are chosen so that

$$T_0 = B(-Q)^{-1/4} \cos\left(\bar{\xi} - \frac{\pi}{4}\right), \tag{6.16}$$

where B is related to a through α. (The actual relation is not needed).

7. Solution for $\eta = 0(1)$

The two-variable method is now applied to (5.3) for $\eta = 0(1)$. In this case $Nf_0' \gg (1/9)\, f_0^2 > 0$; a balance is achieved in (5.3) by writing

$$Z'' + Nf_0'Z = 0. \tag{7.1}$$

The solutions are rapidly oscillatory. A suitable fast variable is

$$\tilde{\eta} = \sqrt{N} \int_0^\eta \sqrt{f_0'(t)}\, dt.$$

The slow variable is η. Then (7.1) becomes

$$\frac{d^2Z}{d\tilde{\eta}^2} + \frac{1}{2\sqrt{N}} \frac{f_0''}{f_0'^{3/2}} \frac{dZ}{d\tilde{\eta}} + Z = 0. \tag{7.2}$$

As before, the leading term of the two-variable expansion,

$$Z(\tilde{\eta}; N) = S_0(\tilde{\eta}, \eta) + \frac{1}{\sqrt{N}} S_1(\tilde{\eta}, \eta) + 0\left(\frac{S_0}{N}\right),$$

is

$$S_0 = C_0(\eta)\cos\tilde{\eta} + D_0(\eta)\sin\tilde{\eta}. \tag{7.3}$$

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Since $Z = 0$ when $\tilde{\eta} = 0$, $C_0 = 0$. Again S_0 and S_1 may depend on N but $S_1/(S_0\sqrt{N}) = o(1)$. When $D_0(\eta)$ is chosen to satisfy this condition — by demanding homogeneity in the equation for S_1 — it is found that

$$S_0 = Df_0'^{-1/4}\cos\left(\frac{\pi}{2} - \tilde{\eta}\right), \tag{7.4}$$

where the constant D may depend on N.

8. Matching the solutions. The eigenvalue formula

The approximations T_0 and S_0 for Z are found to match provided N is suitable restricted. These approximations are

$$Z \sim Df_0'^{-1/4}\cos\Theta, \quad \eta = 0(1), \tag{8.1}$$

$$Z \sim B[\xi(1-\xi^3) + \Delta(\xi)]^{-1/4}\cos\Phi, \quad n \gg 1, \tag{8.2}$$

where $\Delta(\xi) = 0(\varepsilon^{2/3}) = 0([6N]^{-2/3})$ and

$$\Theta = \frac{\pi}{2} - \sqrt{N}\int_0^\eta \sqrt{f_0'(t)}\,dt$$

$$\Phi = 6N\int_\xi^{1+\delta} \sqrt{t(1-t^3) + \Delta(t)}\,dt - \frac{\pi}{4}.$$

Since $f_0 \sim \frac{1}{2}\eta^2 + 0(1)$, the coefficients of the cosines in (8.1) and (8.2) match as $\eta \to \infty$ (less rapidly than $[36N]^{1/3}$) and $\xi \to 0$. The arguments of the cosines must also match. For $\eta \gg 1$,

$$\Theta \sim \frac{\pi}{2} + \Lambda\sqrt{N} - \frac{2}{3}\sqrt{N}\eta^{3/2} + 0(-\exp),$$

where the precise order of the exponentially small error term is not required and $\Lambda = -0.380$, by an integration using the numerical solution for f_0 obtained by Rott and Hakkinen (1965). Again

$$\Phi \sim 6N\int_\xi^1 \sqrt{t(1-t^3)}\,dt - \frac{\pi}{4} + 0([6N]^{-2/3}).$$

$$\sim \left\{N\pi - \frac{\pi}{4} - \frac{2}{3}\sqrt{N}\eta^{3/2} + \frac{1}{324\sqrt{N}}\eta^{9/2} + 0(N^{-3/2})\right\} + 0([6N]^{-2/3}).$$

For matching of Θ and Φ, N must satisfy

$$N\pi - \frac{\pi}{4} = \frac{\pi}{2} - 0.380\sqrt{N} + s\pi, \tag{8.3}$$

from which the asymptotic formula for large eigenvalues $n(= N - 1/3)$ is

$$n = s - 0.121s^{1/2} + 0.424 + 0(s^{-1/2}) \tag{8.4}$$

where s is a large positive integer.

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9. The Goldstein inner eigenfunction problem

The Goldstein wake solution, valid as $x^* \to 0$, must match with the asymptotic form of the solution valid in Stewartson's triple deck as $\varepsilon^{-3}x^* \to \infty$. Formally, the same eigenfunction problem (3.10) to (3.12) arises for the sublayer region. However, the function f_0 in this case satisfies (3.7) with $C_0 = 0$ corresponding to the zero pressure gradient used by Goldstein. Here $f_0 \sim \frac{1}{2}(\eta + \lambda)^2$ for $\eta \gg 1$, where $\lambda = 0.33206$. For the inner region of Hakkinen and O'Neil there is a pressure gradient and for this case, as noted earlier, $f_0 \sim \frac{1}{2}\eta^2 + a_{02}$. This difference naturally leads to different eigenvalues and in fact necessitates a complete departure from Stewartson's procedure of writing $E = f_0' H(\eta)$ in the early stages of the work above. Note also that in Stewartson's (1957) problem it was necessary to consider an additional region for $\eta \ll 1$.

Now the eigensolutions for the sublayer as $\varepsilon^{-3}x^* \to \infty$ are identical with those described by Tillett (1968) for the boundary layer in a free jet of liquid emerging from a two-dimensional channel. An asymptotic formula for the eigenvalues has been found by Price (1968):

$$n = s + 0.126 s^{1/2} + 0.425 + 0(s^{-1/2}), \tag{9.1}$$

which is similar to (8.4). The difference in the sign and value of the coefficient of $s^{1/2}$ together with the small difference in the constant terms may be traced to the different behaviours at infinite of the functions f_0. The first eigenvalue is again $n = 1/3$ for the Tillett and Goldstein problems.

10. Conclusion

As anticipated, the approximate form of E'' for $\eta = 0(1)$ is rapidly oscillatory and is therefore much larger than the term E ignored in the approximation procedure. Again no inconsistency arises for $\eta \gg 1$ through the omission of this term. Formula (8.4) suggests that the eigenvalues are infinite in number. Although (8.3) and (8.4) are most accurate for large eigenvalues, crude estimates of the first four eigenvalues are found from (8.3) and compared with those found exactly ($n = k + 1/3$) for the lienarized problem (Capell (1972)):

linearized result	0.333, 1.333, 2.333, 3.333;
non-linearized result	0.323, 1.264, 2.227, 3.190.

The agreement with the first eigenvalue (1/3 for both problems) is encouraging. It is clear that the expansions of Hakkinen and O'Neil must be modified in an obvious way. The changes associated with the first inner eigensolutions are qualitatively the same as in the linearized case (Capell (1972)).

As remarked in section 9, the work of Price provides the asymptotic formula for large eigenvalues in the sublayer as the Goldstein inner region is approached.

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References

H. A. Antosiewicz (1964), *Bessel functions of fractional order*, Handbook of mathematical functions with formulas, graphs and mathematical tables, 435–578 (edited by Milton Abramowitz and Irene A. Stegun. National Bureau of Standards Applies Mathematics Series 55. United States Department of Commerce, 1964; reprinted Dover, New York, 1965).

S. N. Brown (1968), 'An asymptotic expansion for the eigenvalues arising in perturbations about the Blasius solution'. *Appl. Sci. Res.* **19**, 111–119.

S. N. Brown and K. Stewartson (1965), 'On similarity solutions of the boundary-layer equations with algebraic decay', *J. Fluid Mech.* **23**, 673–687.

K. Capell (1972), 'Asymptotic analysis of a linearized trailing edge flow', *Bull. Austral. Math. Soc.* **6**, 327–347.

J. D. Cole (1968), *Perturbation methods in applied mathematics*. (Waltham, Massachusetts; Blaisdell).

S. Goldstein (1930), 'Concerning some solutions of the boundary layer equations in hydrodynamics', *Proc. Cambridge Philos. Soc.* **26**, 1–30.

R. J. Hakkinen and Elizabeth O'Neil (1967), *On the merging of uniform shear flows at a trailing edge*, (Douglas Aircraft Co. Report DAC-60862).

P. A. Libby and H. Fox (1963), 'Some perturbation solutions in laminar boundary-layer theory. Part 1. The momentum equation'. *J. Fluid Mech.* **17**, 433–449.

A. F. Messiter (1970), 'Boundary-layer flow near the trailing edge of a flat plate', *SIAM J. Appl. Math.* **18**, 241–257.

J. P. Price (1968), *Ph. D. thesis*, (University College, London (1968)).

N. Rott and R. J. Hakkinen, (1965) *Numerical solutions for merging shear flows*, (Douglas Aircraft Co. Report SM-47809).

K. Stewartson (1957), 'On asymptotic expansions in the theory of boundary layers', *J. Math. and Physics* **36**, 173–191.

K. Stewartson (1968), 'On the flow near the trailing edge of a flat plate', *Proc. Roy. Soc. London Ser.* A **306**, 275–290.

K. Stewartson (1969), 'On the flow near the trailing edge of a flat plat II', *Mathematika* **16**, 106–121.

J. P. K. Tillett (1968), 'On the laminar flow in a free jet of liquid at high Reynolds number'. *J. Fluid Mech.* **32**, 273–292.

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