Can. J. Math. Vol. XXIX, No. 3, 1977, pp. 498–527

VARIETIES OBEYING HOMOTOPY LAWS

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The algebraic structure of a topological algebra $\mathscr{A}$ influences its topological structure in a way which is profound but not well understood. (See § 7 below for various examples.) Here we examine this influence rather generally, and give a fairly complete analysis of one of the many forms it can take, namely, the influence of the identities of $\mathscr{A}$ on the group identities obeyed by the homotopy group (or groups of the components) of $\mathscr{A}$. For $\mathscr{V}$ a *variety* (i.e. class of algebras defined by identities), and λ a *group law*, we say that $\mathscr{V}$ *obeys* λ *in homotopy* if and only if every arc-component of every topological algebra in $\mathscr{V}$ has fundamental group obeying λ. Our investigation of this relation was inspired by the much earlier results of Schreier [**44**], who proved in 1924 that topological groups have commutative homotopy (strengthened versions are due to Cartan, Pontrjagin and Hopf), and Wallace [**52**], who proved in 1953 that topological lattices are homotopically trivial (see also [**12**] and [**8**]).

Our main theorem (3.2 below) states that $\mathscr{V}$ *obeys* λ *in homotopy if and only if every group in the idempotent reduct of* $\mathscr{V}$ *obeys* λ. As a corollary, we see that for fixed λ, "$\mathscr{V}$ obeys λ in homotopy" is a Malcev-definable (see [**46**], [**40**] or [**3**]) property of $\mathscr{V}$. The hard part of the theorem is constructing a topological algebra in $\mathscr{V}$ whose fundamental group may fail to obey λ. We do this via Świerczkowski's method of topologizing free algebras, which we explain in § 2.

Our main theorem leads us to the purely algebraic relation, "every group in $\mathscr{V}$ obeys λ," which we study in §§ 5 and 6. We establish (in 5.2) for idempotent $\mathscr{V}$ that *all groups in* $\mathscr{V}$ *are commutative if and only if* $\mathscr{V}$ *contains no nontrivial projection algebra*, i.e. algebra with $\geqq 2$ elements, each of whose operations is a projection. As a corollary, we see that *if* $\mathscr{V}$ *obeys any non-trivial group law in homotopy, then* $\mathscr{V}$ *obeys* $xy = yx$ *in homotopy*. Moreover, in § 5 we get various sufficient conditions for all groups in $\mathscr{V}$ to be commutative, notably the *congruence-modularity* of $\mathscr{V}$ (generalizing Schreier's result above), and for all groups in $\mathscr{V}$ to be zero, notably the *congruence-distributivity* of $\mathscr{V}$ (generalizing Wallace's result above). In § 6, we continue this analysis in the spirit of module theory.

Many of these results were announced in [**49**].

I wish to thank A. Bateson, R. McKenzie and C. Pitzer for helpful discussions on this subject, and especially B. Banaschewski for his active encouragement and interest. Special thanks go to J. Mycielski and S. A. Morris for telling me about Świerczkowski's method [**45**], which was essential to the main

Received May 24, 1976, and in revised form, November 4, 1976.

https://doi.org/10.4153/CJM-1977-054-9 Published online by Cambridge University Press results here. This research was sponsored by the University of Colorado, the Australian-American Educational Foundation and the University of New South Wales.

1. Definitions and preliminaries on homotopy. We assume a modest familiarity with the basics of ("universal") algebra, especially the theory of *varieties* (classes of algebras defined by identities). (At least one should know what a *free algebra* is—see e.g. [**20**].) An operation F of an algebra $\mathfrak{A}$ is *idempotent if and only if*

$$\mathfrak{A} \models F(x, \ldots, x) = x.$$

A *variety* $\mathscr{V}$ *is idempotent* if and only if every operation of every algebra in $\mathscr{V}$ is idempotent. The *idempotent reduct* of an algebra $\mathfrak{A}$ is the algebra $\mathfrak{A}_0$ which has the same universe as $\mathfrak{A}$ and whose operations are precisely the idempotent operations of $\mathfrak{A}$ (either fundamental or defined by a term). The *idempotent reduct* of a variety $\mathscr{V}$ is the variety generated by $(\mathfrak{F}_{\mathscr{V}}(\aleph_0))_0$ (in an appropriate similarity type). (Here $\mathfrak{F}_{\mathscr{V}}(\aleph_0)$ denotes the $\mathscr{V}$-free algebra on $\aleph_0$ generators.)

A topological algebra is a structure $\mathscr{A} = (A, \mathscr{T}, F_t)_{t \in T}$ with $(A, \mathscr{T})$ a topological space, and each F_t a continuous operation defined on a finite power of $(A, \mathscr{T})$, with the usual product topology, $F_t : A^{n_t} \to A$. If $\mathscr{V}$ is a variety, $\mathscr{A}$ is as above, and $(A, F_t)_{t \in T} \in \mathscr{V}$, then we will sometimes simply write $\mathscr{A} \in \mathscr{V}$. Sometimes without further mention we let $A, B, \ldots$ denote the universe of $\mathfrak{A}, \mathfrak{B}, \ldots$.

A *group in* $\mathscr{V}$ is an algebra $(\mathfrak{A}, \cdot, ^{-1})$, with $\mathfrak{A} \in \mathscr{V}$, $(A, \cdot, ^{-1})$ a group, and $\mathfrak{A} \times \mathfrak{A} \xrightarrow{\cdot} \mathfrak{A}$ and $\mathfrak{A} \xrightarrow{-1} \mathfrak{A}$ being $\mathscr{V}$-homomorphisms. (The subject of groups in a variety—or, more generally, in a category—has been discussed in general terms; see e.g. [**19**; **18**, p. 61; **42**, § 3.6]. But in detail, very little was known about groups in $\mathscr{V}$ prior to §§ 5, 6 below.)

Occasionally (in § 5) we will need to express our results with the more general notion of *groupoid in* $\mathscr{V}$. A *groupoid* (in the sense of Ehresmann—see [**51**]) is a (small) category in which every morphism is an isomorphism. A groupoid is *connected* if and only if all its objects are isomorphic (and connected groupoids are sometimes known as *Brandt groupoids*). Of course a group may be thought of as a one-object groupoid. (We omit repetition of the axioms of category theory—see e.g. [**18**] or [**42**]. For groupoid theory, also see [**23**].) A groupoid may be denoted $(G; D, \circ, ^{-1})$, where $D \subseteq G \times G$ is the domain of the binary composition operator, $\circ$: $D \to G$, and $^{-1}$: $G \to G$ is the (everywhere defined) operation of forming inverses. Finally, a *groupoid in* $\mathscr{V}$ is a structure $(\mathfrak{G}, D, \circ, ^{-1})$ with $\mathfrak{G} \in \mathscr{V}$, $(G, D, \circ, ^{-1})$ a groupoid, D a subuniverse of $\mathfrak{G} \times \mathfrak{G}$ and $D \xrightarrow{\circ} G$ and $G \xrightarrow{-1} G$ being $\mathscr{V}$-homomorphisms.

Our chief example of an Ehresmann groupoid is the *path groupoid* $P(A)$ of an arbitrary topological space A, which we now define. First take

$$P_0(A) = \{\gamma : [0, 1] \to A : \gamma \text{ continuous}\},$$

where [0, 1] is the unit interval, and define, for $\gamma, \delta \in P_0(A)$,

$$\gamma \sim \delta,$$

i.e. γ and δ are *homotopic*, if and only if $\gamma(0) = \delta(0)$, $\gamma(1) = \delta(1)$ and there exists a continuous map

$$\Gamma: [0, 1]^2 \to A$$

such that

$$\begin{aligned} \gamma(x) &= \Gamma(0, x) \\ \delta(x) &= \Gamma(1, x) \\ \Gamma(t, 0) &= \gamma(0) = \delta(0) \quad \text{and} \\ \Gamma(t, 1) &= \gamma(1) = \delta(1), \end{aligned}$$

whenever $0 \leqq x, t \leqq 1$. It is now well known and easy to check that if we define

$$P(A) = P_0(A)/\sim$$

$$D = \{(\gamma, \delta): \gamma(1) = \delta(0)\}$$

$$\gamma \circ \delta(t) = \begin{cases} \gamma(2t) & 0 \leqq t \leqq 1/2 \\ \delta(2t-1) & 1/2 \leqq t \leqq 1, \end{cases} \quad \text{and} \quad \gamma^{-1}(t) = \gamma(1-t)$$

(all modulo $\sim$), then $(P(A), D, \circ, {}^{-1})$ is a groupoid. It is a connected groupoid if and only if A is arc-connected.

We next observe that if A is the underlying space of a topological algebra $\mathscr{A}$, then $P_0(A)$ obviously has the structure of a subalgebra of $A^{[0,1]}$ and $\sim$ is a congruence on this subalgebra, and so as an algebra,

$$P(\mathscr{A}) \in \mathbf{HSP}\mathscr{A}.$$

In fact, one can easily verify that *the path groupoid $P(\mathscr{A})$ is a groupoid in $\mathscr{V}$* (for every $\mathscr{V}$ with $\mathscr{A} \in \mathscr{V}$).

The *fundamental group of a space A with basepoint $a \in A$* is, by definition,

$$\pi_1(A, a) = \text{Hom}(a, a) \text{ in the category } P(A)$$

(which may be viewed as the automorphism group of a). And if $\mathscr{A}$ is a topological algebra and if $F_t(a, a, \ldots, a) = a$ for every operation F_t (in particular, *if $\mathscr{A}$ is idempotent*), *then $\pi_1(\mathscr{A}, a)$ is a substructure of $(P(\mathscr{A}); D, \circ, {}^{-1}; F_t)_{t \in T}$, i.e. $\pi_1(\mathscr{A}, a)$ is a group in $\mathscr{V}$* (whenever $\mathscr{A} \in \mathscr{V}$).

We say that a *group law λ holds* (identically) *in a groupoid* $\mathfrak{G}$ (or that $\mathfrak{G}$ *obeys* λ) if and only if Hom (a, a) obeys λ for every object a of $\mathfrak{G}$. The following proposition is evident from the above remarks.

PROPOSITION 1.1. *For any variety $\mathscr{V}$ and any group law λ,* (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(1) *Every group in the idempotent reduct of $\mathscr{V}$ obeys λ.*

(2) *Every groupoid in $\mathscr{V}$ obeys λ.*

(3) *$\mathscr{V}$ obeys λ in homotopy.*

It is one task of the next two sections to establish the equivalence of (1), (2) and (3), to yield our main result (Theorem 3.2). These three conditions are *not* equivalent to

(4) *Every group in $\mathscr{V}$ obeys λ.*

For consider the variety ("*H-spaces*") given by the laws

$$F(e, x) = F(x, e) = x.$$

An easy argument attributed to Hopf (see, e.g. [**37**, p. 141]) shows that any group in this variety is commutative (see e.g. [**19**, p. 94] or [**42**, p. 154]). But a topological algebra in this variety can have arc-components with non-commutative homotopy, as we can see by taking A to be a disconnected union $\{e\} \cup B$, where B is any space with non-commutative homotopy, and defining multiplication by

$$\begin{array}{c|c|c|} & e & c \in B \\ \hline e & e & c \\ \hline b \in B & b & e \\ \hline \end{array}\;.$$

We have only this weak replacement for Proposition 1.1 for groups obeying λ. Some converses of 1.2 have been developed by Ann Bateson (not yet published).

PROPOSITION 1.2. *If every group in $\mathscr{V}$ obeys λ, $\mathscr{A}$ is a topological algebra in $\mathscr{V}$ and a is a one-element subalgebra of $\mathscr{A}$, then $\pi_1(\mathscr{A}, a)$ obeys λ.*

For *higher homotopy* (which one may ignore and still appreciate this paper) we follow [**24**, p. 287] and define $P_n(A)$ ($n = 2, 3, \ldots$) by taking $\sim$-classes of maps

$$\gamma: [0, 1] \rightarrow A^{S^{n-1}}$$

(where S^{n-1} is the $(n - 1)$-sphere) such that for each fixed $x \in S^{n-1}$, $t \mapsto [\gamma(t)](x)$ is homotopically trivial. Proceeding just as before, *$P_n(\mathscr{A})$ is a groupoid in $\mathscr{V}$ whenever $\mathscr{A} \in \mathscr{V}$*. And we may define the nth *homotopy group of A with basepoint $a \in A$* to be

$$\pi_n(A, a) = \text{Hom}\,(e, e) \text{ in } P_n(A), \text{ where } e \text{ is the object}$$
$$\gamma \text{ with } [\gamma(x)](t) = a \text{ for all } x \text{ and } t.$$

If $\mathscr{A}$ is idempotent, then $\pi_n(\mathscr{A}, a)$ is a group in $\mathscr{V}$ (whenever $\mathscr{A} \in \mathscr{V}$), and, it is well known that *$\pi_n(A, a)$ is always a commutative group for $n \geqq 2$.* We omit the obvious analog of 1.1 for higher homotopy groups.

2. Topologies on free algebras. The next theorem is largely a corollary of Świerczkowski [**45**]; for groups it goes back to Markov [**38**]. For any variety $\mathscr{V}$

and metric space (X, d_0) and $\alpha, \beta \in \mathfrak{F}_{\mathscr{V}}(X)$ (the $\mathscr{V}$-free algebra on X), we define

$$d(\alpha, \beta) = \inf \left\{ \sum_{i=1}^{n} d_0(x_i, y_i) \right\},$$

where $\langle x_i \rangle$, $\langle y_i \rangle$ range over finite sequences in X such that for some terms $F_1, F_2, \ldots, F_m$

$$\begin{aligned} \alpha &= F_1(x_1, x_2, \ldots, x_n) \\ F_1(y_1, y_2, \ldots) &= F_2(x_1, x_2, \ldots) \\ F_2(y_1, y_2, \ldots) &= F_3(x_1, x_2, \ldots) \\ &\cdot \\ &\cdot \\ &\cdot \\ F_m(y_1, y_2, \ldots) &= \beta. \end{aligned}$$

THEOREM 2.1. *d is a metric on $\mathfrak{F}_{\mathscr{V}}(X)$ extending d_0, and all operations of $\mathfrak{F}_{\mathscr{V}}(X)$ are d-continuous.*

Remark. It is entirely possible to have d infinite (corresponding to $\inf \emptyset$ in the definition, i.e. the case where no $F_1, \ldots, F_m$ exist). Nonetheless d is still a metric in the obvious extended sense, and still induces a topology in the usual way.

Proof. We will proceed directly, although we may also observe that Świerczkowski's proof in [**45**] deals with what amounts to an adequate family of pseudometrics, and that if this family is a singleton, then Świerczkowski's procedure makes $\mathfrak{F}_{\mathscr{V}}(X)$ into a metric space. It is evident that d is symmetric and satisfies the triangle inequality. And so to see that d is a metric, we need only show that if $d(\alpha, \beta) = 0$, then $\alpha = \beta$. Before doing this, we prove a lemma which will be useful in the sequel.

First, for finite $S \subseteq X^2$, define the *diameter of S*,

$$\text{diam }(S) = \sum \{d_0(x, y) \colon (x, y) \in S\}.$$

Note that if diam $(S) < \epsilon$, then $d_0(x, y) < \epsilon$ for all $(x, y) \in \theta_S$, the equivalence relation on X generated by S. Let $[F]$ denote the subalgebra (of $\mathfrak{F}_{\mathscr{V}}(X)$) generated by any subset F, and for $\mu \colon X \to X$, let $\bar{\mu}$ denote the unique endomorphism of $\mathfrak{F}_{\mathscr{V}}(X)$ extending μ.

LEMMA 2.2. *If F is a finite subset of X, $\alpha \in [F]$ and $d(\alpha, \beta) < \epsilon < d_0(x, y)$ for all $x \neq y$ in F, then there exists $\mu \colon X \to X$ mapping F identically, with $\mu^2 = \mu$, $\bar{\mu}(\beta) = \alpha$ and* ker $\mu = \theta_S$ *for some finite $S \subseteq X^2$ with* diam $(S) < \epsilon$.

Proof. Given $d(\alpha, \beta) < \epsilon$, we may take $F_1, F_2, \ldots, x_1, x_2, \ldots, y_1, y_2, \ldots$ as in the definition of d, with $\sum_{i=1}^{n} d_0(x_i, y_i) < \epsilon$. We take

$$S = \{(x_1, y_1), \ldots, (x_n, y_n)\}.$$

Clearly diam $(S) < \epsilon$, and so $d_0(x, y) < \theta$ for all $(x, y) \in \theta_S$, which means in particular that θ_S is trivial on $F \times F$. Now it is clearly enough to take any

$\mu: X \to X$ which selects one representative from each θ_S-class and is the identity on F.

Returning to the proof of Theorem 2.1, we take $d(\alpha, \beta) = 0$ and let $F \subseteq X$ be any finite subset with $\alpha, \beta \in [F]$, and take positive $\epsilon < d_0(x, y)$ for all $x \neq y$ in F. Taking μ as given by Lemma 2.1, we obviously have $\alpha = \bar{\mu}(\beta) = \beta$. And thus d is a metric.

To see that d extends d_0, note that obviously $d(x, y) \leqq d_0(x, y)$ for $x, y \in X$. If we had $d(x, y) < d_0(x, y)$, then we could take ϵ with $d(x, y) < \epsilon < d_0(x, y)$ and apply Lemma 2.1 with $F = \{x, y\}$ to obtain $x = \mu(y) = y$ (a contradiction).

To check continuity of an n-ary operation F of $\mathfrak{F}_{\mathscr{V}}(X)$, if for $1 \leqq i \leqq n$ we have

$$\begin{aligned} \alpha_i &= F_1{}^i(x_1{}^i, x_2{}^i, \ldots) \\ F_1{}^i(y_1{}^i, y_2{}^i, \ldots) &= F_2{}^i(x_1{}^i, x_2{}^i, \ldots) \\ &\;\;\vdots \\ F_m{}^i(y_1{}^i, y_2{}^i, \ldots) &= \beta_i \end{aligned}$$

with $\sum_j d_0(x_j{}^i, y_j{}^i) < \epsilon/n$ for each i, then

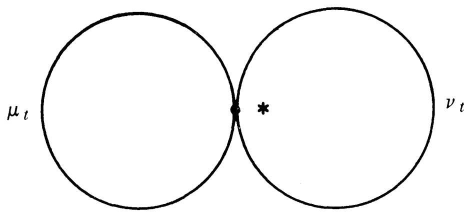
$$\begin{aligned} F(\alpha_1, \alpha_2, \ldots) &= F(F_1{}^1(x_1{}^1, \ldots), F_1{}^2(x_1{}^2, \ldots), \ldots) \\ &\;\;\vdots \\ F(F_m{}^1(y_1{}^1, \ldots), F_m{}^2(y_1{}^2, \ldots), \ldots) &= F(\beta_1, \beta_2, \ldots) \end{aligned}$$

with $\sum_{ij} d_0(x_j{}^i, y_j{}^i) < \epsilon$.

3. The equivalence of the algebraic and topological problems. For simplicity, we state and prove the next theorem in detail only for the commutative law

$$x_1 x_2 x_1{}^{-1} x_2{}^{-1} = 1.$$

By ∞, we mean the one-dimensional compact metrizable topological space which is a join of two circles, $[0, 1] \times \{0, 1\}/(0, 0) = (0, 1) = (1, 0) = (1, 1)$. For convenience, we will let μ_t denote $(t, 0)$ and ν_t denote $(t, 1)$ $(0 \leqq t \leqq 1)$. And $*$ will denote $\mu_0 = \mu_1 = \nu_0 = \nu_1$.



This space should be given the metric d_0 of two circles of radius 1 embedded in a plane. It is to this space that we will apply the results of § 2. We will state without proof the analogous result for an arbitrary group law

$$\tau(x_1, \ldots, x_k) = 1,$$

replacing ∞ by a join of k circles. (See Theorem 3.2.)

If $\gamma = (\gamma_1, \ldots, \gamma_M) \in \infty^M$ and α is an M-ary term in the language of $\mathscr{V}$, then by $\alpha^{\mathscr{V}}(\gamma)$ we mean $\alpha(\gamma_1, \ldots, \gamma_M) \in F_{\mathscr{V}}(\infty)$. Since $\mathscr{V}$ will be clear from the context, we usually write $\alpha(\gamma)$ for $\alpha^{\mathscr{V}}(\gamma)$.

THEOREM 3.1. *For any variety $\mathscr{V}$ the following conditions are equivalent:*

1) *Every group in the idempotent reduct of $\mathscr{V}$ is commutative.*

2) *$\mathscr{V}$ obeys $xy = yx$ in homotopy,*

3) *$\pi_1(\mathfrak{F}_{\mathscr{V}}(\infty), *)$ is commutative.*

4) *There exists an integer M and a triangulation of the unit square into triangles R, and for each R a continuous function γ_R: $R \to \infty^M$ and an M-ary term α_R such that*

(i) $\alpha_R(\gamma_R(P)) = \alpha_S(\gamma_S(P))$ *in $\mathfrak{F}_{\mathscr{V}}(\infty)$ for any P on the overlap of R and S;*

(ii) $\alpha_R(\gamma_R(P)) = *$ *for $P = (0, s)$, $(1, s)$ or $(t, 1)$; and*

(iii) $$\alpha_R(\gamma_R(t, 0)) = \begin{cases} \mu_{4t} & (0 \leqq t \leqq \frac{1}{4}) \\ \nu_{4t-1} & (\frac{1}{4} \leqq t \leqq \frac{1}{2}) \\ \mu_{3-4t} & (\frac{1}{2} \leqq t \leqq \frac{3}{4}) \\ \nu_{4-4t} & (\frac{3}{4} \leqq t \leqq 1). \end{cases}$$

5) *Same as 4, with each γ_R linear (relative to some triangulation of ∞^M).* (4 and 5 are illustrated at the end of this section.)

6) *There exist integers M, N, M-ary terms α_{ij} $(1 \leqq i, j \leqq 4N)$ and σ_{ij}, τ_{ij}, π_{ij}, $\rho_{ij} \in \infty^M$ such that*

(i) *for $1 \leqq i \leqq 4N$, $1 \leqq j < 4N$, if $\sim$ is the smallest equivalence relation on ∞ with $\sigma_{ij}^k \sim \tau_{ij}^k$ $(1 \leqq k \leqq M)$ and $\rho_{i,j+1}^k \sim \pi_{i,j+1}^k$ $(1 \leqq k \leqq M)$, then $d_0(u, v) < 1/8$ whenever $u \sim v$.*

(ii) *for $1 \leqq i < 4N$, $1 \leqq j \leqq 4N$, if $\sim$ is the smallest equivalence relation on ∞ with $\pi_{ij}^k \sim \tau_{ij}^k$ $(1 \leqq k \leqq M)$ and $\rho_{i+1,j}^k \sim \sigma_{i+1,j}^k$ $(1 \leqq k \leqq M)$, then $d_0(u, v) < 1/8$ whenever $u \sim v$.*

(iii) *the following equations hold in $F_{\mathscr{V}}(\infty)$:*

$$\alpha_{11}(\rho_{11}) = *$$

$$\alpha_{1i}(\sigma_{1i}) = \alpha_{1,i+1}(\rho_{1,i+1}) = \begin{cases} \mu_{i/N} & (1 \leqq i \leqq N) \\ \nu_{(i-N)/N} & (N + 1 \leqq i \leqq 2N) \\ \mu_{(3N-i)/N} & (2N + 1 \leqq i \leqq 3N) \\ \nu_{(4N-i)/N} & (3N + 1 \leqq i \leqq 4N) \end{cases}$$

$$\alpha_{1,4N}(\sigma_{1,4N}) = *$$

$$\alpha_{j1}(\rho_{j1}) = \alpha_{j1}(\pi_{j1}) = \alpha_{j,4N}(\sigma_{j,4N}) = \alpha_{j,4N}(\tau_{j,4N}) = *$$

$$\alpha_{4N,i}(\pi_{4N,i}) = \alpha_{4N,i}(\tau_{4N,i}) = *$$

$$\alpha_{ij}(\tau_{ij}) = \alpha_{i,j+1}(\pi_{i,j+1}) = \alpha_{i+1,j}(\sigma_{i+1,j}) = \alpha_{i+1,j+1}(\rho_{i+1,j+1}) \quad (1 \leqq i, j < 4N).$$

Proof. Via

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 5 \arrow[ur] \arrow[dr] & & 2 \arrow[dd] \\ & 4 \arrow[ur] & \\ 6 \arrow[uu] & & 3 \arrow[ll] \end{tikzcd}$$

Clearly $2 \Rightarrow 3$ and $5 \Rightarrow 4$ *a fortiori*, and $1 \Rightarrow 2$ by Proposition 1.1.

$4 \Rightarrow 2$. We will see that the terms and maps of 4 provide universal formulas for commutativity of homotopy. Let $\langle A, \mathscr{T}, F_t \rangle_{t \in T}$ be a topological algebra in $\mathscr{V}$ and let $\Phi, \Psi : [0, 1] \rightarrow A$ with $\Phi(0) = \Phi(1) = \Psi(0) = \Psi(1) = a$. Clearly there exists continuous $\theta \colon \infty \rightarrow A$ with $\Phi(t) = \theta(\mu_t)$ and $\Psi(t) = \theta(\nu_t)$ $(0 \leqq t \leqq 1)$. Now for each triangle R consider

$$(*) \quad R \xrightarrow[\gamma_R]{} \infty^M \xrightarrow[\theta^M]{} A^M \xrightarrow[\alpha_R^A]{} A.$$

One easily checks that this map is continuous; since (i) represents a universally valid equation in $\mathscr{V}$ and $\langle A, F_t \rangle_{t \in T} \in \mathscr{V}$, we see that the functions defined by (*) agree on the common boundary of overlapping triangles R and S. Thus the functions defined by (*) altogether define a continuous function

$$\lambda \colon [0, 1]^2 \rightarrow A.$$

Conditions 4(ii) and 4(iii) now easily imply that λ is a homotopy between $\Phi\Psi\Phi^{-1}\Psi^{-1}$ and the constant map a. Thus $\pi_1(\mathscr{A}, a)$ is commutative.

$3 \Rightarrow 6$. In $\mathfrak{F}_{\mathscr{V}}(\infty)$ consider the loop $\gamma \colon [0, 1] \rightarrow F_{\mathscr{V}}(\infty)$ given by

$$\gamma(t) = \begin{cases} \mu_{4t} & (0 \leqq t \leqq \frac{1}{4}) \\ \nu_{4t-1} & (\frac{1}{4} \leqq t \leqq \frac{1}{2}) \\ \mu_{3-4t} & (\frac{1}{2} \leqq t \leqq \frac{3}{4}). \\ \nu_{4-4t} & (\frac{3}{4} \leqq t \leqq 1). \end{cases}$$

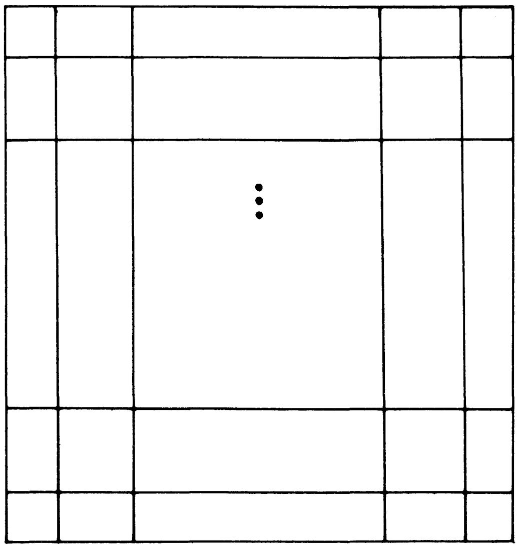
By the hypothesis 3, γ, being of the form $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$, is null-homotopic. Thus there exists a continuous function

$$\Gamma \colon [0, 1]^2 \rightarrow F_{\mathscr{V}}(X)$$

such that

$$\Gamma(0, t) = \gamma(t)$$
$$\Gamma(1, t) = \Gamma(s, 0) = \Gamma(s, 1) = *.$$

Express each $\beta \in \mathfrak{F}_{\mathscr{V}}(\infty)$ as $F(x_1, \ldots, x_k)$ with $x_1, \ldots, x_k \in \infty$; let $\epsilon = \epsilon(\beta) > 0$ be the smallest of $1/64$ and all positive distances $d_0(x_i, x_j)$ $(1 \leqq i < j \leqq k)$. Around each $p \in [0, 1]^2$, by the continuity of Γ, there exists an open rectangle U_p such that $\Gamma(U_p) \subseteq$ the $\epsilon(\Gamma(p))$-neighborhood of $\Gamma(p)$. And clearly we can arrange that U_p contains no edge points (respectively, no corners) unless p is an edge point (respectively, a corner). From the resulting open cover of $[0, 1]^2$ select a finite subcover. By taking subrectangles smaller than the width of any overlap, we may arrive at a covering of the following form



with overlapping open rectangles b_{ij} $(0 \leqq i, j \leqq 4N)$ and with $(i/4N, j/4N) \in b_{ij}$ $(0 \leqq i, j \leqq 4N)$. Moreover, for each b_{ij} there exists $\beta_{ij} \in F_{\mathscr{V}}(\infty)$ such that $\Gamma[b_{ij}] \subseteq$ the $\epsilon(\beta_{ij})$-neighborhood of β_{ij}, and moreover β (corner) $= *$, β (top edge point) $\in \infty$ and β (any other edge point) $= *$. By adjustments of not more than $1/64$, we may assume that

$$\beta_{0i} = \begin{cases} \mu_{i/N} & (1 \leqq i \leqq N) \\ \nu_{(i-N)/N} & (N + 1 \leqq i \leqq 2N) \\ \mu_{(3N-i)/N} & (2N + 1 \leqq i \leqq 3N) \\ \nu_{(4N-i)/N} & (3N + 1 \leqq i \leqq 4N) \end{cases}$$

After this adjustment, we may assume that all $\epsilon(\beta) < 1/32$. Now define $\theta_{ij} \in [0, 1]^2$ $(0 \leqq i, j \leqq 4N + 1)$ as follows:

$$\begin{aligned} \theta_{00} &= (0, 0) \\ \theta_{0,4N+1} &= (0, 1) \\ \theta_{4N+1,0} &= (1, 0) \\ \theta_{4N+1,4N+1} &= (1, 1) \\ \theta_{0,j}(0 < j \leqq 4N) &= (0, \alpha) \in b_{0,(j-1)} \cap b_{0j} \text{ (for any } \alpha) \\ & \quad \text{(and other boundary values similarly defined)} \\ \theta_{ij}(0 < i, j \leqq 4N) &= \text{any point in the overlap of } b_{ij}, b_{i-1,j}, b_{i,j-1}, b_{i-1,j-1}. \end{aligned}$$

Taking M as large as necessary to accommodate all i and j, we know that for each i, j there exists a term α_{ij} such that

$$\Gamma(\theta_{ij}) = \alpha_{ij}(\lambda_{ij}^1, \ldots, \lambda_{ij}^M)$$

for some $\lambda_{ij}^1, \ldots, \lambda_{ij}^M \in \infty$. Since $\Gamma(\theta_{ij})$ lies in the $\epsilon(\beta_{ij})$-neighborhood of β_{ij}, i.e. since we have $d(\beta_{ij}, \Gamma(\theta_{ij})) < \epsilon < d(x_i, x_j)$ $(1 \leqq i < j \leqq k)$ for $\beta_{ij} = F(x_1, \ldots, x_k)$, and with $\epsilon < 1/32$, we may apply Lemma 2.2 to see the existence of idempotent $\mu_{ij}: \infty \to \infty$ with $\alpha_{ij}(\mu_{ij}(\lambda_{ij}^1), \ldots, \mu_{ij}(\lambda_{ij}^M)) = \beta_{ij}$ and ker μ_{ij} generated by a finite subset $S \subseteq X^2$ of diameter $\leqq 1/32$. We now define $\tau_{ij}^k = \mu_{ij}(\lambda_{ij}^k)$. Since μ_{ij} is idempotent, $(\tau_{ij}^k, \lambda_{ij}^k) \in \ker \mu_{ij} = \theta_S$. The M-tuples ρ_{ij}, π_{ij} and σ_{ij} are obtained similarly, using the facts that $\theta_{ij} \in b_{i-1,j-1}$, $b_{i,j-1}$ and $b_{i-1,j}$, respectively, i.e., we obtain

$$\begin{aligned} \beta_{i-1,j-1} &= \alpha_{ij}(\rho_{ij}) \\ \beta_{i,j-1} &= \alpha_{ij}(\pi_{ij}) \\ \beta_{i-1,j} &= \alpha_{ij}(\sigma_{ij}) \\ \beta_{ij} &= \alpha_{ij}(\tau_{ij}). \end{aligned}$$

Now equations 6 (iii) are immediate. Condition 6(i) (similarly 6(ii)) follows from the trivial observation that diam $(S \cup T) \leqq$ diam $S +$ diam T and the fact that the equivalence relation described in 6(i) is a subset of the join of four equivalence relations, each generated by a set of diameter $\leqq 1/32$.

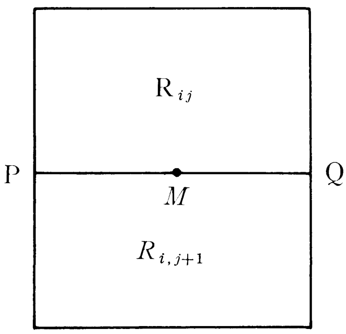
6 $\Rightarrow$ 5. Let terms α_{ij}, π_{ij}, ρ_{ij}, σ_{ij}, τ_{ij} $(1 \leqq i, j \leqq 4N)$ be as specified in 6. Partition the unit square into rectangles

$$R_{ij} = \left[\frac{i-1}{4N}, \frac{i}{4N}\right] \times \left[\frac{j-1}{4N}, \frac{j}{4N}\right] \quad (1 \leqq i, j \leqq 4N),$$

and simply define $\alpha_{R_{ij}} = \alpha_{ij}$ of 6. It is enough to define piecewise linear continuous functions

$$\gamma_{ij}: R_{ij} \to \infty^M$$

so that (i), (ii) and (iii) hold. We do this by first defining γ_{ij} on the boundary of R_{ij} and then later making a piecewise linear extension. Let us first do this in detail for an internal horizontal edge $\overline{PQ}$.



https://doi.org/10.4153/CJM-1977-054-9 Published online by Cambridge University Press

We know from 6(iii) that

$$(*)\quad \begin{cases} \alpha_{ij}(\sigma_{ij}) = \alpha_{i,j+1}(\rho_{i,j+1}) \\ \alpha_{ij}(\tau_{ij}) = \alpha_{i,j+1}(\pi_{i,j+1}). \end{cases}$$

We now let $\sim$ be the smallest equivalence relation on ∞ with $\sigma_{ij}{}^k \sim \tau_{ij}{}^k$ and $\rho_{i,j+1}{}^k \sim \pi_{i,j+1}{}^k$ for all k. Take θ: $\infty \to \infty$ such that

$$\theta(u) \sim u$$
$$\theta(u) = \theta(v) \Rightarrow u \sim v$$

for all $u, v \in \infty$. Let M denote the midpoint of $\overline{PQ}$, and define the functions γ_{ij} and $\gamma_{i,j+1}$ along $\overline{PQ}$ as follows:

$$\begin{aligned} \gamma_{ij}(P) &= (\sigma_{ij}{}^1, \ldots, \sigma_{ij}{}^M) \\ \gamma_{ij}(M) &= (\theta(\sigma_{ij}{}^1), \ldots, \theta(\sigma_{ij}{}^M)) \\ \gamma_{ij}(Q) &= (\tau_{ij}{}^1, \ldots, \tau_{ij}{}^M) \\ \gamma_{i,j+1}(P) &= (\rho_{i,j+1}{}^1, \ldots, \rho_{i,j+1}{}^M) \\ \gamma_{i,j+1}(M) &= (\theta(\rho_{i,j+1}{}^1), \ldots, \theta(\rho_{i,j+1}{}^M)) \\ \gamma_{i,j+1}(Q) &= (\pi_{i,j+1}{}^1, \ldots, \pi_{i,j+1}{}^M), \end{aligned}$$

with γ_{ij} and $\gamma_{i,j+1}$ extended linearly (along shortest paths) to the entire segment $\overline{PQ}$. We next check condition (i), namely that $\alpha_{ij}(\gamma_{ij}(R)) = \alpha_{i+1,j}(\gamma_{i+1,j}(R))$ for any point R of the segment $\overline{PQ}$. For $R = P$ or $R = Q$, this follows from (*), and again for $R = M$, this follows from (*) by substitution under θ. We next check (i) when $R \in \overline{MQ}$. To begin, we claim that

$$\tau_{ij}{}^k = \tau_{ij}{}^s \Rightarrow \theta(\sigma_{ij}{}^k) = \theta(\sigma_{ij}{}^s)$$

for all k, s. To see this, note that if the lefthand side holds, then

$$\sigma_{ij}{}^k \sim \tau_{ij}{}^k = \tau_{ij}{}^s \sim \sigma_{ij}{}^s \;\therefore\; \sigma_{ij}{}^k \sim \sigma_{ij}{}^s,$$

and so according to the choice of θ made above,

$$\theta(\sigma_{ij}{}^k) = \theta(\sigma_{ij}{}^s).$$

Similarly, one can establish that

$$\pi_{i,j+1}{}^k = \pi_{i,j+1}{}^s \Rightarrow \theta(\rho_{i,j+1}{}^k) = \theta(\rho_{i,j+1}{}^s), \quad \text{and}$$
$$\tau_{ij}{}^k = \pi_{i,j+1}{}^s \Rightarrow \theta(\sigma_{ij}{}^k) = \theta(\rho_{i,j+1}{}^s).$$

In other words, we have established that *for any equality between any two entries of* $\gamma_{ij}(Q)$ *and* $\gamma_{i,j+1}(Q)$, *the corresponding two entries of* $\gamma_{ij}(M)$ *and* $\gamma_{i,j+1}(M)$ *are also equal.* But then clearly by linearity this italicized statement holds with M replaced by R, and so the equality

$$\alpha_{ij}(\gamma_{ij}(R)) = \alpha_{i,j+1}(\gamma_{i,j+1}(R))$$

is a substitution instance of the second equation in (*), and hence valid.

The proof for $R \in \overline{PM}$ is similar but easier, using the statement obtained from the recent italicized statement by replacing Q by P.

We skip a detailed description of the definition of γ_{ij} along an internal vertical edge, since it is entirely analogous. For $\overline{PQ}$ one of the top edges, we define

$$\gamma_{1j}(P) = \rho_{1j}$$
$$\gamma_{1j}(Q) = \pi_{1j}$$

and extend linearly, as before. All the other external edges are handled similarly, and one may easily check that conditions (i), (ii) and (iii) have been made valid.

It remains to provide a piecewise linear extension of each individual γ_{ij} to all of R_{ij}. It is of course enough to work with each component

$$\gamma_{ij}^s: R_{ij} \to \infty$$

individually. Returning to the relation $\sim$ defined above, we note that, since $\theta(\pi_{ij}^s) \sim \pi_{ij}^s$, we have, by 6(i),

$$d_0(\gamma_{ij}^s(M), \gamma_{ij}^s(P)) \leqq 1/8.$$

Proceeding in this way around the boundary ∂R_{ij} of the square R_{ij}, we see that

$$\operatorname{diam} \gamma_{ij}^s[\partial R_{ij}] \leqq 1/2.$$

Thus, since each circle of ∞ has diameter 2 in the usual metric, we see that

$$\gamma_{ij}^s[\partial R_{ij}] \subseteq X$$

(a space homeomorphic to the connected union of two crossing segments in the plane). One easily checks that a piecewise linear continuous

$$\gamma_{ij}^s: \partial R_{ij} \to X$$

always has a piecewise linear continuous extension to all of R_{ij}.

$5 \Rightarrow 1$. We first observe that each α_R of 5 is idempotent, since (ii) says that at least one α_R is idempotent, and (i) says that

$$\alpha_R(x, x, \ldots, x) = \alpha_S(x, x, \ldots, x)$$

holds identically in $\mathscr{V}$ whenever R and S are adjacent triangles. And so it will be enough to show that groups in the $\{\alpha_R\}$-reduct of $\mathscr{V}$ are commutative. Let $\alpha, \beta \in$ such a group G; we want to prove $\alpha\beta = \beta\alpha$. Choose intervals $A, B \subseteq \infty$, one in each of the two circles, such that for every edge e of every R, and $1 \leqq s \leqq M$

$$A \subseteq \gamma_R^s[e] \quad \text{or} \quad A \cap \gamma_R^s[e] = \emptyset; \text{ and}$$
$$B \subseteq \gamma_R^s[e] \quad \text{or} \quad B \cap \gamma_R^s[e] = \emptyset.$$

For an oriented edge e of a triangle R, and $1 \leqq s \leqq M$, define

$$g_R^s(e) = \begin{cases} \alpha & \text{if } \gamma_R^s \restriction e \text{ traces out } A, \text{ forwards} \\ \alpha^{-1} & \text{if } \gamma_R^s \restriction e \text{ traces out } A, \text{ in reverse} \\ \beta & \text{if } \gamma_R^s \restriction e \text{ traces out } B, \text{ forwards} \\ \beta^{-1} & \text{if } \gamma_R^s \restriction e \text{ traces out } B, \text{ in reverse} \\ 1 & \text{otherwise,} \end{cases}$$

and define

$$L_R(e) = \alpha_R{}^G(g_R{}^1(e), \ldots, g_R{}^M(e)) \in G.$$

We now will establish

(1) $L_R(e) = L_Q(e)$ if Q and R are adjacent along e; and

(2) $L_R(e_1) \cdot L_R(e_2) \cdot L_R(e_3) = 1$ in G, if e_1, e_2, e_3 are the three edges of a a triangle, oriented cyclically.

To establish (1), consider the $2M$ functions $\gamma_R{}^1 \restriction e, \ldots, \gamma_R{}^M \restriction e, \gamma_Q{}^1 \restriction e, \ldots, \gamma_Q{}^M \restriction e$. Since these are linear functions, there exists $P \in e$ such that two of these functions are equal if and only if their corresponding values at P are equal. And so if any two members of

$$\{\gamma_R{}^1(P), \ldots, \gamma_R{}^M(P), \gamma_Q{}^1(P), \ldots, \gamma_Q{}^M(P)\}$$

are equal, then the corresponding members of

$$\{g_R{}^1(e), \ldots, g_R{}^M(e), g_Q{}^1(e), \ldots, g_Q{}^M(e)\}$$

are equal. But this means that we may substitute into the $\mathscr{V}$-identity

$$\alpha_R(\gamma_R{}^1(P), \ldots, \gamma_R{}^M(P)) = \alpha_Q(\gamma_Q{}^1(P), \ldots, \gamma_Q{}^M(P))$$

from (i) to obtain

$$\begin{aligned} L_R(e) &= \alpha_R(g_R{}^1(e), \ldots, g_R{}^M(e)) \\ &= \alpha_Q(g_Q{}^1(e), \ldots, g_Q{}^M(e)) \\ &= L_Q(e). \end{aligned}$$

To prove (2), since $(\gamma_R{}^i \restriction e_1)(\gamma_R{}^i \restriction e_2)(\gamma_R{}^i \restriction e_3)$ is a path shrinkable to a point in ∞ $(1 \leqq i \leqq M)$, we have

$$g_R{}^s(e_1) g_R{}^s(e_2) g_R{}^s(e_3) = 1 \quad (1 \leqq s \leqq M)$$

and so

$$\begin{aligned} &L_R(e_1) L_R(e_2) L_R(e_3) \\ &\quad = \alpha_R{}^G(g_R{}^1(e_1), \ldots, g_R{}^M(e_1)) \alpha_R{}^G(g_R{}^1(e_2), \ldots, g_R{}^M(e_2)) \\ &\qquad\qquad \alpha_R{}^G(g_R{}^1(e_3), \ldots, g_R{}^M(e_3)) \\ &\quad = \alpha_R{}^G(g_R{}^1(e_1) g_R{}^1(e_2) g_R{}^1(e_3), \ldots, g_R{}^M(e_1) g_R{}^M(e_2) g_R{}^M(e_3)) \\ &\qquad\qquad \text{(by definition of "group in } \mathscr{V}\text{")} \\ &\quad = \alpha_R{}^G(1, \ldots, 1) = 1, \end{aligned}$$

establishing (2). Now (ii) tells us that also

$$L_R(e) = 1$$

if e is any outer edge, except for $e \subseteq J = [0, 1] \times \{0\}$, from which we deduce, using (1), (2) and standard cancellation arguments, that in G

$$\prod_{e \subseteq J} L_R(e) = 1.$$

But from (iii) we immediately see that

$$\prod_{e \subseteq J} L_R(e) = \alpha\beta\alpha^{-1}\beta^{-1}. \qquad Q.E.D.$$

THEOREM 3.2 *For fixed group law* λ *and equational class* $\mathscr{V}$, *the following conditions are equivalent:*

1. *Every group in the idempotent reduct of* $\mathscr{V}$ *obeys* λ.
2. $\mathscr{V}$ *obeys* λ *in homotopy.*
3. $\pi_1(\mathfrak{F}_{\mathscr{V}}(X))$ *obeys* λ, *where* X *is a join of circles, one for every variable appearing in* λ.
4. *There exists an integer* M *and a triangulation of the unit square into triangles* R *and for each* R *a continuous function* γ_R: $\bar{R} \to X^M$ (X a join of circles, one for each variable in λ) *and an* M*-ary term* α_R *such that*
   (i) $\alpha_R(\gamma_R(P)) = \alpha_S(\gamma_S(P))$ *in* $F_{\mathscr{V}}(X)$ *for any* P *on the overlap of* R *and* S;
   (ii) $\alpha_R(\gamma_R(P)) \in X$ *for* $P \in \partial[0, 1]^2$, *the boundary of the unit square; and*
   (iii) *the mapping* $\partial[0, 1]^2 \to X$ *given by the various* $\alpha_R \circ \gamma_R$, *via* (i) *and* (ii), *represents the word* $\bar{\beta}$ *in* $\pi_1(X)$ (*where* λ *is the law* $\beta = 1$, *and* $\bar{\beta}$ *is obtained by replacing each variable of* β *by the corresponding loop in* X).

Note that Theorem 3.2 implies, e.g., the well known fact that $\pi_1(S^1) = Z$. (Taking $\mathscr{V}$ to be a variety with no operations.)

COROLLARY 3.3. "$\mathscr{V}$ *obeys* λ *in homotopy" is a Malcev-definable property of* $\mathscr{V}$.

Recall that this means that this property of $\mathscr{V}$ is equivalent to the existence of terms obeying one of a family $\{\Sigma_n\}$ of finite sets of identities. For more information, see [**46**]; the main theorem of [**46**] immediately yields Corollary 3.3. The finite sets of identities appearing in 6(iii) of 3.1 would form a Malcev condition were it not for the requirement that Σ_{n+1} should follow from Σ_n (after suitably interpreting the function symbols of Σ_n). The pleasanter identities appearing in Corollary 5.3 below fail to be a Malcev condition, for the same reason. The proof in [**46**] effectively converts these identities (either 6(iii) or 5.3) into $\{\Sigma_n\}$, but this conversion is not practical.

We will illustrate Condition 4 of Theorems 3.1 and 3.2 for the variety $\mathscr{V}$ given by the laws

$$\begin{aligned} x &= p_1(x, y, y) \\ p_1(x, x, y) &= p_2(x, y, y) \\ p_2(x, x, y) &= p_3(x, y, y) \\ p_3(x, x, y) &= y. \end{aligned}$$

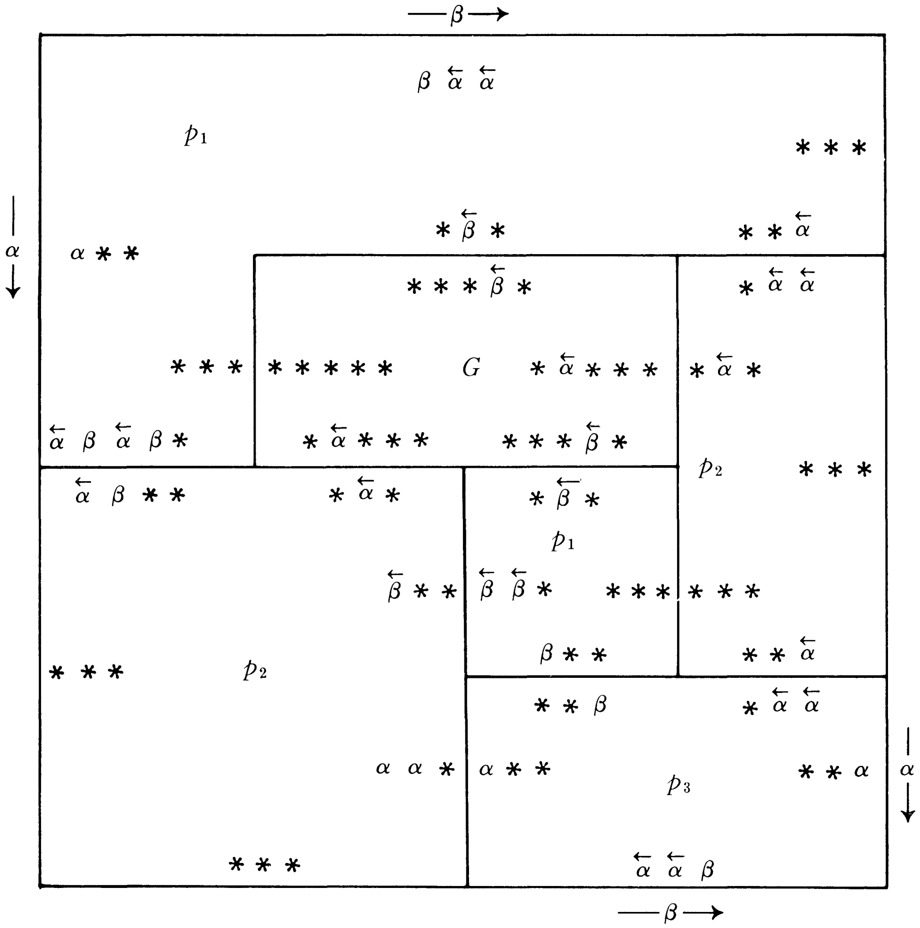
(By [**21**] these laws imply the 4-permutability of congruences in $\mathscr{V}$, and conversely, must hold in $\mathscr{W}$, under some interpretation of p_1, p_2, p_3, if $\mathscr{W}$ has 4-permutable congruences; i.e. they form a Malcev condition for 4-permutability.) The remarks after 5.4 below immediately imply that $\mathscr{V}$ obeys $xy = yx$ in homotopy. To see Condition 4, besides p_1, p_2 and p_3, we need only one term:

$$G(x_1, x_2, x_3, x_4, x_5) = p_1(p_2(x_1, x_2, x_3), x_4, x_5).$$

Then one easily checks that we also have the laws

$$\begin{aligned} G(x, x, x, y, x) &= P_1(x, y, x) \\ G(x, y, x, x, x) &= P_2(x, y, x). \end{aligned}$$

The diagram below provides a universal scheme for defining a homotopy between $\alpha\beta$ and $\beta\alpha$. Regarding α and β as the two loops of a bouquet of two circles, denoted ∞, we define maps of the five regions below into ∞^3 (∞^5 in the case of the single G-region) which agree with the indicated maps along the boundaries; this is possible since the indicated maps are obviously homotopic to a point in each of the three (resp. five) co-ordinates. Then we apply the indicated operations in any topological algebra, and the above mentioned laws give continuity on overlaps. But clearly the map on the boundary is the commutator $[\beta, \alpha]$.



(N.b. Maps α and β are thought of as winding from left to right or top to bottom. The reverse maps are denoted $\overleftarrow{\alpha}$ and $\overleftarrow{\beta}$. $*$ is the base point.).

4. Higher homotopy groups. Let us say that $\mathscr{V}$ *obeys* λ *in n-homotopy* if and only if $\pi_n(\mathscr{A}, a)$ obeys λ for every topological algebra $\mathscr{A} \in \mathscr{V}$ and every

$a \in \mathscr{A}$. As is well known, the higher homotopy groups $\pi_n(\mathscr{A}, a)$ $(n \geqq 2)$ are all commutative, and so any group law is equivalent in π_n to one of the form $x^m = 1$. In what follows, S^n denotes the n-sphere, i.e. the boundary of an $(n + 1)$-ball.

THEOREM 4.1. *For any $n \geqq 2$, any variety $\mathscr{V}$ and any Abelian group law $x^m = 1$, the following conditions are equivalent:*

1. *Every group in the idempotent reduct of $\mathscr{V}$ obeys $x^m = 1$.*
2. *$\mathscr{V}$ obeys $x^m = 1$ in n-homotopy.*
3. *$\pi_n(\mathfrak{F}_{\mathscr{V}}(S^n), *)$ obeys $x^m = 1$.*
4. *There exists an integer M, a triangulation of the unit $(n + 1)$-cube $[0, 1]^{n+1}$ and for each $(n + 1)$-simplex R of this triangulation a continuous function $\gamma_R: \bar{R} \to (S^n)^M$ and an M-ary term α_R such that*
   (i) *$\alpha_R(\gamma_R(P)) = \alpha_S(\gamma_S(P))$ in $F_{\mathscr{V}}(S^n)$ for any P on the common boundary of adjacent simplices R and S;*
   (ii) *for $P \in \partial[0, 1]^{n+1}$, the boundary of the $(n + 1)$-cube, $\alpha_R(\gamma_R(P)) \in S^n$; and*
   (iii) *the mapping $\partial[0, 1]^{n+1} \to S^n$ given by the various $\alpha_R \circ \gamma_R$, via* (i) *and* (ii), *is of degree m.*
5. *Same as 4, with each γ_R linear (in some triangulation of S^n).*

Sketch of proof. Much the same as that of Theorems 3.1 and 3.2 above. It is convenient to make a sixth condition parallel to 3.1(6), which we have not stated because it seems complicated; notably, in place of $\{\pi_{ij}, \sigma_{ij}, \rho_{ij}, \tau_{ij}\}$ one has 2^{n+1} elements of $(S^n)^M$. Now the proof is exactly like that of 3.1; the greatest difference being in $5 \Rightarrow 1$, which we examine briefly. Again one begins by establishing the universal validity of

$$\alpha_R(x, \ldots, x) = x$$

in $\mathscr{V}$ for each R. Now to see

$$\alpha^m = 1$$

for any member α of a group G in $\mathscr{V}$, we select an n-simplex A in a triangulation of S^n so that every boundary n-simplex of $[0, 1]^n$ either maps onto A via any component $\gamma_R{}^s$ or misses A altogether. Having oriented A and all simplices R of the given triangulation, define for e an n-simplex of the triangulation, and e a face of R,

$$g_R{}^s(e) = \begin{cases} \alpha & \text{if } \gamma_R{}^s \restriction e \text{ traces out } A \text{ preserving orientation} \\ \alpha^{-1} & \text{if } \gamma_R{}^s \restriction e \text{ traces out } A \text{ reversing orientation} \\ 1 & \text{otherwise,} \end{cases}$$

and define

$$L_R(e) = \alpha_R{}^G(g_R{}^1(e), \ldots, \alpha_R{}^M(e)) \in G.$$

And now the proof continues as before, except that in place of (2) in the proof of 5 $\Rightarrow$ 1 above, we simply write

$$\prod L_R(e) = 1$$

where the product ranges over all faces e of a fixed $(n + 1)$-simplex R. Since there is no natural order to these multiplicands, we need to know that G is commutative. But this follows immediately from Corollary 5.2 below, since we know 5 $\Rightarrow$ 2.

COROLLARY 4.2. *For $m, n \geqq 1$, $\mathscr{V}$ obeys λ in m-homotopy if and only if $\mathscr{V}$ obeys λ in n-homotopy.*

Proof. If $m, n \geqq 2$, the corollary is immediate. If $\mathscr{V}$ obeys λ in 1-homotopy, then $\mathscr{V}$ obeys λ in n-homotopy by Theorem 3.2 and 4.1. Finally, if $\mathscr{V}$ obeys λ in n-homotopy, then all groups in $\mathscr{V}$ are commutative (by 5.2 below), and so λ is equivalent in fundamental groups in $\mathscr{V}$ to a law $x^m = 1$, and we may apply Theorem 4.2.

Cartan proved [**7**] that $\pi_2(G) = 0$ for G any Lie group, and Browder [**4**] extended this result to a wide class of connected H-spaces. (I thank S. Schiffman for informing me of these facts.) But it obviously fails for groups in general, by Theorem 4.1, since there exist non-trivial groups in groups. Similarly, Harper proved [**22**] that if H is a connected H-space which is a finite complex, then $\pi_4(H)$ obeys the law $x^2 = 1$. But by Theorem 4.1, Harper's theorem obviously fails for H an arbitrary topological group.

5. The groups (and groupoids) in a variety. Theorem 3.2 essentially tells us that to know what homotopy laws hold for a variety $\mathscr{V}$, it is enough to answer the purely algebraic question of what group laws hold in all groups in $\mathscr{V}$. Naturally we would like to know as much as possible about the class of groups which can be made into groups in $\mathscr{V}$, not merely about their identities. This section and the next form a preliminary investigation of this topic, about which little seems to be known. Nonetheless our main results are about group laws, especially 5.1 and its corollary 5.2, which give complete information about which varieties have commutative homotopy. Of course every result in this section is automatically a result about homotopy groups of topological algebras.

We first give two atypical examples: varieties $\mathscr{V}$ for which the groups in $\mathscr{V}$ are completely known. if $\mathscr{V}$ is given by the laws

$$x^2 = x$$
$$x(yz) = xz = (xy)z$$
$$F(F(x)) = x$$
$$F(xy) = F(y)F(x)$$

then the groups in $\mathscr{V}$ are precisely the squares G^2 of all groups G. (On which we may take $(a, b)(c, d) = (a, d)$ and $F((a, b)) = (b, a)$.) (By an easy argument—see e.g. Evans [**13**] or [**48**, p. 268], *q.v.* also for kth powers.) And a well known easy argument establishes that the groups G in the variety $\mathscr{W}$ defined by

$$F(fx, gx) = x$$

$$fFxy = x \quad gFxy = y$$

are precisely those with $G \cong G^2$.

We now turn to the main algebraic results of this article.

THEOREM 5.1 ($\mathscr{V}$ idempotent). *If there exists a non-commutative group $\mathfrak{G}$ in $\mathscr{V}$, then there exists a nontrivial group $\mathfrak{H} = (H, \circ, {}^{-1}, K_t)_{t \in T}$ in $\mathscr{V}$ with each K_t a projection operation. In fact, $\mathfrak{H}$ may be taken as a quotient of $\mathfrak{G}$.*

Proof. We first note that it is enough to prove the Theorem for T finite This is easily seen by adding a unary predicate symbol to stand for a normal. subgroup of G, together with a name for each element of G, and applying the compactness theorem of first order logic.

We next show how to reduce the proof to the case of singleton T. (Our device is well known—see e.g. Padmanabhan and Quackenbush [**41**].) We replace any two operations F_1, F_2 by the single operation

$$F(x_1, \ldots, x_{mn}) = F_1(F_2(x_1, \ldots, x_n), F_2(x_{n+1}, \ldots, x_{2n}), \ldots, F_2(\ldots, x_{mn}))$$

(where F_1 is m-ary and F_2 is n-ary). By idempotence, F_1 and F_2 can be recovered from F as

$$F_1(x_1, \ldots, x_m) = F(x_1, \ldots, x_1, x_2, \ldots, x_2, x_3, \ldots \ldots, x_m) \quad \text{and}$$

$$F_2(x_1, \ldots, x_n) = F(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots \ldots, x_n),$$

and so both hypothesis and conclusion of the theorem are invariant under this replacement.

And so we may assume that $\mathfrak{G} = \langle G, \cdot, {}^{-1}, F \rangle$, with G a non-commutative group and $F\colon G^n \to G$ a homomorphism obeying the law $F(x, \ldots, x) = x$. For $i = 1, \ldots, n$ we define

$$N_i = \{F(1, \ldots, 1, \underset{i}{x}, 1, \ldots, 1) : x \in G\}.$$

From the laws

$$y^{-1}F(1, \ldots, 1, x, 1, \ldots, 1)y = F(1, \ldots, 1, y^{-1}x\, y, 1, \ldots, 1)$$

$$x = F(x, 1, \ldots, 1) \cdot F(1, x, 1, \ldots) \cdot \ldots \cdot F(1, \ldots, 1, x)$$

(which follow from idempotence), we deduce

(1) each N_i is a normal subgroup of G;

(2) $N_1 N_2 \ldots N_n = G$.

Now it is evident that, for each i

(3) every member of N_i commutes with every member of N_j for $j \neq i$.

From (3) we easily deduce that, for each i

(3′) every member of N_i commutes with every member of $N_1 \ldots \hat{N}_i \ldots N_n$. (Where ^ indicates a deletion.) We claim that for some i, $N_1 \ldots \hat{N}_i \ldots N_n \neq G$. Otherwise, by (3′), each N_i commutes with G, and so by (2), G is commutative, a contradiction.

And so without loss of generality $N = N_2 \ldots N_n$ is a proper normal subgroup of G. We calculate:

$$\begin{aligned} F(x_1, x_2, \ldots, x_n) &= F(x_1, x_1, \ldots, x_1) \cdot F(1, x_1^{-1}x_2, \ldots, x_1^{-1}x_n) \\ &= x_1 w \quad (w \in N), \end{aligned}$$

which obviously implies that congruence modulo N is an F-congruence and that F becomes a projection in the (non-trivial) quotient G/N.

COROLLARY 5.2 ($\mathscr{V}$ idempotent). *Every group in $\mathscr{V}$ is commutative if and only if $\mathscr{V}$ contains no nontrivial projection algebras.*

Proof. ($\Rightarrow$) immediate; ($\Leftarrow$) by Theorem 5.1.

Corollary 5.2 fails for non-idempotent $\mathscr{V}$, as we can see by considering any non-commutative group and F: $G^2 \to G$ given by $F(x, y) = 1$ (a constant). Then $(G, F) \in \mathscr{V}$ = the variety defined by $Fxy = Fyx$, but clearly $\mathscr{V}$ has no non-trivial projection algebras.

COROLLARY 5.3 ($\mathscr{V}$ idempotent). *Every group in $\mathscr{V}$ is commutative if and only if $\mathscr{V}$ has an idempotent term $F(x_1, \ldots, x_N)$ obeying the laws $F(\sigma_1) = F(\tau_1), \ldots, F(\sigma_n) = F(\tau_n)$, where each σ_i, τ_i is a substitution $\{x_1, \ldots, x_N\} \to \{x, y\}$, such that for each j $(1 \leqq j \leqq N)$, there exists k $(1 \leqq k \leqq n)$ with $\sigma_k(x_j) = x$ and $\tau_k(x_j) = y$ (or vice versa).*

Sketch of proof. Suppose we have finitely many equations (*) $F_i(x, \ldots, x) = x$ and (**) $\alpha_j = \beta_j$ (in the operations F_i) which rule out projection algebras. Following the Padmanabhan-Quackenbush method in the proof of 5.1, we construct a single N-ary term F from which each term α_j, β_j can be recovered by substituting only variables. Using a new N-ary operation symbol $\bar{F}$, take the finite set Σ of all identities $\bar{F}(x_{i_1}, x_{i_2}, \ldots) = \bar{F}(x_{j_1}, x_{j_2}, \ldots)$ $(1 \leqq i_k, j_k \leqq N)$ which would be consequences of (*) (**) if $\bar{F}$ were replaced by the term F. It is easy to check that Σ prevents $\bar{F}$ from being a projection. Conversion of Σ to the required form is now straightforward by some obvious substitutions. Conversely, it is apparent that these identities rule out projection algebras.

The known examples led me to conjecture the next corollary early in this investigation, but I could not prove it until much later. It is analogous to a conjecture† of Nation and McKenzie that congruence lattices of algebras in a variety which all obey some non-trivial lattice law must obey the modular law (see [**39**]). It is reminiscent of Wagner's theorem [**50**] that ordered rings which obey any extra law must obey the commutative law.

†*Added in proof.* This conjecture has been shown false by S. V. Polin.

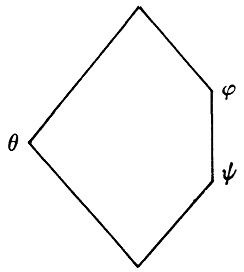
COROLLARY 5.4. *If $\mathscr{V}$ obeys any non-trivial group law in homotopy, then $\mathscr{V}$ obeys $xy = yx$ in homotopy.*

Proof. By 3.2, we may assume $\mathscr{V}$ idempotent. If $\mathscr{V}$ fails to obey $xy = yx$, then $\mathscr{V}$ contains a nontrivial projection algebra, hence (by **S** and **P**) a projection algebra of power $2^{\aleph_0}$. Since *any* topologization of a projection algebra is legitimate, we easily find a topological algebra in $\mathscr{V}$ with homotopy group not obeying λ.

Moreover, Corollary 5.2 (or 5.3) immediately tells us that if $\mathscr{V}$ obeys many of the familiar Malcev conditions, such as modularity or k-permutability of congruences, then groups in $\mathscr{V}$ are commutative. (See [**46**] for references to these and other Malcev conditions.) For a projection algebra has *all* equivalence relations as congruences, and hence makes the familiar Malcev conditions fail. All we need to see is that the Malcev condition in question admits the formation of idempotent reducts, and this is often obvious from the specific Malcev conditions involved, e.g. those of Day [**11**] for congruence-modularity. Thus it follows directly from Corollary 5.2 that groups in congruence-modular varieties are commutative. We present here a proof, due to B. Banaschewski, of a stronger ("local") result.

THEOREM 5.5. *If $(G, \circ, ^{-1}, F_t)_{t\in T}$ is a non-commutative group-algebra, then $(G, F_t)_{t\in T}{}^2$ has non-modular congruence lattice.*

Proof. We will show that $(G, F_t)_{t\in T}{}^2$ has a sublattice of congruences



with $\varphi \neq \psi$. We take

$(a_1, b_1)\theta(a_2, b_2)$ if and only if $a_1 = a_2$
$(a_1, b_1)\varphi(a_2, b_2)$ if and only if $b_1 = b_2$
$(a_1, b_1)\psi(a_2, b_2)$ if and only if $b_1 = b_2$ and $a_1b_1a_1^{-1}b_1^{-1} = a_2b_2a_2^{-1}b_2^{-1}$.

One easily sees that $\varphi \wedge \theta$ is the identity, and since

$$(a_1, b_1)\theta(a_1, 1)\psi(a_2, 1)\theta(a_2, b_2)$$

is always true, $\psi \vee \theta = G^4$. Thus we have the pictured sublattice. To see that $\varphi \neq \psi$, take $ab \neq ba$ and observe that

$$((a, b), (1, b)) \in \varphi - \psi.$$

Banaschewski used the same method to verify the author's conjecture that if $\mathscr{V}$ is a congruence-modular variety and $(A, \mathscr{T}, F_t)_{t \in T}$ is a topological algebra in $\mathscr{V}$, then for $a \in A$, the natural action of $\pi_1(A, a)$ on $\pi_n(A, a)$ [the higher-dimensional analog of an inner automorphism] is always trivial. (See [**24**].) We omit the details.

Let us say that $\mathscr{V}$ has a *factorable congruences* if and only if every congruence θ on any $\mathfrak{A} \times \mathfrak{B} \in \mathscr{V}$ is given by $(a, b)\theta(a', b')$ if and only if $a\varphi a'$ and $b\psi b'$ for some congruences φ, ψ on $\mathfrak{A}$, $\mathfrak{B}$. For more information see [**17**] or [**46**, Theorem 5.5].

THEOREM 5.6. *If $\mathscr{V}$ has factorable congruences, then every commutative connected groupoid in $\mathscr{V}$ obeys the group law* $(x = 1)$.

Proof. Let $\mathfrak{G} = (G, D, \circ, {}^{-1}, F_t)_{t \in T}$ be a commutative connected groupoid in $\mathscr{V}$, and define H to be the set of automorphisms in $\mathfrak{G}$, i.e. [iso]-morphisms with the same domain and co-domain. It is clear that H is a subuniverse of $(G, F_t)_{t \in T}$, and so we may define $\mathfrak{H} = (H, F_t)_{t \in T}$. We claim that the following relation θ is a congruence on $\mathfrak{H}^2$:

$$(\alpha_1, \beta_1)\theta(\alpha_2, \beta_2) \text{ if and only if there exist } x_i, y_i, z_i \in G \ (i = 1, 2), \text{ with}$$
$$x_i y_i z_i \text{ an identity } (i = 1, 2), \text{ and } x_1\alpha_1 y_1 \beta_1^{-1} z_1 = x_2 \alpha_2 y_2 \beta_2^{-1} z_2.$$

Symmetry is immediate; reflexivity follows from the connectedness of G; and the fact that θ is preserved by the operations F_t is immediate from the definition of "groupoid in $\mathscr{V}$." It remains to check transitivity. And so we assume that $(\alpha_1, \beta_1)\theta(\alpha_2, \beta_2)\theta(\alpha_3, \beta_3)$, i.e.

$$x_1\alpha_1 y_1\beta_1^{-1}z_1 = x_2\alpha_2 y_2\beta_2^{-1}z_2 \quad \text{and} \quad x_4\alpha_2 y_4\beta_2^{-1}z_4 = x_3\alpha_3 y_3\beta_3^{-1}z_3,$$

where $x_i y_i z_i$ is an identity $(1 \leqq i \leqq 4)$. Define x_5, y_5, z_5 via

$$\begin{aligned} x_5 &= x_2 y_2 z_4 x_3 \\ y_5 &= y_3 \\ z_5 &= z_3 x_4 y_4 z_2. \end{aligned}$$

(A little checking is required to make sure that these products exist.) One easily checks that $x_5 y_5 z_5$ is an identity, and then we compute

$$\begin{aligned} x_5\alpha_3 y_5\beta_3^{-1}z_5 &= x_2 y_2 z_4 (x_3\alpha_3 y_3\beta_3^{-1}z_3) x_4 y_4 z_2 \\ &= x_2 y_2 z_4 (x_4\alpha_2 y_4\beta_2^{-1}z_4) x_4 y_4 z_2 \\ &= x_2 y_2 (y_4^{-1}\alpha_2 y_4)\beta_2^{-1}(z_4 x_4 y_4) z_2 \\ &= x_2 y_2 \beta_2^{-1}(y_4^{-1}\alpha_2 y_4) z_2 \quad \text{(by commutativity)} \\ &= x_2 (y_2\beta_2^{-1}y_4^{-1})\alpha_2 y_4 z_2 \\ &= x_2\alpha_2 (y_2\beta_2^{-1}y_4^{-1}) y_4 z_2 \quad \text{(by commutativity)} \\ &= x_2\alpha_2 y_2\beta_2^{-1}z_2 \\ &= x_1\alpha_1 y_1\beta_1^{-1}z_1. \end{aligned}$$

Thus $(\alpha_1, \beta_1)\theta(\alpha_3, \beta_3)$, establishing transitivity. And so by congruence-factorability, $\theta = \varphi \times \psi$ for some φ and ψ. One easily sees that for all $\alpha, \beta \in H$,

$$(\alpha, \alpha)\theta(\beta, \beta),$$

and so clearly $\varphi = \psi = H^2$, and so $\theta = H^4$. Thus for 1 a fixed identity in H and all $\alpha \in H$,

$$(1, 1)\theta(\alpha, 1), \quad \text{i.e.}$$
$$x_1y_1z_1 = x_2\alpha y_2z_2, \quad \text{i.e.}$$
$$\alpha = y_2z_2x_2, \quad \text{an identity.}$$

Problem 5.7. Is the hypothesis of commutativity needed in 5.6? (We know that the connectedness assumption is essential—see after 5.9 below.)

For *groups* in $\mathscr{V}$, the next corollary is much easier; see e.g. [**42**, p. 155].

COROLLARY 5.8. *If $\mathscr{V}$ is the variety of rings with unit, or more generally, if $\mathscr{V}$ obeys the laws*

$$x + 0 = 0 + x = x$$
$$x \cdot 0 = 0$$
$$x \cdot 1 = x,$$

then every connected groupoid in $\mathscr{V}$ obeys the group law $x = 1$.

Proof. The facts about H-spaces mentioned just after Proposition 1.1 tell us (using the first equations) that such a connected groupoid is commutative. These equations imply congruence-factorability, and so we may apply Theorem 5.6.

The next corollary is immediate from the last corollary and the results of § 1.

COROLLARY 5.9. *Every arcwise connected topological ring with unit has trivial homotopy groups $\pi_n (n = 1, 2, \ldots)$.*

Arcwise connectedness is essential here, as the following example of B. Banaschewski shows. We take $(Z, +, -, \cdot)$ to be the ring of integers with the discrete topology, and $(T, +)$ the circle group, i.e. real numbers modulo 1, with the usual topology. Consider

$$R = (T \times Z, +, \circ),$$

where

$$([x], m) + ([y], n) = ([x + y], m + n), \quad \text{and}$$
$$([x], m) \cdot ([y], n) = ([nx + my], mn)$$

(with $[x]$ denoting the class of x modulo 1). One easily checks that R is a topological ring with unit $([0], 1)$. A similar example (not fully a ring, but generating a congruence-factorable variety) can be based on $\mathbf{SO}(3) \times Z$. (For a general description of constructions of this sort, see the introduction of [**16**].)

The following theorem is related to but much easier than Theorem 5.6 above.

THEOREM 5.10. *If $\mathscr{V}$ has factorable congruences, then every group in $\mathscr{V}$ is trivial.*

Proof. If $(G, \circ, ^{-1}, F_t)_{t \in T}$ is a group in $\mathscr{V}$, then we define a congruence θ on $(G, F_t)_{t \in T}{}^2$ as follows:

$$(a_1, a_2)\theta(b_1, b_2) \text{ if and only if } a_1a_2^{-1} = b_1b_2^{-1}.$$

One immediately checks that θ is a congruence which is not factorable.

COROLLARY 5.11. *If $\mathscr{V}$ is congruence-distributive, then every group in $\mathscr{V}$ is trivial.*

Proof. Immediate from A. Hales' observation that congruence-distributivity implies congruence-factorability (see [**17**]).

It is important to realize that this last result applies (via 3.2) to homotopy groups of topological algebras in $\mathscr{V}$, since every congruence-distributive variety has congruence-distributive idempotent reduct (as follows from the Malcev conditions for congruence-distributivity [**29**]). This contrasts with the more intricate situation for congruence-factorability in 5.6–5.9 and the example following 5.9.

We close this section with two further results restricting groups in $\mathscr{V}$. First say that $\mathscr{V}$ is *k-indecomposable* [**6**] if and only if no $\mathfrak{A} \in \mathscr{V}$ can be written as a union $\mathfrak{A} = \mathfrak{A}_1 \cup \ldots \cup \mathfrak{A}_k$ with each $\mathfrak{A}_i$ a proper subalgebra. The next theorem and its proof can be generalized, but for simplicity we state only this case. Z denotes the group of integers.

THEOREM 5.12. *If $\mathfrak{G}$ is a groupoid in a 3-indecomposable variety, then not every object in $\mathfrak{G}$ has automorphism group Z.*

Proof. Supposing, to the contrary, that all such groups are Z, we take

$$\mathfrak{A} = \{(\alpha, \beta) \in G^2: ((\exists \text{ object } B)\ \alpha, \beta \in \text{Aut } B\}.$$

Clearly $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3$ where

$$\mathfrak{A}_1 = \{(\alpha, \beta) \in \mathfrak{A}: \alpha \text{ is even}\},$$
$$\mathfrak{A}_2 = \{(\alpha, \beta) \in \mathfrak{A}: \beta \text{ is even}\}, \quad \text{and}$$
$$\mathfrak{A}_3 = \{(\alpha, \beta) \in \mathfrak{A}: \alpha + \beta \text{ is even}\}.$$

Let us say that $\mathscr{V}$ is *essentially non-unary* if and only if no non-trivial subvariety of $\mathscr{V}$ is equivalent to a unary variety.

THEOREM 5.13. *If $\mathscr{V}$ is essentially non-unary, then no group in $\mathscr{V}$ is finite, non-commutative and simple.*

Proof. Follows readily from the easy fact of group theory that if G is finite, non-commutative and simple, then every homomorphism $G^n \to G$ depends on only one coordinate. (See [**27**, Theorem 9.12b, p. 51].)

6. The Abelian groups in a variety. Here we introduce some special methods which apply to the *Abelian* groups in a variety. As we have seen, there are many varieties $\mathscr{V}$ for which these methods will apply to all groups in $\mathscr{V}$.

For any variety $\mathscr{V}$ we define a ring with unit $R_{\mathscr{V}}$ as follows. For each n-ary operation F we take non-commuting free generators ("indeterminates") $u_1^F, \ldots, u_n^F$, and we define

$$R_{\mathscr{V}} = Z[u_i^F\colon \text{all } F, \text{all } i]/\Sigma,$$

where Σ is the ideal generated by all $\omega_1 - \omega_2$ obtained as follows. We first recursively define the linearization $\bar{\tau}$ of a $\mathscr{V}$-term via

$$\bar{x}_i = 1 \cdot x_i$$

$$\overline{F(\sigma_1, \ldots, \sigma_n)} = u_1^F \bar{\sigma}_n + \ldots + u_n^F \bar{\sigma}_n.$$

Now let the above ω_1 and ω_2 be obtained as the coefficients of an arbitrary variable x_j in the terms τ_1 and τ_2, where $\tau_1 = \tau_2$ is any identity of $\mathscr{V}$. (It is enough to let $\{\tau_1 = \tau_2\}$ contain an equational axiomatization of $\mathscr{V}$.)

For any $R_{\mathscr{V}}$-module $\mathfrak{M}$ we define $\mathfrak{A}^{\mathfrak{M}}$ to be the algebra with universe M and operations $F(x_1, \ldots, x_n) = u_1^F x_1 + \ldots + u_n^F x_n$ (for each F in the type of $\mathscr{V}$), together with $+$ and $-$.

THEOREM 6.1. *The mapping $\mathfrak{M} \mapsto \mathfrak{A}^{\mathfrak{M}}$ is a one-one correspondence between the variety of unital $R_{\mathscr{V}}$-modules and the variety of Abelian groups in $\mathscr{V}$. Thus these two varieties are equivalent.*

The proof is straightforward and omitted. Though easy, the theorem gives us a valuable viewpoint, especially since we can sometimes identify the ring $R_{\mathscr{V}}$.

For example, if $\mathscr{V}$ is defined by

$$xx = x$$

$$xy = yx,$$

then 5.2 tells us that all groups in $\mathscr{V}$ are commutative. Linearization of these equations yields

$$(\alpha + \beta)x = x$$

$$\alpha x + \beta y = \alpha y + \beta x,$$

and so $R_{\mathscr{V}}$ is $Z[\alpha, \beta]/\alpha = \beta$, $\alpha + \beta = 1$, which is isomorphic to the ring $Z[1/2]$ of rationals with denominator a power of 2. And so the groups in $\mathscr{V}$ are precisely the uniquely 2-divisible Abelian groups.

Semilattices have the above two laws and the associative law $x(yz) = (xy)z$, which when linearized yields

$$\alpha x + \beta\alpha y + \beta^2 z = \alpha^2 x + \alpha\beta y + \beta z.$$

Thus $R_{\mathscr{S}}$, for $\mathscr{S}$ the variety of semilattices, is the quotient of the above ring $R_{\mathscr{V}} \cong Z[1/2]$ by the smallest ideal I containing $(\alpha - \alpha^2)$. Clearly I also contains $4(\alpha - \alpha^2) = 2 \cdot (2\alpha) - (2\alpha)^2 = 2 - 1 = 1$, and hence $R_{\mathscr{S}} = 0$. Hence we have proved

THEOREM 6.2. *Semilattices obey the law* $x = 1$ *in homotopy.*

(Closely related results were proved by Anderson and Ward [**2**] and Brown [**5**]. For instance, Brown's argument essentially shows that all groups are trivial in the variety given by the laws $x^2 = x$, $0 \cdot x = x \cdot 0 = 0$; this implies our 6.2 for *compact connected* semilattices.)

For another example, if $\mathscr{V}$ is given by the laws

$$F(x, x) = x$$
$$F(F(x, y), F(y, x)) = y,$$

then again 5.2 says that groups in $\mathscr{V}$ are commutative, and our ring equations are

$$\alpha + \beta = 1$$
$$\alpha^2 + \beta^2 = 0$$
$$2\alpha\beta = 1.$$

Clearly these equations are satisfied if we take $\alpha = \frac{1}{2}(1 + i)$ and $\beta = \frac{1}{2}(1 - i)$ in $Z[1/2, i]$, the ring of rational complex numbers with denominators a power of 2. In fact one may easily check that this correspondence establishes an isomorphism $R_{\mathscr{V}} \cong Z[1/2, i]$. And so all *squares* A^2 of uniquely 2-divisible Abelian groups A can be groups in $\mathscr{V}$ by taking

$$i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(as an endomorphism of A^2). But clearly e.g. Q^{2n+1} cannot. A cyclic group of finite odd order m can be a group in $\mathscr{V}$ if and only if -1 is a quadratic residue (mod m). (I thank Ann Bateson for simplifying the description of this $R_{\mathscr{V}}$.)

Theorem 6.1 tells us that our problem verges into module theory; and various problems present themselves, of which we mention only one.

Problem 6.3. Characterize those classes of Abelian groups which can appear as all underlying groups of R-modules for a fixed ring R.

(This is clearly a *spectrum* problem in the general sense outlined in [**14**].)

It should be remarked that of course the variety of Z/mZ modules gives an example of a variety obeying the group law $x^m = 1$ in homotopy, but no further group laws.

7. The spaces in a variety. While the main results in §3 and §5 have told us a lot about the laws obeyed by the homotopy groups of spaces which can be underlying spaces of topological algebras in $\mathscr{V}$ (briefly, "*spaces in* $\mathscr{V}$"), we naturally would like to have more information about and a better description of the spaces in $\mathscr{V}$, not just their homotopy groups. Apart from the homotopy information (which is greatly extended in forthcoming work of Ann Bateson), very little is known in general. But some suggestive examples have occurred sporadically in the literature, making it clear, for instance, that understanding homotopy is far from enough. In this section we review a few of these examples, because they are probably not well known as a *collection* of examples, and because the very divergence of methods of proof calls for a better understanding: is there a general method (or theory) which subsumes several of these examples? We cannot begin to give such a theory, but we give one new result (7.7 below) and a possible new viewpoint ("Malcev conditions").

7.1 First note that if $\mathscr{V}$ and $\mathscr{W}$ are as in the beginning of § 5, then the spaces in $\mathscr{V}$ are simply the *squares* A^2 of arbitrary spaces A, and the spaces in $\mathscr{W}$ are those A with $A^2 \cong A$. (And the proofs are completely analogous.) In fact the general theory of "kth power varieties" tells us that the spaces in $\mathscr{V}^{[k]}$ (see [**48**, § 0] for definitions) are precisely the kth powers of spaces in $\mathscr{V}$.

7.2 If $\mathscr{V}$ is the *product* of varieties $\mathscr{U}$ and $\mathscr{W}$ (see [**46**, § 0] or [**43**]), then the spaces in $\mathscr{V}$ are precisely the products of spaces in $\mathscr{V}$ with spaces in $\mathscr{W}$. (The proof is apparent from these two references.)

7.3 The variety $\mathscr{H}$ of "*H-spaces*" given by $ex = x = xe$ excludes all spheres except S^1, S^3 and S^7 (Adams [**1**]), and all spaces in $\mathscr{H}$ have at least one arc-component with commutative homotopy (see § 1). *Associative H*-spaces (i.e. *semigroups with unit* or *monoids*) exclude S^7 (James [**28**]), Bing's "house with two rooms" (see [**35**, p. 141]) and certain 3-manifolds [**34**]. For further concrete geometric examples of spaces in and not in various related varieties (semigroups with 0 and 1, semilattices with or without 0 and 1), see [**32**; **33**; **35** and **36**].

7.4 As is well known, a topological group has a completely regular and homogeneous space, and a compact uncountable topological group has power $\geqq 2^{\aleph_0}$. (This last property holds for a wide class of varieties [**47**, § 3]). In varieties of rings obeying $x^n = x$ (for some fixed n), the only compact spaces are products of finite spaces [**30**].

7.5 If $\mathscr{V}$ is defined by the identities

$$F(f(x), x, y) = x$$
$$F(x, x, y) = y,$$

then obviously no non-trivial space in $\mathscr{V}$ has the fixed point property. The more stringent equations

$$F(f^k(x), x, y) = x$$
$$F(x, x, y) = y$$

rule out any space in which some iterate of every function must have a fixed point, e.g. any finite complex of non-zero Euler characteristic (see [**31**, p. 111]).

7.6 There exists a space A least likely to be a space in a variety. H. Cook exhibited [**10**] a compact connected metric space A such that every map $A \to A$ is a constant or the identity, from which it easily follows that every finitary operation $A^n \to A$ is either a constant or a projection. Hence if A is a space in $\mathscr{V}$, then every space is a space in $\mathscr{V}$.

We now give a theorem characterizing spaces (among simplicial spaces) in the variety of a "majority function."

THEOREM 7.7 *For A a simplicial complex, the following conditions are equivalent*:
(i) $\pi_n(A) = 0 \quad (n = 1, 2, 3, \ldots)$,
(ii) *there exists a continuous ternary operation F: $A^3 \to A$ obeying the laws*

$$F(x, x, y) = F(x, y, x) = F(y, x, x) = x.$$

Proof. (ii) $\Rightarrow$ (i) by Corollary 5.11, since the laws (ii) are well known to imply congruence distributivity. Conversely, given (i), let us construct F obeying these laws. Let B be the subspace of A^3,

$$B = \{(x, y, z): x = y \text{ or } y = z \text{ or } z = x\};$$

clearly we may assume that B is a subcomplex of A^3. Now one easily defines

$$F_0: B \to A \quad \text{as}$$

$$F_0(x, y, z) = \begin{cases} x & \text{if } x = y \\ y & \text{if } y = z \\ z & \text{if } z = x, \end{cases}$$

(obviously a consistent and exhaustive set of conditions yielding a continuous function). We will be done if we can find F: $A^3 \to A$ with $F \restriction B = F_0$. But the obstructions to extending to $A^{3(n)} \cup B$ are cochains with coefficients in $\pi_n(A) = 0$, and hence the extension exists (see Hilton and Wylie [**24**, Chap. 7]).

As an alternate viewpoint on the question of what spaces can be in a variety $\mathscr{V}$, we can select a property P of spaces and examine

$$K(P) = \{\mathscr{V}: \text{every space in } \mathscr{V} \text{ satisfies } P\}.$$

One easily checks that if P is a productive property of spaces, then $K(P)$ satisfies all the conditions of [**46**] which are necessary and sufficient for Malcev-definability, except perhaps for (v): if Σ defines a variety in $K(P)$, then so does some finite subset of Σ. (If $P(A)$ says, e.g., "A has commutative homotopy," then $K(P)$ is indeed Malcev-definable, by §§ 3 and 5.) One of the most interesting cases would be when $P(A)$ means "A is not homeomorphic to B," for a fixed product-indecomposable space B. I.e., we have

$$K_B = \{\mathscr{V}: B \text{ is not a space in } \mathscr{V}\}.$$

We know almost nothing about this K_B, even for very simple B such as the unit interval. But we have managed to show that (v) above does *not* hold for K_B with B = unit interval. For take Σ to be the laws of lattice theory (in $\wedge$, $\vee$) with 0 and 1 together with

$$a_i \wedge a_{i+1} = a_i \quad (i = 0, 1, 2, \ldots)$$
$$\left.\begin{array}{l} f(a_{2j}) = 0 \\ f(a_{2j+1}) = 1 \end{array}\right\} \quad (j = 0, 1, 2, \ldots).$$

Essentially well known connectedness arguments (see e.g. [**9**; **15** or **33**]) tell us that $\wedge$ and $\vee$ must be the ordinary min and max (or dually). From this it is easy to see that Σ cannot be modeled on the unit interval, but every finite subset can be. Still, we can ask the following question (for definitions, see [**46**]).

Problem 7.8 For compact B, is K_B *weakly* Malcev-definable?

The notion of "weak Malcev condition" is refined in [**40**] and [**3**], leading, of course, to refined versions of this problem. It would be very interesting to see explicit weak Malcev conditions for K_B taken as e.g. the unit interval.

Added in proof. For some further results on spaces in varieties, see my abstracts in the Notices of the American Mathematical Society *23* (1976), p. A-577 and *24* (1977) (June, to appear).

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