ON MAPPING SEQUENCES

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Introduction

In 1958 D. Puppe developed the theory of *Abbildungsfolge* and applied it to the study of Hilton formula and his *sphärenähnliche Mannigfaltigkeiten* [12].[1] We shall be concerned here with a dual situation.

In Part I we introduce the *mapping sequence* and discuss some applications of it. Let $f: X \to Y$ be a map[2] of one topological space to another. We associate with it the following two sequences which are homotopically equivalent to each other (§2, Th. 2)

$$\mathfrak{M}f: \cdots \to \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{\Omega If} \Omega E_f \xrightarrow{\Omega Pf} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{If} E_f \xrightarrow{Pf} X \xrightarrow{f} Y,$$

$$\mathfrak{P}f: \cdots \to E_{P^5 f} \xrightarrow{P^6 f} E_{P^4 f} \xrightarrow{P^5 f} E_{P^3 f} \xrightarrow{P^4 f} E_{P^2 f} \xrightarrow{P^3 f} E_{Pf} \xrightarrow{P^2 f} E_f \xrightarrow{Pf} X \xrightarrow{f} Y,$$

where $Pf: E_f \to X$ is the fibering induced by f from the contractible fibre space over Y, If the injection of the fibre to the total, and Ω the loop functor. Furthermore, it will be shown in §4 that they are homotopically invariant in the sense that homotopically equivalent sequences are obtained when f is altered within its homotopy class. $\mathfrak{M}f$ induces for any space V an exact sequence

$$(\mathfrak{M}f)_*: \cdots \to \pi(V, \Omega X) \xrightarrow{(\Omega f)_*} \pi(V, \Omega Y) \xrightarrow{(If)_*} \pi(V, E_f) \xrightarrow{(Pf)_*} \pi(V, X) \xrightarrow{f_*} \pi(V, Y),$$

where $\pi(V, X)$ denotes the set of homotopy classes of maps $V \to X$. This is reduced to the usual exact sequence of homotopy groups in case V is a sphere and f an inclusion.

It will turn out in §5 that $\mathfrak{M}f$, together with its invariance, gives some delicate informations about homotopy equivalences. For instance, we prove that the fibre of a contractible fibering has the same homotopy type as the

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1) Numbers in square brackets refer to the papers listed at the end.

2) By the term "map" is always meant a continuous mapping.

loop space of its base. This is a variant of a theorem of Samelson [13].

In Part II we deal with an interrelation between our mapping sequence and Puppe's Abbildungsfolge. For any triple $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $g \circ f \simeq 0$, we construct the *connecting diagram* which is the basic machinary of our later investigation. Using this we define, for any map $f : X \to Y$, suspensions

$$\sigma^* : \pi(C_f, V) \to \pi(SE_f, V)$$

and

$$\sigma_* : \pi(V, E_f) \to \pi(V, \Omega C_f)$$

which may be regarded as an extension of usual cohomology and homotopy suspensions. Here C_f is a mapping cone of f, and S denotes the suspension functor.

We will prove isomorphism theorems concerning σ^* and σ_* (§9), as an application of which we present a detailed exposition of the Postnikov system in line with a treatment in [6]. Finally, it will be shown that the connecting diagram allows us to give a direct description of functional cohomology operations.

Part I. Mapping sequence and homotopy equivalence

1. Preliminaries

1.1. We begin with some notations and conventions to be used here.

A fixed base point will always be chosen in each space and denoted by a subscript 0: thus $x_0 \in X$, $a_0 \in A$. All maps and homotopies are to carry base points to such. The identity map of X on itself is denoted by 1_X or simply by 1. Given two maps $f_1, f_2 : X \to Y$, $f_1 \simeq f_2$ means the existence of a homotopy between them. The fact that there exists a homotopy equivalence $\varphi : X \to Y$ is expressed by $\varphi : X \equiv Y$. Let f, f', φ, ψ be maps such that φ and ψ are homotopy equivalences. Suppose the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

In this case we say that f and f' are homotopically equivalent and we denote

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by $f \equiv f'$.

The set of all homotopy classes of maps $X \to Y$ will be denoted by $\pi(X, Y)$, which contains the distinguished element 0, namely, the homotopy class of the constant map $0 : X \to Y$. The homotopy class of a map $f : X \to Y$ is denoted by $[f]$. Any map $g : Y \to Z$ induces a mapping $g_* : \pi(X, Y) \to \pi(X, Z)$ by the rule $g_*[f] = [g \circ f]$.

For any subspaces A and B of X we now define $E_{A,B}(X)$ to be the path space $\{\alpha : I \to X \mid \alpha(0) \in A,\ \alpha(1) \in B\}$ with compact open topology, where I is the unit interval $[0, 1]$. $E_{\{x_0\},X}(X)$ and $E_{\{x_0\},\{x_0\}}(X)$ are abbreviated by EX, ΩX respectively. We shall denote the constant path at a point $x \in X$ by e_x. For any path $\alpha : I \to X$ let $\alpha_{a,b} : I \to X$ $(0 \leqq a \leqq b \leqq 1)$ be a path defined by $\alpha_{a,b}(s) = \alpha(a + (b-a)s)$ for $0 \leqq s \leqq 1$. Also, let α^{-1} be the inverse of α, which is given by $\alpha^{-1}(s) = \alpha(1-s)$ for $s \in I$. By $\alpha \cdot \beta$ is meant the composition of $\alpha : I \to X$ with $\beta : I \to X$ such that $\alpha(1) = \beta(0)$.

1.2. A map $p : X \to B$ is called a *fibering* if it possesses the covering homotopy property for all spaces. As indicated in W. Hurewicz [7], $p : X \to B$ is a fibering if, and only if, there exists a continuous function

$$\Lambda : \{(x, \alpha) \mid x \in X,\ \alpha \in E_{B,B}(B),\ p(x) = \alpha(1)\} \to E_{X,X}(X),$$

called a *path lifting function* for p, subject to the requirements $p\Lambda(x, \alpha) = \alpha$ and $\Lambda(x, \alpha)(1) = x$.

By setting

$$\lambda(x, \alpha) = \Lambda(x, \alpha)(0),$$

we obtain a map

$$\lambda : \{(x, \alpha) \mid x \in X,\ \alpha \in E_{B,B}(B),\ p(x) = \alpha(1)\} \to X,$$

with the following properties:

(i) $p\lambda(x, \alpha) = \alpha(0)$,

(ii) the map $x \to \lambda(x, e_{p(x)})$, $x \in X$, is homotopic to $1_X : X \to X$ via a homotopy which moves points along fibres.

λ is said to be a *lifting function* for p.[3]

[3] It should be remarked that a map with such a λ may fail to be a fibre map in the sense of Serre, as shown by the retraction $I \vee I \to I$. But it is sufficient for later purpose to postulate the existence of such a λ.

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Now let $f : X \to Y$ be an arbitrary map. We define $Pf : E_f \to X$ as follows:

$$E_f = \{(x, \beta) \mid x \in X,\ \beta \in EY,\ f(x) = \beta(1)\},$$
$$Pf(x, \beta) = x.$$

Then $(Pf)^{-1}(x_0) = x_0 \times \Omega Y$. We define the injection $If : \Omega Y \to E_f$ by

$$If(\beta) = (x_0, \beta) \qquad \text{for } \beta \in \Omega Y.$$

Remark. In case f is an inclusion then $E_f = E_{y_0, X}(Y)$. For a constant map $f(X) = y_0$, we have $E_f = X \times \Omega Y$.

Let $g : B' \to B$ be a map and let $p : X \to B$ be a fibering. It is readily verified that if we define $p' : X' \to B'$ by

$$X' = \{(b', x) \mid b' \in B',\ x \in X,\ g(b') = p(x)\},$$
$$p'(b', x) = b',$$

then $p' : X' \to B'$ is also a fibering. In particular, we have

LEMMA 1. *$Pf : E_f \to X$ is a fibering whose lifting function and path lifting function are given as follows:*

$$\lambda((x, \beta), \alpha) = (\alpha(0), \beta \cdot (f\alpha)^{-1})$$
$$\text{for } (x, \beta) \in E_f,\ \alpha : I \to X \text{ with } \alpha(1) = x,$$
$$\Lambda((x, \beta), \alpha)(s) = (\alpha(s), \gamma_s)$$
$$\text{for } 0 \leqq s \leqq 1,$$

where $\gamma_s : I \to Y$ is defined by

$$\gamma_s(\tau) = \begin{cases} \beta\left(\dfrac{2\tau}{1+s}\right) & \text{for } 0 \leqq \tau \leqq \dfrac{1+s}{2}, \\ f\alpha(2(1-\tau)+s) & \text{for } \dfrac{1+s}{2} \leqq \tau \leqq 1. \end{cases}$$

Also, we have

LEMMA 2. *$E_f \xrightarrow{Pf} X \xrightarrow{f} Y$ induces for any space V an exact sequence*

$$\pi(V, E_f) \xrightarrow{(Pf)_*} \pi(V, X) \xrightarrow{f^*} \pi(V, Y).$$

Proof. Consider a family of maps $h_t : E_f \to Y$ defined by $h_t(x, \beta) = \beta(1-t)$ for $x \in X$, $0 \leqq t \leqq 1$. This gives rise to a homotopy between $h_0 = f \circ Pf$ and $h_1 = 0$, and thus $f \circ Pf \simeq 0$.

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Conversely, given $g : V \to X$ such that $f_*[g] = 0$, we can find a homotopy $k_t : V \to Y$ with $k_0 = 0$, $k_1 = f \circ g$. For each point $v \in V$ let $\beta(v)$ be the path in Y given by $\beta(v)(s) = k_s(v)$, $0 \leqq s \leqq 1$. Then if we define

$$k(v) = (g(v),\ \beta(v)) \qquad \text{for } v \in V,$$

we obtain a map $k : V \to E_f$ such that $Pf \circ k = g$, which proves our assertion.

2. Mapping sequence

2.1. Suppose $f : X \to Y$ is a fibering with fibre $F = f^{-1}(y_0)$ and let $i : F \to X$ be the injection. Using the lifting function λ for f we shall define $\Phi : E_f \to F$, $\Psi : E_f \to F$ by

$$\Phi(x) = (x,\ e_{y_0}) \qquad \text{for } x \in F,$$
$$\Psi(x,\ \beta) = \lambda(x,\ \beta) \qquad \text{for } x \in X,\ \beta \in EY \text{ with } f(x) = \beta(1).$$

Then Ψ is well-defined, since $f\lambda(x,\ \beta) = y_0$ because of the property (i) of the lifting function λ (cf. 1.2).

The following theorem plays a crucial role in our later development.

THEOREM 1. *Φ and Ψ are mutually inverse homotopy equivalences and, in addition, the following diagram is commutative up to homotopy*

$$\begin{array}{ccc} F & \xrightarrow{\ i\ } & X \\ {\scriptstyle\Phi}\downarrow\uparrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle 1_X} \\ E_f & \xrightarrow[\ Pf\ ]{} & X. \end{array}$$

Proof. By definition we have $\Psi \circ \Phi(x) = \lambda(x,\ e_{y_0})$, $\Phi \circ \Psi(x,\ \beta) = (\lambda(x,\ \beta),\ e_{y_0})$. Obviously $\Psi \circ \Phi \simeq 1_F$ by the property (ii) of λ (cf. 1.2). On the other hand, using the path lifting function Λ for f, a homotopy

$$((x,\ \beta),\ t) \to (\Lambda(x,\ \beta)(t),\ \beta_{0,t}), \qquad (0 \leqq t \leqq 1)$$

yields $\Phi \circ \Psi \simeq 1_{E_f}$. Since $Pf \circ \Phi = i$, it follows at once that $i \circ \Psi \simeq Pf$, and this concludes the proof.

2.2. We have already seen in Lemma 1 that $Pf : E_f \to X$ is a fibering for any map $f : X \to Y$. This fact enables us to apply Theorem 1 to Pf instead of f, and we obtain the homotopy commutative diagram

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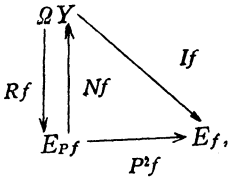
(1)



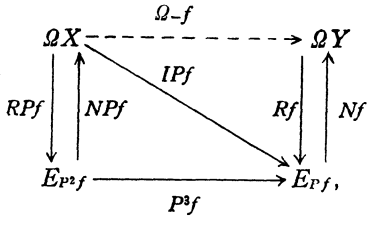
where vertical maps are homotopy equivalences and $P^n f$ $(n \geqq 2)$ is defined inductively to be $P(P^{n-1}f)$. Here we may identify E_{Pf} with $\{(\beta, \alpha) \mid \beta \in EY, \alpha \in EX, \beta(1) = f\alpha(1)\}$. Then Rf and Nf are determined by

$$Rf(\beta) = (\beta, e_{x_0}) \qquad \text{for } \beta \in \Omega Y,$$
$$Nf(\beta, \alpha) = \beta \cdot (f\alpha)^{-1} \qquad \text{for } (\beta, \alpha) \in E_{Pf},$$

and, moreover, $P^2 f(\beta, \alpha) = (\alpha(1), \beta)$ for $(\beta, \alpha) \in E_{Pf}$.

Replacing f by Pf in the triangle (1), we get the following diagram

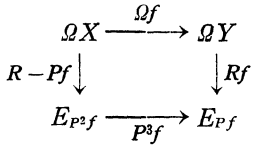
(2)



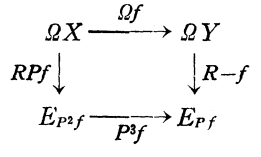
where $\Omega_- f : \Omega X \to \Omega Y$ is the map given by $\Omega_- f(\alpha) = (f\alpha)^{-1}$ for $\alpha \in \Omega X$. Since we have $(Nf) \circ IPf(\alpha) = Nf(e_{y_0}, \alpha) = e_{y_0} \cdot (f\alpha)^{-1}$ for $\alpha \in \Omega X$, homotopy-commutativity holds in (2).

Let $\sigma_X : \Omega X \to \Omega X$ and $\sigma_Y : \Omega Y \to \Omega Y$ be involutions given by inversions of loops. We set $R_- f = Rf \circ \sigma_Y$, $R_- Pf = RPf \circ \sigma_X$. Then we have homotopy-commutative diagrams

(3)



(4)



Consider the following diagram

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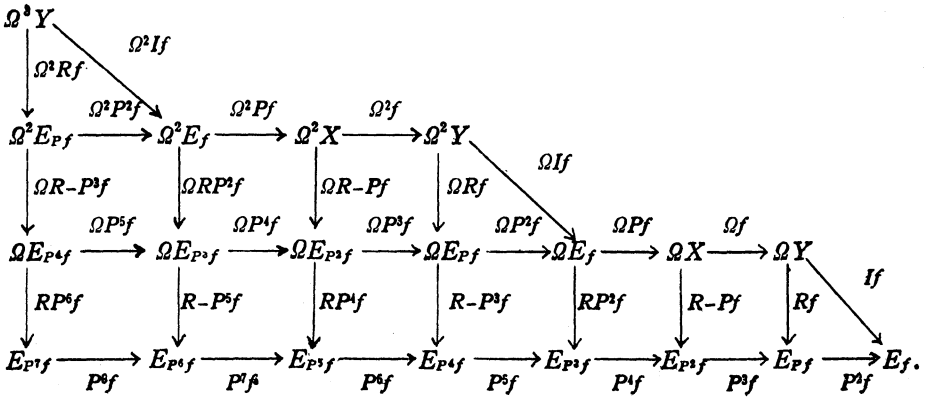

On the lowest ladder homotopy commutativity follows from (1), (3) and (4), as is readily seen. We note that other ladders are obtained from the lowest one by applying the loop functor. Every vertical map is a homotopy equivalence. For brevity we write

$$R_n f = \begin{cases} RP^n f & \text{for even } n, \\ R\text{-}P^n f & \text{for odd } n. \end{cases}$$

Then we see immediately that vertical equivalences are given by

$$R_{3n-3} f \circ \cdots \circ \Omega^{n-2} R_3 f \circ \Omega^{n-1} Rf : \Omega^n Y \equiv E_{P^{3n-2} f},$$
$$R_{3n-2} f \circ \cdots \circ \Omega^{n-2} R_4 f \circ \Omega^{n-1} R_1 f : \Omega^n X \equiv E_{P^{3n-1} f},$$
$$R_{3n-1} f \circ \cdots \circ \Omega^{n-2} R_5 f \circ \Omega^{n-1} R_2 f : \Omega^n E_f \equiv E_{P^{3n} f}.$$

The results obtained above are summarized as follows.

THEOREM 2. *The sequence*

$$\mathfrak{M} f : \cdots \to \Omega^2 E_f \xrightarrow{\Omega^2 Pf} \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{\Omega If} \Omega E_f \xrightarrow{\Omega Pf} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{If} E_f \xrightarrow{Pf} X \xrightarrow{f} Y$$

is homotopically equivalent to the sequence obtained by iterated construction:

$$\mathfrak{P} f : \cdots \xrightarrow{P^8 f} E_{P^6 f} \xrightarrow{P^7 f} E_{P^5 f} \xrightarrow{P^6 f} E_{P^4 f} \xrightarrow{P^5 f} E_{P^3 f} \xrightarrow{P^4 f} E_{P^2 f} \xrightarrow{P^3 f} E_{Pf} \xrightarrow{P^2 f} E_f \xrightarrow{Pf} X \xrightarrow{f} Y.$$

We refer to the sequence $\mathfrak{M} f$ above as the *mapping sequence* of f.

Combining Th. 2 with Lemma 2 gives rise to the following

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COROLLARY 2.1. *The mapping sequence* $\mathfrak{M}f$ *induces for any space* V *an exact sequence*

$$(\mathfrak{M}f)_* : \cdots \to \pi(V, \Omega E_f) \xrightarrow{(\Omega Pf)_*} \pi(V, \Omega X) \xrightarrow{(\Omega f)_*} \pi(V, \Omega Y)$$
$$\xrightarrow{(If)_*} \pi(V, E_f) \xrightarrow{(Pf)_*} \pi(V, X) \xrightarrow{f_*} \pi(V, Y).$$

COROLLARY 2.2. *If* $f : X \to Y$ *is a fibering with fibre* F, *and if* $i : F \to X$, *the injection, has a left homotopy inverse* (*e.g.* F *is a retract of* X), *then the fibering* $\Omega f : \Omega X \to \Omega Y$ *admits a cross-section.*

Proof. By the assumption above we have $0 = \mathrm{Ker}\ (Pf)_* = \mathrm{Im}\ (If)_*$, since $i \equiv Pf$ by Th. 1. Thus $(\Omega f)_*$ is onto and hence Ωf has a right homotopy inverse which may be modified into a cross-section.

3. Relative mapping sequence

3.1. It would be natural to expect that the mapping sequence may be relativized.

Let $X \xrightarrow{h} Y \xrightarrow{g} Z$ be a *triple* with $f = g \circ h$ Consider the diagram

$$\begin{array}{ccccccccccccc}
\cdots \to & \Omega^2 E_g & \xrightarrow{\Omega d} & \Omega E_h & \xrightarrow{\Omega l} & \Omega E_f & \xrightarrow{\Omega k} & \Omega E_g & \xrightarrow{d} & E_h & \xrightarrow{l} & E_f & \xrightarrow{k} & E_g \\
& \downarrow 1 & & \downarrow\uparrow \Omega\varphi \ \ \Omega\psi & & \downarrow 1 & & \downarrow 1 & & \downarrow\uparrow \varphi \ \ \psi & & \downarrow 1 & & \downarrow 1 \\
\cdots \to & \Omega^2 E_g & \xrightarrow[\Omega Ik]{} & \Omega E_k & \xrightarrow[\Omega Pk]{} & \Omega E_f & \xrightarrow[\Omega k]{} & \Omega E_g & \xrightarrow[Ik]{} & E_k & \xrightarrow[Pk]{} & E_f & \xrightarrow[k]{} & E_g,
\end{array}$$

where the maps are set as follows:

$k(x, \beta) = (h(x), \beta)$ for $x \in X$, $\beta \in EZ$ with $\beta(1) = f(x)$,

$l(x, \alpha) = (x, g\alpha)$ for $x \in X$, $\alpha \in EY$ with $\alpha(1) = h(x)$,

$d(\gamma, \tilde{\beta}) = (x_0, \gamma)$ for $\gamma \in \Omega Y$, $\tilde{\beta} \in \Omega EZ$ with $g\gamma(t) = \tilde{\beta}(1, t)$

$\psi((x, \beta), (\alpha, \alpha')) = (x, \alpha)$ for $(x, \beta) \in E_f$, $(\alpha, \alpha') \in EE_g$ with $h(x) = \alpha(1)$, $\beta(s) = \alpha'(s, 1)$,

$\varphi(x, \alpha) = ((x, g\alpha), (\alpha, \alpha''))$ for $x \in X$, $\alpha \in EY$ with $\alpha(1) = h(x)$,

in which $\alpha'' \in EEZ$ is given by

$$\alpha''(s, t) = \begin{cases} g\alpha(t) & \text{for } s \geqq t, \\ g\alpha(s) & \text{for } s \leqq t. \end{cases}$$

It follows at once from these that $Pk \circ \varphi = l$, $\psi \circ \varphi = 1$, $\psi \circ Ik = d$.

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Next, we shall show that $\varphi \circ \psi$ is homotopic to $1 : E_k \to E_k$. The above definitions lead to

$$\varphi \circ \psi((x, \beta), (\alpha, \alpha')) = ((x, g\alpha), (\alpha, \alpha'')) \text{ for}$$
$$h(x) = \alpha(1),\ \beta(s) = \alpha'(s, 1),\ g\alpha(t) = \alpha'(1, t).$$

Define a family of maps $\gamma_\tau : I^2 \to Z (0 \leqq \tau \leqq 1)$ by the rule

$$\gamma_\tau(s, t) = \begin{cases} \alpha'(\min(1, s + 2\tau - 2\tau t), t) & \text{for } s \geqq t, \\ \alpha'(s, t - 2\tau + 2\tau s) & \text{for } s \leqq t,\ t - 2\tau + 2\tau s \geqq s, \\ \alpha'(\min(1, 2s - t + 2\tau - 2\tau s), s) & \text{for } s \leqq t,\ t - 2\tau + 2\tau s \leqq s, \end{cases}$$

and define $\beta_\tau \in EZ$ by setting $\beta_\tau(s) = \gamma_\tau(s, 1)$. Then the homotopy given by

$$((x, \beta), (\alpha, \alpha')) \to ((x, \beta_\tau), (\alpha, \gamma_\tau)) \in E_k, \qquad (0 \leqq \tau \leqq 1),$$

coincides with $1 : E_k \to E_k$ for $\tau = 0$ and with $\varphi \circ \psi$ for $\tau = 1$, which proves that φ is a homotopy equivalence with ψ as a homotopy inverse, and therefore the diagram above is commutative up to homotopy.

Now we state our results as follows.

THEOREM 3. *The sequence constructed for any triple* $f = g \circ h$

$$(5) \qquad \to \Omega^2 E_g \xrightarrow{\Omega d} \Omega E_h \xrightarrow{\Omega l} \Omega E_f \xrightarrow{\Omega k} \Omega E_g \xrightarrow{d} E_h \xrightarrow{l} E_f \xrightarrow{k} E_g$$

is homotopically equivalent to the "absolute" mapping sequence $\mathfrak{M}k$ *of* $k : E_f \to E_g$ *and thus induces for any space* V *an exact sequence*[4]

$$\cdots \to \pi(V, \Omega E_g) \xrightarrow{d_*} \pi(V, E_h) \xrightarrow{l^*} \pi(V, E_f) \xrightarrow{k_*} \pi(V, E_g).$$

The sequence (5) above is said to be the *relative mapping sequence* of a triple $f = g \circ h$. Note that when Z consists of a single point the sequence (5) is reduced to $\mathfrak{M}h$.

3.2. As an illustration we shall derive some exact sequences found in [9].

Suppose $(X;\ A, B)$ be a triad. Let $\Delta : X \to X \times X$ be the diagonal map, and let $\Delta' : A \cap B \to A \times B$ be the map determined by Δ. We denote by π the projection $A \times B \to A$. Then the composite $\pi \circ \Delta'$ is the injection $i : A \cap B \to A$. Let $j : E_{\Delta'} \to E_\Delta$ be the inclusion.

W. S. Massey [9] defined $\pi_n(A/B)$ to be the set of all homotopy classes of

[4] Note that this corresponds to a sequence in [6], Prop. 2.3.

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maps $(S^n ; E_+^n, E_-^n) \to (A \cup B ; A, B)$, where E_+^n, E_-^n are northern and southern hemispheres respectively. As is easily seen, the following natural isomorphisms hold:

$$\begin{aligned} \pi_{n+1}(X ; A, B) &\approx \pi(S^{n-1}, E_j), \\ \pi_n(A/B) &\approx \pi(S^{n-1}, E_{\Delta'}), \\ \pi_n(X) &\approx \pi(S^{n-1}, E_\Delta), \\ \pi_{n-1}(B) &\approx \pi(S^{n-1}, E_\pi). \end{aligned}$$

Hence we have exact sequences

$$\begin{aligned} (\mathfrak{M} j)_* : \cdots \to \pi_{n+1}(A/B) &\to \pi_{n+1}(X) \\ &\to \pi_{n+1}(X ; A, B) \to \pi_n(A/B) \to \pi_n(X), \\ (\mathfrak{M} \Delta')_* : \cdots \to \pi_n(A) + \pi_n(B) &\to \pi_n(A/B) \\ &\to \pi_{n-1}(A \cap B) \to \pi_{n-1}(A) + \pi_{n-1}(B), \\ \cdots \to \pi_{n+1}(A, A \cap B) &\to \pi_n(B) \to \pi_n(A/B) \\ &\to \pi_n(A, A \cap B) \to \pi_{n-1}(B), \end{aligned}$$

where the last sequence is obtained by applying Th. 3 to the triple $i = \pi \circ \Delta'$.

4. Invariance theorem

4.1. Suppose we are given the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ X' & \xrightarrow[f']{} & Y' \end{array} \tag{6}$$

which is commutative up to homotopy. Such a pair of maps (φ, ψ) will be called a *transformation* of f to f'. With this transformation, together with a fixed homotopy Φ such that $\Phi_0 = \varphi \circ f$ and $\Phi_1 = f' \circ \psi$, we associate a map

$$\chi_0 = E(\varphi, \psi ; \Phi) : E_f \to E_{f'}$$

by the rule

$$\chi_0(x, \beta) = (\psi(x), \beta') \text{ for } x \in X, \beta \in EY \text{ with } f(x) = \beta(1),$$

where $\beta' \in EY'$ is given by

$$\beta'(s) = \begin{cases} (\varphi\beta)(2s) & \text{for } 0 \leqq s \leqq \frac{1}{2}, \\ \Phi_{2s-1}(x) & \text{for } \frac{1}{2} \leqq s \leqq 1. \end{cases}$$

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In case (6) is strictly commutative, i.e. Φ can be chosen to be constant, $E(\varphi, \psi ; \Phi)$ is denoted simply by $E(\varphi, \psi)$.

Then commutativity holds in the diagram

$$\begin{array}{ccc} E_f & \xrightarrow{Pf} & X \\ {\scriptstyle \chi_0}\downarrow & & \downarrow{\scriptstyle \psi} \\ Ef' & \xrightarrow[Pf']{} & X' \end{array} .$$

Thus, by proceeding in such a manner, we obtain a transformation $(\cdots, \chi_2, \chi_1, \chi_0, \psi, \varphi) : \mathfrak{P}f \to \mathfrak{P}f'$, where $\chi_n = E(\chi_{n-2}, \chi_{n-1})$, $n \geqq 2$ and $\chi_1 = E(\psi, \chi_0)$. On the other hand, simple calculation shows that $\chi_0 \circ If \simeq If' \circ \Omega\varphi$. Hence this yields a transformation $(\cdots, \Omega\psi, \Omega\varphi, \chi_0, \psi, \varphi) : \mathfrak{M}f \to \mathfrak{M}f'$. These two transformations are related to each other, as stated in

LEMMA 3. *In the transformations above, $\mathfrak{M}f \to \mathfrak{M}f'$ and $\mathfrak{P}f \to \mathfrak{P}f'$, the corresponding maps are homotopically equivalent to each other; more precisely, homotopy equivalences obtained in* 2.2 *yield $\Omega^n\chi_0 \equiv \chi_{3n}$, $\Omega^n\psi \equiv \chi_{3n-1}$, $\Omega^n\varphi \equiv \chi_{3n-2}$.*

4.2. Let (φ, ψ) be as before, and let (φ', ψ') be a transformation of f' to f'' with a fixed homotopy Φ' such that $\Phi'_0 = \varphi' \circ f'$ and $\Phi'_1 = f'' \circ \psi'$. We define a homotopy $(\Phi' \circ \Phi) : \varphi' \circ \varphi \circ f \simeq f'' \circ \psi' \circ \psi$ by setting

$$(\Phi' \circ \Phi)_t = \begin{cases} \varphi' \circ \Phi_{2t} & \text{for } 0 \leqq t \leqq \dfrac{1}{2}, \\ \Phi'_{2t-1} \circ \psi & \text{for } \dfrac{1}{2} \leqq t \leqq 1. \end{cases}$$

Then it is readily verified that the following homotopy holds:

LEMMA 4. $E(\varphi' \circ \varphi, \psi' \circ \psi ; (\Phi' \circ \Phi)) \simeq E(\varphi', \psi' ; \Phi') \circ E(\varphi, \psi ; \Phi)$.

LEMMA 5. *Let (φ, ψ) be a transformation of f to f' with a fixed homotopy Φ, and let $\varphi \simeq \bar{\varphi}$, $\psi \simeq \bar{\psi}$. Then there exists a homotopy $\bar{\Phi} : \bar{\varphi} \circ f \simeq f' \circ \bar{\psi}$ such that $E(\bar{\varphi}, \bar{\psi} ; \bar{\Phi}) \simeq E(\varphi, \psi ; \Phi)$.*

The proof of Lemma 5 may be proceeded in the same manner as in [12], 2.5, B) and thus is omitted.

These two Lemmas yield the next result.

LEMMA 6. *Let (φ, ψ) be a transformation of f to f' with a fixed homotopy Φ. If φ and ψ are homotopy equivalences, so is $\chi_0 = E(\varphi, \psi ; \Phi) : E_f \to E_{f'}$.*

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From this we can derive a invariance theorem for mapping sequences.

THEOREM 4. *If in the transformation* $\mathfrak{M}f \to \mathfrak{M}f'$ *or* $\mathfrak{P}f \to \mathfrak{P}f'$ *two consecutive maps are homotopy equivalences, so are the following ones. In particular, if* $f \equiv f'$, *then each map in this transformation is a homotopy equivalence.*

COROLLARY 4.1. (Invariance of mapping sequence) *Let* $f, f' : X \to Y$ *be homotopic. Then* $\mathfrak{M}f$ *and* $\mathfrak{P}f$ *are homotopically equivalent to* $\mathfrak{M}f'$ *and* $\mathfrak{P}f'$ *respectively.*

COROLLARY 4.2. *If* $f : X \to Y$ *is nullhomotopic, then* $\mathfrak{M}f$ *coincides up to homotopy equivalences with the next sequence*

$$\cdots \to \Omega X \times \Omega^2 Y \xrightarrow{\Omega\pi} \Omega X \xrightarrow{0} \Omega Y \xrightarrow{i} X \times \Omega Y \xrightarrow{\pi} X \xrightarrow{0} Y,$$

where π *and* i *denote the projection and injection respectively; especially, if* $f : X \to Y$ *is an inessential fibering with fibre* F, *then* $F \equiv X \times \Omega Y$.

COROLLARY 4.3. *If* $f : X \to Y$ *is homotopically equivalent to* $\Omega f' : \Omega X' \to \Omega Y'$ *for some* $f' : X' \to Y'$, *then* $E_f \equiv \Omega E_{f'}$.

Proof. It follows from Th. 2 that $f \equiv \Omega f' \equiv P^3 f'$. Hence the above Lemma 6 gives homotopy equivalence $E_f \equiv E_{P^3 f'}$. Thus, again in view of Th. 2, we have the desired conclusion $E_f \equiv \Omega E_{f'}$, since $E_{P^3 f'} \equiv \Omega E_{f'}$.

5. Applications to homotopy equivalences

5.1. The following relations between $f : X \to Y$ and $f_* : \pi(V, X) \to \pi(V, Y)$ are used freely.

(i) f is nullhomotopic if and only if f_* vanishes for every space V.

(ii) f has a right homotopy inverse if and only if f_* is onto for each V.

(iii) If f has a left homotopy inverse, then Ker $f_* = 0$ for any V.

(iv) (a partial converse to (iii)). If f_* has kernel zero for every V, then Ωf admits a left homotopy inverse.

Proof of (*iv*). Our hypothesis, together with the exactness of $(\mathfrak{M}f)_*$, implies that $Pf \simeq 0$. Accordingly, Cor. 4.2 asserts that $\mathfrak{M}Pf$ is equivalent to the sequence

$$\to \Omega E_f \xrightarrow{0} \Omega X \xrightarrow{i} E_f \times \Omega X \xrightarrow{\pi} E_f \xrightarrow{0} X,$$

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where i and π are the injection and projection respectively. Hence we see from Th. 2 that $i \equiv IPf \equiv P^2Pf = P^3f \equiv \Omega f$. This shows that Ωf, like i, has a left homotopy inverse.

The mapping sequence provides a useful tool for establishing various homotopy equivalences, as will be shown in what follows.

5.2. Let us consider the injection $i : X \vee Y \to X \times Y$, whose mapping sequence is written

$$\cdots \to \Omega(X \vee Y) \xrightarrow{\Omega i} \Omega(X \times Y) = \Omega X \times \Omega Y \xrightarrow{Ii} E_i \xrightarrow{Pi} X \vee Y \xrightarrow{i} X \times Y.$$

First of all, we note

LEMMA 7. $Ii \simeq 0$.

Proof. Let us introduce the following subspaces of ΩX and ΩY:

$$\bar{\Omega}X = \left\{\alpha \in \Omega X \mid \alpha\left(\left[0, \frac{1}{2}\right]\right) = x_0\right\},$$

$$\underline{\Omega}Y = \left\{\beta \in \Omega Y \mid \beta\left(\left[\frac{1}{2}, 1\right]\right) = y_0\right\}.$$

Then injections $\bar{\Omega}X \to \Omega X$ and $\underline{\Omega}Y \to \Omega Y$ are evidently homotopy equivalences with homotopy inverses $\alpha \to e_{x_0} \cdot \alpha$ and $\beta \to \beta \cdot e_{y_0}$ respectively. Consider now the commutative diagram

$$\begin{array}{ccc} \bar{\Omega}X \times \underline{\Omega}Y & \xrightarrow{j} & \Omega X \times \Omega Y \\ \sigma \downarrow & & \downarrow Ii \\ E(X \vee Y) & \longrightarrow & E_i = E_{(x_0, y_0), X \vee Y}(X \times Y), \end{array}$$

where σ is given by $\sigma(\alpha, \beta) = \alpha \times \beta$, the product path of α and β, and the other maps are all injections. Since $E(X \vee Y)$ is contractible, it follows from commutativity that $Ii \circ j \simeq 0$. j, as the product of homotopy equivalences, is a homotopy equivalence. Therefore, it follows $Ii \simeq 0$, as we wish to prove.

Combining this lemma with Cor. 4.2 we obtain an equivalence $E_{Ii} \equiv \Omega X \times \Omega Y \times \Omega E_i$, while by virtue of Th. 2 we see that $Ii \equiv P^2 i$, $E_{P^2 i} \equiv \Omega(X \vee Y)$. Combining these results we conclude

PROPOSITION 1. $\Omega(X \vee Y) \equiv \Omega X \times \Omega Y \times \Omega E_{(x_0, y_0), X \vee Y}(X \times Y)$.

This may be regarded as a result dual to Hilton formula ([12], Th. 15).

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5.3. The following is, in a sense, a substitute for Sugawara's Lemma ([16], p. 118) in case X is not a CW-complex.

PROPOSITION 2. *Let X be an H-space with H-structure $\mu : X \times X \to X$ and let $f, g : X \times X \to X$ be defined by*

$$\begin{aligned} f(x_1, x_2) &= (\mu(x_1, x_2), x_1) \\ g(x_1, x_2) &= (\mu(x_1, x_2), x_2) \end{aligned} \qquad \text{for } x_1, x_2 \in X.$$

Then Ωf and Ωg are both homotopy equivalences. In particular, if X is a CW-complex such that $X \times X$ is also a CW-complex, then f and g are homotopy equivalences.

Proof. Let $\pi_1, \pi_2 : X \times X \to X$ be projections onto the first and second factors respectively. Since $\pi_2 \circ f = \pi_1$, Th. 3 yields the sequence

$$(7) \qquad \cdots \to \Omega E_{\pi_1} \xrightarrow{\Omega k} \Omega E_{\pi_2} \longrightarrow E_f \longrightarrow E_{\pi_1} \xrightarrow{k} E_{\pi_2},$$

where k is given by $k((x_1, x_2), \alpha) = ((\mu(x_1, x_2), x_1), \alpha)$ for $x_1, x_2 \in X$, $\alpha \in EX$ with $\alpha(1) = x_1$.

Let $k' : E_{\pi_1} \to E_{\pi_2}$ be defined by $k'((x_1, x_2), \alpha) = ((\mu(x_0, x_2), x_0), e_{x_0})$. Then k is homotopic to k' via a homotopy defined by

$$((x_1, x_2), \alpha) \to ((\mu(\alpha(1-t), x_2), \alpha(1-t)), \alpha_{0,1-t}), \quad 0 \leqq t \leqq 1.$$

Consider now the following commutative diagram

$$\begin{array}{ccc} E_{\pi_1} & \xrightarrow{k'} & E_{\pi_2} \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\eta} \\ X & \xrightarrow[k'']{} & X, \end{array}$$

where the maps are set as follows,

$$\begin{aligned} &\xi((x_1, x_2), \alpha) = x_2, && \text{for } x_1, x_2 \in X,\ \alpha, \beta \in EX \text{ with} \\ &&& \alpha(1) = x_1,\ \beta(1) = x_2, \\ &\eta((x_1, x_2), \beta) = x_1, \\ &k''(x) = \mu(x_0, x) && \text{for } x \in X. \end{aligned}$$

We see that ξ and η are both homotopy equivalences and that k'' is homotopic to 1_X by the condition imposed upon H-structure, so that k and Ωk are homotopy equivalences. It follows from exactness of the sequence induced by

(7) that $\pi(V, E_f) = 0$ for each space V. This implies that E_f is contractible to a point. Upon examination of $\mathfrak{M}f$ one sees that Ωf is a homotopy equivalence; similarly for g.

The latter statement of the proposition follows at once by application of a well-known theorem of J.H.C. Whitehead [17] to the first assertion.

It is readily verified that if the above map $f : X \times X \to X$ has a right homotopy inverse, then X admits a right homotopy inversion. Thus the following is an immediate conclusion.

COROLLARY. (Sugawara [16]) *Let X be an H-space which is a CW-complex and let $X \times X$ be also a CW-complex. Then X has right and left homotopy inversions.*

5.4. In the sequel we shall establish various equivalences related to fiberings. To this end we need the following lemma which is dual to Lemma 8 of D. Puppe ([12], 3.4).

LEMMA 8. *Let $f : X \to Y$ be any map such that $Pf : E_f \to X$ admits a left homotopy inverse $l : X \to E_f$. Let $\psi : X \to Y \times E_f$ be determined by f and l, i.e. $\psi(x) = (f(x), l(x))$ for $x \in X$. Then $\Omega\psi : \Omega X \to \Omega Y \times \Omega E_f$ is a homotopy equivalence.*

Proof. It is clear from the definition of ψ that the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle 1_Y} \\ Y \times E_f & \xrightarrow[\pi]{} & Y \end{array}$$

is commutative. Here π denotes the projection on the first factor. Thus it induces a transformation of $\mathfrak{M}f$ to $\mathfrak{M}\pi$. Observe that the projection $\rho : E_\pi = E_f \times EY \to E_f$ is a homotopy equivalence since EY is contractible.

Consider the map

$$\chi_0 = E(1_Y, \psi) : E_f \to E_\pi = E_f \times EY,$$

whose definition leads to the following calculation

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$$(\rho \circ \chi_0)(x, \beta) = \rho(\psi(x), \beta \cdot e_{f(x)}) \qquad \text{for } x \in X,\ \beta \in EY \text{ with } f(x) = \beta(1),$$
$$= \rho(l(x), \beta \cdot e_{f(x)})$$
$$= l(x)$$
$$= l \circ Pf(x, \beta).$$

Therefore, our assumption implies $\rho \circ \chi_0 \simeq 1_{Ef}$. It follows from the previous remark that χ_0 is a homotopy equivalence. Hence the transformation $\mathfrak{M}f \to \mathfrak{M}\pi$ contains two consecutive maps χ_0 and $\Omega 1_Y = 1_{\Omega Y}$ both of which are homotopy equivalences. Thus, in view of Th. 4, we see that $\Omega\psi : \Omega X \to \Omega(Y \times E_f)$ is also a homotopy equivalence. This completes the proof.

Combining this Lemma with Th. 1 then gives

PROPOSITION 3. (cf. I.M. James and J.H.C. Whitehead [8]) *Let $f : X \to Y$ be a fibering with fibre F, and let F be a retract of X. Let $r : X \to F$ be a retraction. Then the map $\psi : X \to Y \times F$ defined by $\psi(x) = (f(x), r(x))$ induces a homotopy equivalence $\Omega\psi : \Omega X \equiv \Omega(Y \times F)$. In particular, if X and Y are pathwise connected spaces dominated by CW-complexes, then ψ is a homotopy equivalence.*

PROPOSITION 4. *Let $f : X \to Y$ be a fibering which admits a cross-section, and let F be its fibre. Then $\Omega^2 X \equiv \Omega^2(Y \times F)$.*

Proof. Since $\Omega f : \Omega X \to \Omega Y$ also admits a cross-section, the exactness of $(\mathfrak{M}f)_*$ implies Ker $(Pf)_* = 0$. By virtue of (iv) in 5.1, it results that ΩPf admits a left homotopy inverse. Since $\Omega Pf \equiv P^3 Pf = P^4 f = P(P^3 f)$ by Th. 2, we can apply Lemma 8 to $P^3 f : E_{P^2 f} \to E_{Pf}$. Then we have $\Omega E_{P^2 f} \equiv \Omega(E_{Pf} \times E_{P^3 f})$. We note that $E_{P^3 f} \equiv \Omega E_f \equiv \Omega F$, $E_{P^2 f} \equiv \Omega X$, $E_{Pf} \equiv \Omega Y$ on account of Th. 1 and 2. We thus see that $\Omega^2 X \equiv \Omega^2(Y \times F)$.

Following Peterson and Thomas [11], we shall say that a fibering $f : X \to Y$ with fibre F is ***principal*** if there exist maps

$$\mu : F \times X \to X, \qquad h : \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\} \to F$$

satisfying the conditions:

(i) $f \circ \mu(x_1, x_2) = f(x_2)$ for $x_1 \in F$, $x_2 \in X$,

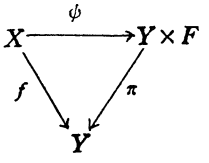
(ii) $\mu | F \times F$ gives an H-structure of F,

(iii) $(x_1, x_2) \to \mu(h(x_1, x_2), x_1)$ is homotopic to $(x_1, x_2) \to x_2$ via a homotopy

which moves image-points along fibers.[5]

Then we can strengthen Prop. 4 for principal fiberings as follows.

PROPOSITION 5. *Suppose that $f : X \to Y$ is a principal fibering with fibre F and that it admits a cross-section. Then there exists a commutative triangle*



such that $\Omega\psi : \Omega X \equiv \Omega(Y \times F)$, where π is the projection.

Proof. We denote by $s : Y \to X$ a cross-section, and let μ, h be as above. We now define $\xi : X \to F$ by

$$\xi(x) = \mu(h(s \circ f(x), x), x_0) \qquad \text{for } x \in X.$$

Since $\xi(x) = \mu(h(x_0, x), x_0)$ for $x \in F$, it follows from the condition (iii) of principal fiberings that $\xi : X \to F$ is a left homotopy inverse of the injection $F \to X$. It follows from Th. 1 that $Pf : E_f \to X$ also admits a left homotopy inverse. This fact enables us to apply Lemma 8. Indeed, if we define $\psi : X \to Y \times F$ by $\psi(x) = (f(x), \xi(x))$, we see that $\Omega\psi$ is a homotopy equivalence, which proves our assertion.

Finally we shall prove

PROPOSITION 6. *Let $f : X \to Y$ be a fibering such that the fibre F is contractible to a point in X. Then $\Omega Y \equiv F \times \Omega X$.*

Proof. Since $F \equiv E_f$ by Th. 1, our assumption implies that $Pf : E_f \to X$ is nullhomotopic. Therefore, we see from Cor. 4.2 that $E_{Pf} \equiv E_f \times \Omega X$. With reference to Th. 1 and 2 we have $E_f \equiv F$ and $E_{Pf} \equiv \Omega Y$, which lead to the desired conclusion.

Upon examination, one sees easily that homotopy equivalences of Prop. 6

$$\eta : F \times \Omega X \to \Omega Y, \qquad \kappa : \Omega Y \to F \times \Omega X$$

are given, using a contraction $\phi_s : F \to X$ such that $\phi_1 = 0$, by

[5] This condition is more restrictive than the one given in [11].

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$$\eta(x,\ \alpha) = \begin{cases} f \circ \Phi_{2s}(x) & \text{for } 0 \leqq s \leqq \frac{1}{2}, \\ f\alpha(2-2s) & \text{for } \frac{1}{2} \leqq s \leqq 1, \end{cases}$$

$$\kappa(\beta) = (\lambda(x_0,\ \beta),\ \alpha'),$$

where $\alpha' \in \Omega X$ is determined by

$$\alpha'(s) = \begin{cases} \Lambda(x_0,\ \beta)(1-2s), & 0 \leqq s \leqq \frac{1}{2}, \\ \Phi_{2s-1}\lambda(x_0,\ \beta), & \frac{1}{2} \leqq s \leqq 1. \end{cases}$$

Here we denote a lifting function and path lifting function for f by λ, Λ respectively.

COROLLARY (cf. Spanier and J.H.C. Whitehead [15]). *Under the same situation as in Proposition* 6, *the fibre F is an H-space.*

COROLLARY. *If $f : X \to Y$ is a fibering with the contractible total X, then $\eta : F \to \Omega Y$ defined by $[\eta(x)](s) = f \circ \Phi_s(x)$ for $x \in F$, $0 \leqq s \leqq 1$ is a (strict) homotopy equivalence, where Φ_s denotes a contraction of X.*

The latter corollary is a variant of a result due to H. Samelson [13].

PART II. MAPPING SEQUENCES AND SUSPENSIONS

6. Preliminaries

6.1. We start by recalling all the basic definitions and results stated in [12] in so far as they are necessary for the application we have in view.

Given a map $f : X \to Y$, let C_f be the mapping cone of f, the space obtained from $CX \cup Y$ by identifying $(x,\ 1)$ with $f(x)$, where CX denotes the cone over X. We denote by S the reduced suspension functor. With these notations, it is known that the sequence

$$\mathfrak{A}f : X \xrightarrow{f} Y \xrightarrow{P'f} C_f \xrightarrow{Qf} SX \xrightarrow{Sf} SY \xrightarrow{SP'f} SC_f \to \cdots$$

has the same properties as $\mathfrak{M}f$, where the maps involved are defined in the following fashion:

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$$P'f(y) = y \in Y \subset C_f \qquad \text{for } y \in Y,$$
$$Qf(x, s) = (x, s) \qquad \text{for } (x, s) \in CX,\ Qf(y) = (x_0, 1) \text{ for } y \in Y.$$

Next, given a transformation from $f : X \to Y$ to $f' : X' \to Y'$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

with a fixed homotopy Φ such that $\Phi_0 = \varphi \circ f$, $\Phi_1 = f' \circ \phi$, we construct the map

$$\chi'_0 = C(\phi, \varphi\ ;\ \Phi) : C_f \to C_{f'}$$

by setting

$$\chi'_0(y) = \varphi(y) \qquad \text{for } y \in Y \subset C_f,$$
$$\chi'_0(x, s) = \begin{cases} (\phi(x), 2s) & \text{for } x \in X,\ 0 \leqq s \leqq \dfrac{1}{2}, \\ \Phi_{2-2s}(x) & \text{for } x \in X,\ \dfrac{1}{2} \leqq s \leqq 1. \end{cases}$$

Then we have (D. Puppe [12], Lemma 7)

LEMMA 9. *If φ and ϕ are both homotopy equivalences, so is χ'_0.*

6.2. Following Eckmann and Hilton [4], we shall say that $f : X \to Y$ is a *cofibering* if it has the homotopy lowering property for all spaces, i.e., if, for $g : X \to Z$, $G : Y \to Z$ with $g = G \circ f$, each homotopy of g can be obtained by composing f with some homotopy of G. The quotient space $Y/f(X)$ is called the *cofibre* of f. Then we shall prove

LEMMA 10. *Let $f : X \to Y$ be a map and let M_f be its mapping cylinder. In order that f be a cofibering, it is necessary and sufficiant that there exist a map $\Lambda' : Y \times I \to M_f$ such that $\Lambda'(f(x), t) = (x, t)$, $\Lambda'(y, 1) = y$ for $x \in X$, $y \in Y$, $0 \leqq t \leqq 1$.*

Proof. Suppose f is a cofibering. We define a map $Y \times 1 \to M_f$ by $(y, 1) \to y$. Then the homotopy $X \times I \to M_f$ given by $(x, t) \to (x, t)$ can be lowered to a homotopy $\Lambda' : Y \times I \to M_f$ which is a desired function.

Conversely, let $G : Y \to Z$, $g_t : X \to Z$ be such that $G(f(x)) = g_1(x)$ for $x \in X$, where $0 \leqq t \leqq 1$. Using the above Λ', we define a homotopy

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$$G' \circ \Lambda' : Y \times I \to Z,$$

where $G' : M_f \to Z$ is given by taking $G'(y) = G(y)$, $G'(x, s) = g_s(x)$ for $y \in Y$, $x \in X$, $0 \leqq s \leqq 1$. This proves the sufficiency.

We see from the above lemma that if f is a cofibering then f is necessarily univalent, so that from now on we consider only inclusion cofiberings. With Λ' above, define $\lambda' : Y \to M_f$ by

$$\lambda'(y) = \Lambda'(y, 0) \qquad \text{for } y \in Y.$$

Then this λ', called a *extension function* for f, has the following properties:

(i) $\lambda'(x) = (x, 0)$ for $x \in X$,

(ii) the composition $r \circ \lambda'$ with the retraction $r : M_f \to Y$ is homotopic to the identity 1_Y of Y via a homotopy which sends X into X.

Consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{P'f} & C_f \\ \| & & \Phi' \downarrow \uparrow \Psi' \\ Y & \xrightarrow[p]{} & Y/X, \end{array}$$

where p is the natural projection, Φ' the map $C_f \to C_f/CX = Y/X$ obtained by pinching CX to a point and Ψ' the map induced by $\lambda' : Y \to M_f$. Then we have

LEMMA 11. *Φ' and Ψ' are mutually inverse homotopy equivalences and, furthermore, the above diagram is commutative up to homotopy.* (Puppe [12])

Proof. It follows from (ii) that $\Phi' \circ \Psi' \simeq 1$. On the other hand, the homotopy given by

$$\begin{aligned} (x, s) &\to \Lambda'(x, st), && \text{for } 0 \leqq t \leqq 1,\ (x, s) \in X \times I, \\ y &\to \Lambda'(y, t) && \text{for } 0 \leqq t \leqq 1,\ \in Y \end{aligned}$$

yields a homotopy connecting $\Psi' \circ \Phi'$ with 1_{C_f}. It is obvious that $p = \Phi' \circ P'f$, so our assertion is proved.

The following is easily read from the proof of Puppe's Lemma 6 [12].

LEMMA 12. *$P'f : Y \to C_f$ is a cofibering, whose λ' and Λ' are given by*

$$\lambda'(x, s) = \begin{cases} (x, 2s) \in C_f,\ 0 \leqq s \leqq \dfrac{1}{2},\ (x, s) \in CX, \\ (f(x), 2-2s) \in Y \times I,\ \dfrac{1}{2} \leqq s \leqq 1,\ (x, s) \in CX, \end{cases}$$

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$$\lambda'(y) = (y, 0) \in Y \times I,\ y \in Y \subset C_f,$$

$$\Lambda'((x, s), t) = \begin{cases} \left(x, \dfrac{2s}{1+t}\right), & 0 \leqq s \leqq \dfrac{1+t}{2} \\ (f(x), 2-2s+t), & \dfrac{1+t}{2} \leqq s \leqq 1. \end{cases}$$

$$\Lambda'(y, t) = (y, t).$$

Finally we state the following well known lemma which makes it possible to convert any map into a map of simpler type.

LEMMA 13 ([3] or [1]). *Any map is equivalent to a fibering (or a cofibering).*

Proof. Given any map $f : X \to Y$, let M_f be the mapping cylinder of f and we set

$$Z_f = \{(x, \beta) \mid x \in X,\ \beta \in E_{Y,Y}(Y),\ f(x) = \beta(1)\}.$$

Let $p : Z_f \to Y$, $i : X \to M_f$ be defined by setting $p(x, \beta) = \beta(0)$, $i(x) = (x, 0)$. Then we see at once that p is a fibering with fibre E_f and that i is a cofibering with cofibre C_f, both of which are clearly equivalent to f respectively.

6.3. Let $f : X \to Y$ be a map. We shall define left operations (cf. [12], 4.3)

$$\mu : \Omega Y \times E_f \to E_f, \qquad \mu' : C_f \to SX \vee C_f$$

as follows.

$$\mu(\omega, (x, \beta)) = (x, \omega \cdot \beta) \qquad \text{for } \omega \in \Omega Y,\ x \in X,\ \beta \in EY,$$

$$\mu'(y) = y \qquad \text{for } y \in Y \subset C_f,$$

$$\mu'(x, s) = \begin{cases} (x, 2s) \in SX, & s \leqq \dfrac{1}{2} \\ (x, 2s-1) \in CX, & s \geqq \dfrac{1}{2} \end{cases} \qquad \text{for } (x, s) \in CX \subset C_f$$

These induce natural H- and H'-structures (cf. [4]) $\Omega Y \times \Omega Y \to \Omega Y$, $SX \to SX \vee SX$ which are also denoted by μ, μ'. We have several properties about them. For example,

a) The diagrams

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$$\begin{array}{ccc} \Omega Y \times E_f & \xrightarrow{\mu} & E_f \\ {\scriptstyle 1\times Pf}\downarrow & & \downarrow{\scriptstyle Pf} \\ \Omega Y \times X & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & SX \vee Y \\ {\scriptstyle P'f}\downarrow & & \downarrow{\scriptstyle 1\vee P'f} \\ C_f & \xrightarrow[\mu']{} & SX \vee C_f \end{array}$$

are commutative

b)
$$\begin{array}{ccc} \Omega Y \times \Omega Y \times E_f & \xrightarrow{1\times\mu} & \Omega Y \times E_f \\ {\scriptstyle \mu\times 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ \Omega Y \times E_f & \xrightarrow[\mu]{} & E_f \end{array} \qquad \begin{array}{ccc} C_f & \xrightarrow{\mu'} & SX \vee C_f \\ {\scriptstyle \mu'}\downarrow & & \downarrow{\scriptstyle \mu'\vee 1} \\ SX \vee C_f & \xrightarrow[1\vee\mu']{} & SX \vee SX \vee C_f \end{array}$$

are homotopy-commutative.

Moreover E_f has a principal structure as mentioned in 5.4. Let E_f^* be the reciprocal image of the diagonal under $Pf \times Pf : E_f \times E_f \to X \times X$. If we define $h : E_f^* \to \Omega Y$ by taking

$$h((x, \beta), (x, \beta')) = \beta' \cdot \beta^{-1},$$

we obtain a homotopy commutative diagram

$$\begin{array}{ccc} E_f^* & \xrightarrow{h\times p_1} & \Omega Y \times E_f \\ & {\scriptstyle p_2}\searrow & \downarrow{\scriptstyle \mu} \\ & & E_f \end{array}$$

where $p_i : E_f^* \to E_f$ are defined by $p_i(z_1, z_2) = z_i$, $i = 1, 2$, $z_i \in E_f$. This homotopy $\mu \circ (h \times p_1) \simeq p_2$ can be chosen so as to move image-points along fibres.

7. Connecting diagram[6]

7.1. We shall call a triple $X \xrightarrow{f} Y \xrightarrow{g} Z$ *nullhomotopic* if and only if the composition $g \circ f$ is homotopic to the constant. Given such a triple with a definite homotopy $H_t : X \to Z$ such that $H = g \circ f$, $H_1 = 0$, we form the *connecting diagram of* $X \xrightarrow{f} Y \xrightarrow{g} Z$ which is written as follows.

$$\begin{array}{ccccccccccccccc} \cdots & \xrightarrow{\Omega f} & \Omega Y & \xrightarrow{If} & E_f & \xrightarrow{Pf} & X & \xrightarrow{f} & Y & \xrightarrow{P'f} & C_f & \xrightarrow{Qf} & SX & \xrightarrow{S-f} & SY & \xrightarrow{SP'f} & \cdots \\ & & \| & & {\scriptstyle \eta_{f,g}}\downarrow & & {\scriptstyle \xi_{f,g}}\downarrow & & \| & & \downarrow{\scriptstyle \xi'_{f,g}} & & \downarrow{\scriptstyle \eta'_{f,g}} & & \| & & \\ \cdots & \xrightarrow[\Omega Pg]{} & \Omega Y & \xrightarrow[\Omega-g]{} & \Omega Z & \xrightarrow[Ig]{} & E_g & \xrightarrow[Pg]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[P'g]{} & C_g & \xrightarrow[Qg]{} & SY & \xrightarrow[Sg]{} & \cdots \end{array}$$

[6] This construction was inspired from a discussion in [5].

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in which the maps exhibited are defined by setting

$$\xi_{f,g}(x) = (f(x),\ \gamma),\ \gamma(s) = H_{1-s}(x) \qquad \text{for } x \in X,\ 0 \leqq s \leqq 1,$$

$$\eta_{f,g}(x,\ \beta) = \begin{cases} H_{1-2s}(x) & \text{for } 0 \leqq s \leqq \frac{1}{2},\ (x,\ \beta) \in E_f, \\ g\beta(2-2s) & \text{for } \frac{1}{2} \leqq s \leqq 1,\ (x,\ \beta) \in E_f, \end{cases}$$

$$\xi'_{f,g}(y) = g(y),\ \xi'_{f,g}(x,\ s) = H_{1-s}(x) \qquad \text{for } y \in Y,\ (x,\ s) \in CX,$$

$$\eta'_{f,g}(x,\ s) = \begin{cases} H_{1-2s}(x) \in Z & \text{for } 0 \leqq s \leqq \frac{1}{2}, \\ (f(x),\ 2-2s) \in CY & \text{for } \frac{1}{2} \leqq s \leqq 1. \end{cases}$$

Here, to simplify notations we omit all mention of a nullhomotopy H_t. With these definitions we assert

THEOREM 5. *The connecting diagram above is commutative up to homotopy.*

Proof. Let $G_\tau (0 \leqq \tau \leqq 1)$ be a homotopy $E_f \to E_g$ given by

$$G_\tau(x,\ \beta) = (\beta(\tau),\ \gamma_\tau) \qquad \text{for } x \in X,\ \beta \in EY,\ f(x) = \beta(1),$$

where

$$\gamma_\tau(s) = \begin{cases} H_{(1-2s+\tau)(1+\tau)^{-1}}(x), & 0 \leqq s \leqq 1+\tau/2 \\ g\beta(2-2s+\tau) & 1+\tau/2 \leqq s \leqq 1. \end{cases}$$

Since $G_0(x,\ \beta) = (y_0,\ \eta_{f,g}(x,\ \beta))$, $G_1(x,\ \beta) = \xi_{f,g}(x)$, it follows that $Ig \circ \eta_{f,g} \simeq \xi_{f,g} \circ Pf$.

Similarly, if we consider a homotopy $G'_\tau : C_f \to C_g$ $(0 \leqq \tau \leqq 1)$ defined by

$$G'_\tau(y) = (y,\ \tau)$$

$$G'_\tau(x,\ s) = \begin{cases} H_{(1-2s+\tau)(1+\tau)^{-1}}(x), & 0 \leqq s \leqq (1+\tau)/2, \\ (f(x),\ 2-2s+\tau), & (1+\tau)/2 \leqq s \leqq 1, \end{cases}$$

then we see that $\eta'_{f,g} \circ Qf \simeq P'g \circ \xi'_{f,g}$. The other verifications are straightforward.

$\xi_{f,g}$, $\xi'_{f,g}$, etc. will be called *connecting maps* in the sequel.

7.2. Next we shall determine to what extent the connecting maps are altered by the choice of nullhomotopies of $g \circ f$ or by alterations of f, g within their homotopy classes.

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Let $\bar{\xi}_{f,g}$, $\bar{\xi}'_{f,g}$, etc. be the connecting maps which are constructed by using another nullhomotopy $\bar{H}_t$ of $g \circ f$. It is clear that the correspondence

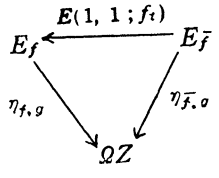
$$(x,\ s) \to \begin{cases} \bar{H}_{1-2s}(x), & 0 \leqq s \leqq \frac{1}{2}, \\ H_{2s-1}(x), & \frac{1}{2} \leqq s \leqq 1. \end{cases}$$

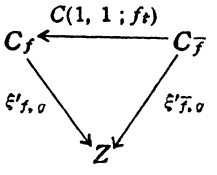
determines two maps $\kappa(\bar{H},\ H) : X \to \Omega Z$, $\kappa'(\bar{H},\ H) : SX \to Z$. It is readily verified that

$$\mu \circ \{\kappa(\bar{H},\ H) \times \bar{\xi}_{f,g}\} \simeq \bar{\xi}_{f,g}, \qquad \mu \circ \{\kappa(\bar{H},\ H) \times \eta_{f,g}\} \simeq \bar{\eta}_{f,g},$$
$$\{\kappa'(\bar{H},\ H) \vee \bar{\xi}'_{f,g}\} \circ \mu' \simeq \bar{\xi}'_{f,g}, \qquad \{\kappa'(\bar{H},\ H) \vee \eta'_{f,g}\} \circ \mu' \simeq \bar{\eta}'_{f,g},$$

where μ and μ' are left operations in E_g, C_g as defined in the previous section.

Secondly, when $\bar{f} \simeq f$, we shall construct $\bar{\xi}_{\bar{f},g}$, etc. by using the nullhomotopy $g \circ f_t$ followed by H_s, where f_t is a homotopy connecting $\bar{f}$ with f. Then we see at once that $\bar{\xi}_{\bar{f},g} \simeq \xi_{f,g}$, $\eta'_{\bar{f},g} \simeq \eta'_{f,g}$ and that the diagrams





are homotopy commutative, where each horizontal map is a homotopy equivalence (cf. Lemmas 6,9). Similarly for $\bar{g} \simeq g$. Thus we have established

THEOREM 6. *The effect of changing a nullhomotopy of $g \circ f$ upon connecting maps is described in terms of the left operation of some element in $\pi(X,\ \Omega Z)$ or $\pi(SX,\ Z)$. When f and g are altered with their homotopy classes, the resulting connecting maps are equivalent to the initial ones.*

7.3. In case $g \circ f$ is just a constant, we may simplify the definition of connecting maps to some degree, i.e., in that case we set

$$\xi_{f,g}(x) = (f(x),\ e_{z_0}), \quad \eta_{f,g}(x,\ \beta) = (g\beta)^{-1},$$
$$\xi'_{f,g}(y) = g(y), \quad \xi'_{f,g}(x,\ s) = z_0, \quad \eta'_{f,g}(x,\ s) = (f(x),\ 1-s).$$

We shall now prove a result corresponding to excision theorems due to Eckmann and Hilton [4]

THEOREM 7. i) *If g is a fibering with fibre X, then $\xi_{f,g}$ and $\eta_{f,g}$ are homo-*

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topy equivalences. ii) *Let f be a cofibering with cofibre Z. Then we have homotopy equivalences $\xi'_{f,g}$ and $\eta'_{f,g}$.*

Proof. The first halves of each assertion are just restatements of Th. 1 and Lemma 11. Therefore it remains to prove that $\eta_{f,g}$ and $\eta'_{f,g}$ are equivalences.

That $\eta_{f,g}$ is homotopic to the composite map

$$E_f \xrightarrow{E(1,\,\xi_{f,g})} E_{Pg} \xrightarrow{Ng} \Omega Z,$$

where Ng is a map as defined in 2.2, follows from the next computation

$$\begin{aligned} Ng \circ E(1,\ \xi_{f,g})\,(x,\ \beta) &= Ng(\xi_{f,g}(x),\ \beta) \\ &= Ng((x,\ e_{z_0}),\ \beta) \\ &= Ng(e_{z_0},\ \beta) = e_{z_0} \cdot (g\beta)^{-1}. \end{aligned}$$

Since, by 2.2 and Lemma 6, $E(1,\ \xi_{f,g})$ and Ng are homotopy equivalences, so is $\eta_{f,g}$, which proves (i).

As regards $\eta'_{f,g}$, consider the composite map

$$SX \xrightarrow{\lambda'} C_{P'f} \xrightarrow{C(1,\,\xi'_{f,g})} C_g,$$

where λ' is the map determined by $\lambda' : C_f \to M_{P'f}$ in Lemma 12. It is easy to show

$$C(1,\ \xi'_{f,g}) \circ \lambda'(x,\ s) = \begin{cases} z_0 \in Z, & 0 \leqq s \leqq \dfrac{1}{2}, \\ (x,\ 2-2s) \in CY, & \dfrac{1}{2} \leqq s \leqq 1, \end{cases}$$

so that it is homotopic to $\eta'_{f,g}$. One sees from Lemmas 9 and 11 that $\eta'_{f,g}$ is a equivalence. This concludes our proof.

8. Suspensions

8.1. We shall give here a definition of suspensions for an arbitrary map $f : X \to Y$ which is substantially a generalization of usual ones, as mentioned in Remarks of 8.1 and 8.2

Before doing so we make a convention. $\pi(SX,\ Y)$ and $\pi(X,\ \Omega Y)$ are in $1-1$ correspondence with each other by the rule $\{\tilde{f}(x)\}(s) = f(x,\ s)$ for $\tilde{f}$: $X \to \Omega Y$, $f : SX \to Y$. In this case we use the notation $[\tilde{f}] \leftrightarrow [f]$.

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Let $v : C_f \to Z$ be given, then we set $u = v \circ P'f$. Note that $X \xrightarrow{f} Y \xrightarrow{u} Z$ and $E_f \xrightarrow{Pf} X \xrightarrow{f} Y$ are nullhomotopic triples. Thus we may consider the diagram

$$\begin{array}{ccccccc}
E_f & \xrightarrow{Pf} & X & \xrightarrow{P'Pf} & C_{Pf} & \xrightarrow{QPf} & SE_f \\
\| & & \| & & \downarrow{\scriptstyle \xi'_{Pf,f}} & & \downarrow{\scriptstyle \eta'_{Pf,f}} \\
E_f & \xrightarrow{Pf} & X & \xrightarrow{f} & Y & \xrightarrow{P'f} & C_f \\
{\scriptstyle \eta_{f,u}}\downarrow & & \downarrow{\scriptstyle \xi_{f,u}} & & \| & & \downarrow{\scriptstyle v} \\
\Omega Z & \xrightarrow[Iu]{} & Eu & \xrightarrow[Pu]{} & Y & \xrightarrow[u]{} & Z.
\end{array}$$

We define the *Eilenberg-MacLane suspension* for f

$$\sigma^* : \pi(C_f, Z) \to \pi(SE_f, Z)$$

by taking $\sigma^*([v]) = [v \circ \eta'_{Pf,f}]$. We also define

$$\bar{\sigma}^* : \pi(C_f, Z) \to \pi(E_f, \Omega Z)$$

by $\bar{\sigma}^*([v]) = [\eta_{f,u}]$. $f = Pu \circ \xi_{f,u}$ may be called the *Postnikov factorization of f with respect to v.*

Since we have

$$v \circ \eta'_{Pf,f}((x, \beta), s) = \begin{cases} u\beta(2s), & 0 \leqq s \leqq \frac{1}{2}, \\ v(x, 2-2s), & \frac{1}{2} \leqq s \leqq 1, \end{cases}$$

$$\{\eta_{f,u}(x, \beta)\}(s) = \begin{cases} v(x, 2s), & 0 \leqq s \leqq \frac{1}{2}, \\ u\beta(2-2s), & \frac{1}{2} \leqq s \leqq 1, \end{cases}$$

we obtain

LEMMA 14. $\bar{\sigma}^*([v]) \leftrightarrow -\sigma^*([v])$.

Remark. In case f is a fibering with fibre F, we shall call

$$(\eta'_{Pf,f} \circ S\Phi)^* : \pi(C_f, Z) \to \pi(SF, Z)$$

the *suspension of the fibering* f, where $\Phi : F \to E_f$ is an equivalence given in Th. 1. In particular, let f be the Serre fibering $EY \to Y$ defined by $\beta \to \beta(1)$ and let $u : Y \to Z$ be given; then we set $v(y) = u(y)$, $v(\beta, s) = u\beta(s)$ for $y \in Y$,

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$\beta \in EY$, $0 \leqq s \leqq 1$, obtaining a map $v : C_f \to Z$. We see then that $\eta_{f,u} \circ \Phi \simeq \Omega u : \Omega Y \to \Omega Z$, where $\Phi : \Omega Y \to E_f$ is an equivalence.

8.2. Given a map $v' : Z \to E_f$, we set $u' = Pf \circ v'$ and we proceed in a dual fashion. Consider the diagram

$$\begin{array}{ccccccc}
Z & \xrightarrow{u'} & X & \xrightarrow{P'u'} & C_{u'} & \xrightarrow{Qu'} & SZ \\
{\scriptstyle v'}\downarrow & & \| & & \downarrow{\scriptstyle \xi'_{u',f}} & & \downarrow{\scriptstyle \eta'_{u',f}} \\
E_f & \xrightarrow{Pf} & X & \xrightarrow{f} & Y & \xrightarrow{P'f} & C_f \\
{\scriptstyle \eta_{f,P'f}}\downarrow & & \downarrow{\scriptstyle \xi_{f,P'f}} & & \| & & \| \\
\Omega C_f & \xrightarrow{-1P'f} & E_{P'f} & \xrightarrow{-PP'f} & Y & \xrightarrow{-P'f} & C_f
\end{array}$$

$f = \xi'_{u',f} \circ P'u'$ may be said to be the *Moore factorization of f with respect to v'*.

We define the *Freudenthal suspension* for f

$$\sigma_* : \pi(Z, E_f) \to \pi(Z, \Omega C_f)$$

by setting $\sigma_*([v']) = [\eta_{f,P'f} \circ v']$. We define also

$$\bar{\sigma}_* : \pi(Z, E_f) \to \pi(SZ, C_f)$$

by $\bar{\sigma}_*([v']) = [\eta'_{u',f}]$. As in the case of the Eilenberg-MacLane suspension, we may obtain

LEMMA 14′. $\bar{\sigma}_*([v']) \leftrightarrow -\sigma_*([v'])$.

Remark. In case in which f is a cofibering with cofibre Y/X, then we have a natural equivalence $\Phi' : C_f \to Y/X$. (Lemma 11). We say that

$$(\Omega\Phi' \circ \eta_{f,P'f})_* : \pi(Z, E_f) \to \pi(Z, \Omega(Y/X))$$

is the *suspension of the cofibering f*. In particular consider $f : X \to CX$ which is the injection. Given $u' : Z \to X$, we define $v' : Z \to E_f$ by $v'(z) = (u'(z), \beta')$ where $\beta' : I \to CX$ is defined by $\beta'(s) = (u'(z), s)$, $0 \leqq s \leqq 1$. It follows at once that $\Phi' \circ \eta'_{u',f} \simeq Su'$, where $\Phi' : C_f \to SX$ is a natural equivalence.

8.3. We shall now establish naturality of suspensions. Let

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \varphi} \\
X' & \xrightarrow[f']{} & Y'
\end{array}$$

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be a transformation as in 6.1. It will be reasonable to consider the diagrams

$$(7)\qquad \begin{array}{ccc} SE_f & \xrightarrow{SE(\varphi,\,\psi\,;\,\Phi)} & SE_{f'} \\ {\scriptstyle \eta'_{Pf,\,f}}\downarrow & & \downarrow{\scriptstyle \eta'_{Pf',\,f'}} \\ C_f & \xrightarrow[C(\psi,\,\varphi\,;\,\Phi)]{} & C_{f'} \end{array} \qquad \begin{array}{ccc} E_f & \xrightarrow{E(\varphi,\,\psi\,;\,\Phi)} & E_{f'} \\ {\scriptstyle \eta_{f,\,P'f}}\downarrow & & \downarrow{\scriptstyle \eta_{f',\,P'f'}} \\ \Omega C_f & \xrightarrow[\Omega C(\psi,\,\varphi\,;\,\Phi)]{} & \Omega C_{f'} \end{array}$$

The following lemma is readily deduced from the definitions involved.

LEMMA 15. *The above diagrams* (7) *are homotopy commutative.*

Next, given maps $\bar{v} : C_{f'} \to Z$, $\tilde{v} : Z \to E_f$, we set

$$v = \bar{v} \circ C(\psi,\, \varphi\,;\, \Phi), \quad \bar{u} = \bar{v} \circ P'f', \quad u = v \circ P'f,$$
$$v' = E(\varphi,\, \psi\,;\, \Phi) \circ \tilde{v}, \quad \tilde{u} = Pf \circ \tilde{v}, \quad u' = Pf' \circ v'.$$

Consider the diagrams

$$(8)\qquad \begin{array}{ccc} E_f & \xrightarrow{E(\varphi,\,\psi\,;\,\Phi)} & E_{f'} \\ & {\scriptstyle \eta_{f,\,u}}\searrow \quad \swarrow{\scriptstyle \eta_{f',\,\bar{u}}} & \\ & \Omega Z & \end{array} \qquad \begin{array}{ccc} & SZ & \\ {\scriptstyle \eta'_{\tilde{u},\,f}}\swarrow & & \searrow{\scriptstyle \eta'_{u',\,f'}} \\ C_f & \xrightarrow[C(\psi,\,\varphi\,;\,\Phi)]{} & \end{array}$$

Observing that $u = \bar{u} \circ \varphi$, $u' = \psi \circ \tilde{u}$, we easily verify

LEMMA 16. *The diagrams* (8) *are homotopy commutative.*

9. Suspension theorems

9.1. We are now in a position to prove an important property concerning suspensions, which is an extension of usual suspension theorems. In the rest of this paper we assume that the spaces to be considered are 1-connected.

THEOREM 8. *Let $f : X \to Y$ be any map and suppose Y is r-connected, E_f s-connected. Then*

$$\sigma^* : H^q(C_f) \to H^q(SE_f) = H^{q-1}(E_f)$$

is an isomorphism for $q \leqq r+s+1$ and a monomorphism for $q = r+s+2$.

Proof. In view of Lemmas 13, 15, 6, 9, we may restrict our attention to the case in which $f : X \to Y$ is an inclusion cofibering. In this case

$$E_f = E_{y_0,\,X}(Y), \qquad C_f = CX \cup Y,$$

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and $\eta'_{Pf,f}$ is given by

$$\eta'_{Pf,f}(\beta, s) = \begin{cases} \beta(2s) \in Y, & 0 \leqq s \leqq \frac{1}{2}, \ \beta \in E_f, \\ (\beta(1), 2-2s) \in CX, & \frac{1}{2} \leqq s \leqq 1, \ \beta \in E_f. \end{cases}$$

We shall form a diagram

$$\begin{array}{ccccccc} & & Qi & & & & \\ SE_f & \longleftarrow & C_i = EY \cup CE_f & \longrightarrow & (C_i, CE_f) & \longleftarrow & (EY, E_f) \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \eta'} \ \downarrow{\scriptstyle \eta''} & & \downarrow{\scriptstyle \eta'''} & & \downarrow{\scriptstyle p} \\ SE_f & & & & & & \\ {\scriptstyle \eta'_{Pf,f}} \searrow & & & & & & \\ & & C_f & \longrightarrow & (C_f, CX) & \longleftarrow & (Y, X), \end{array}$$

where ω is the involution determined by inversion of suspension parameter; $i : E_f \to EY$ the inclusion; η' the composite $\eta'_{Pf,f} \circ \omega \circ Qi$; η'' the map given by $\eta''(\beta) = \beta(1)$ for $\beta \in EY$, $\eta''(\beta, s) = (\beta(1), s)$ for $(\beta, s) \in CE_f$; η''' is induced by η'; p the map defined by $p(\beta) = \beta(1)$ for $\beta \in EY$; other horizontal maps are all inclusions.

Define $G_t (0 \leqq t \leqq 1)$ to be the homotopy such that

$$G_t(\beta) = \beta(t) \qquad \text{for } \beta \in EY,$$

$$G_t(\beta, s) = \begin{cases} \left(\beta(1), \frac{2s}{1+t}\right), & 0 \leqq s \leqq \frac{1+t}{2}, \quad \beta \in E_f, \\ \beta(2-2s+t), & \frac{1+t}{2} \leqq s \leqq 1, \quad \beta \in E_f. \end{cases}$$

Since $G_0 = \eta'$, $G_1 = \eta''$, we have $\eta' \simeq \eta''$. We see at once that the diagram above is commutative.

By passing to cohomolgy it is clear that all horizontal maps, Qi and ω induce isomorphisms. Note that p is a fibre map. Since ΩY is $(r-1)$-connected and (Y, X) is $(s+1)$-connected, it follows from a well known theorem ([14], Th. 1.B) that $p^* : H^q(Y, X) \to H^q(EY, E_f)$ is isomorphic for $q \leqq r+s+1$ and monomorphic for $q = r+s+2$, so that the same is true for $\sigma^* = (\eta'_{Pf,f})^*$, which is what we wanted to prove.

THEOREM 8′. *Let $f : X \to Y$ be any map, and let X and E_f be r- and s-connected respectively. Then*

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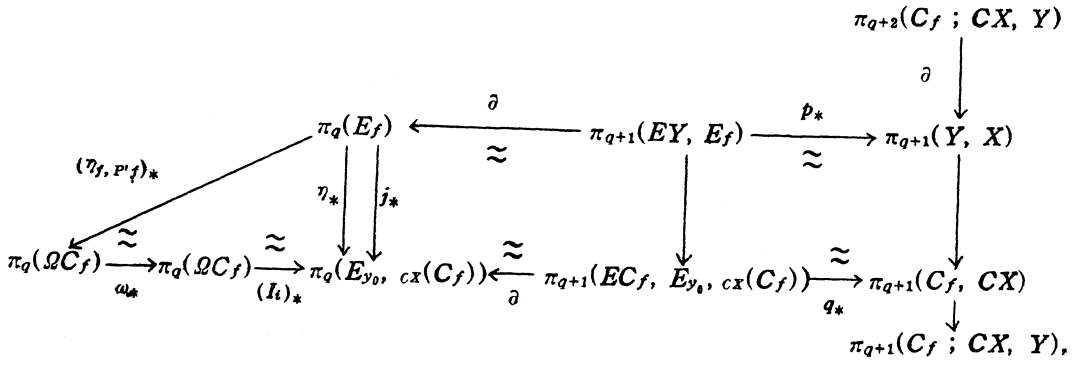
$$\sigma_* : \pi_q(E_f) \to \pi_q(\Omega C_f) = \pi_{q+1}(C_f)$$

is an isomorphism for $q \leqq r+s$ *and an epimorphism for* $q = r+s+1$.

Proof. As before we may assume that f is an inclusion. We consider the diagram



where ω is the involution induced by inverting loops; $\eta = (Ii) \circ \omega \circ \eta_{f,Pf}$, i: $CX \to C_f$ the inclusion; p and q are fibre maps defined by taking the terminal point of paths; other vertical maps are induced by injections.

By proceeding as in the previous theorem, we see that η and j are homotopic to each other. Since (CX, X) is $(r+1)$-connected and (Y, X) $(s+1)$-connected, we have

$$\pi_{q+2}(C_f ; CX, Y) = 0 \qquad \text{for } q \leqq r+s$$

by the triad theorem [2]. Hence it follows that $\sigma_* = (\eta_{f,Pf})_*$ is isomorphic for $q \leqq r+s$ and onto for $q = r+s+1$. This completes the proof.

10. Postnikov decomposition

Let $f : X \to Y$ be any map such that $\pi_q(E_f) = 0$ for $q \leqq n-1$. We abbreviate $\pi_q(E_f)$ by π_q. By a convention made in 9.1 we have $\pi_1(Y) = 0$, so that Th. 8 asserts that $\bar{\sigma}^* : H^q(C_f) \to H^{q-1}(E_f)$ is isomorphic for $q \leqq n+1$. Therefore we can find a map

$$v : C_f \to K(\pi_n, n+1)$$

such that $\bar{\sigma}^*([v]) = [\eta_{f,u}] \in \pi(E_f, \Omega K(\pi_n, n+1)) = H^n(E_f) \approx \mathrm{Hom}(H_n(E_f), \pi_n)$ is the inverse of Hurewicz isomorphism, in which we set $u = v \circ Pf$.

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We have the maps

$$\begin{array}{ccccccc} E_f & \xrightarrow{Pf} & X & \xrightarrow{f} & Y & \xrightarrow{P'f} & C_f \\ \downarrow{\scriptstyle \eta_{f,u}} & & \downarrow{\scriptstyle \xi_{f,u}} & & \| & & \downarrow{\scriptstyle v} \\ \Omega K(\pi_n, n+1) & \xrightarrow[Iu]{} & E_u & \xrightarrow[Pu]{} & Y & \xrightarrow[u]{} & K(\pi_n, n+1). \end{array}$$

Since the relative mapping sequence for the triple $f = Pu \circ \xi_{f,u}$ is written

$$\begin{array}{ccccccc} \Omega E_f & \xrightarrow{\Omega k} & \Omega E_{Pu} & \longrightarrow & E_{\xi_{f,u}} & \longrightarrow & E_f \xrightarrow{k} E_{Pu} \\ & & \| & & & & \| \\ & & K(\pi_n, n-1) & & & & K(\pi_n, n), \end{array}$$

it follows from Th. 3 that $\pi_q(E_{\xi_{f,u}}) \approx \pi_q(E_f)$ for $q \neq n, n-1$. A simple computation shows that $\eta_{f,u}$ coincides with the composite

$$E_f \xrightarrow{k} E_{Pu} \xrightarrow{Nu} \Omega K(\pi_n, n+1),$$

where N_u is an equivalence constructed in 2.2, and thus we have

$$k_* : \pi_n(E_f) \approx \pi_n(E_{Pu}), \quad (\Omega k)_* : \pi_{n-1}(\Omega E_f) \approx \pi_{n-1}(\Omega E_{Pu}).$$

These results show that

$$(9) \qquad \begin{cases} \pi_q(E_{\xi_{f,u}}) = 0 & \text{for } q \leqq n \\ \pi_q(E_{\xi_{f,u}}) \approx \pi_q(E_f) & \text{for } q \geqq n+1 \end{cases}$$

On the other hand, we obtain quickly from $(\mathfrak{M}\xi_{f,u})_*$ that

$$(10) \qquad \begin{cases} (\xi_{f,u})_* : \pi_q(X) \approx \pi_q(E_u) & \text{for } q \leqq n, \\ (\xi_{f,u})_* : \pi_{n+1}(X) \to \pi_{n+1}(E_u) & \text{onto.} \end{cases}$$

Furthermore we have

$$(11) \qquad (Pu)_* : \pi_q(E_u) \approx \pi_q(Y) \qquad \text{for } q \neq n+1, n.$$

This construction is essentially due to Eckmann and Hilton [6]. They call it the *homotopy decomposition* of f.

In case f is a fibering then, by Th. 1, E_f is equivalent to the fibre of f, and thus we see from (9) that we have the *Moore-Postnikov system for* f ([1], p. 911). In particular, when Y is a point we obtain the Postnikov system for the space X.

It X is a point then $E_f = \Omega Y$, so that $Pu : E_u \to Y$ is a fibering in which

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the fibre is $K(\pi_{n+1}(Y), n)$ and moreover Pu is $(n+1)$-connective by (10), (11). This is nothing but the Cartan-Serre-Whitehead technique for killing homotopy groups [3].

11. Functional operations

11.1. In this section we try to make an explicit deduction of the formulas stated in [10], without making use of the universal example.

Suppose given a quadruple $L \xrightarrow{f} K \xrightarrow{h} X \xrightarrow{\theta} Y$ such that $h \circ f \simeq 0$, $\theta \circ h \simeq 0$ which are realized by homotopies H_t, G_t respectively. Using the diagram

$$\begin{array}{ccccccc} & & L & \xrightarrow{f} & K & & \\ & & \downarrow{\scriptstyle \xi_{f,h}} & & \| & & \\ \Omega X & \xrightarrow{Ih} & E_h & \xrightarrow{Ph} & K & \xrightarrow{h} & X \\ \| & & \downarrow{\scriptstyle \eta_{h,\theta}} & & \downarrow{\scriptstyle \xi_{h,\theta}} & & \| \\ \Omega X & \xrightarrow[\Omega-\theta]{} & \Omega Y & \xrightarrow[I\theta]{} & E_\theta & \xrightarrow[P\theta]{} & X \xrightarrow[\theta]{} Y \end{array}$$

we shall define the *functional θ-operation* $\bar{\theta}_f$ by

$$\bar{\theta}_f(h) = [\eta_{h,\theta} \circ \xi_{f,h}].$$

This must be considered as an element of the set of equivalence classes by left operation of $f^*\pi(K, \Omega Y)$ and right operation of $(\Omega\theta)_*\pi(L, \Omega X)$.

Alternatively we consider the maps

$$\begin{array}{ccccccccc} L & \xrightarrow{f} & K & \xrightarrow{P'f} & C_f & \xrightarrow{Qf} & SL & \xrightarrow{S-f} & SK \\ & & \| & & \downarrow{\scriptstyle \xi'_{f,h}} & & \downarrow{\scriptstyle \eta'_{f,h}} & & \| \\ & & K & \xrightarrow{h} & X & \xrightarrow{P'h} & C_h & \xrightarrow{Qh} & SK \\ & & & & \| & & \downarrow{\scriptstyle \xi'_{h,\theta}} & & \\ & & & & X & \xrightarrow[\theta]{} & Y & & \end{array}$$

and we set

$$\theta_f(h) = [\xi'_{h,\theta} \circ \eta'_{f,h}],$$

which is regared as an element of $\pi(SL, Y)$ classified by right operation of $(Sf)^*\pi(SK, Y)$ and left operation of $\theta^*\pi(SL, X)$.

The following gives a relationship between the above two definitions.

PROPOSITION 7. $\bar{\theta}_f(h) \leftrightarrow -\theta_f(h)$ (cf. Th. 5.1 in [10])

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Proof. By a direct calculation based on the definition of connecting maps we have

$$\{\eta_{h,\theta} \circ \xi_{f,h}(x)\}(s) = \begin{cases} G_{1-2s}(f(x)), & 0 \leqq s \leqq \frac{1}{2}, \quad x \in L, \\ \theta H_{2s-1}(x), & \frac{1}{2} \leqq s \leqq 1, \quad x \in L, \end{cases}$$

$$(\xi'_{h,\theta} \circ \eta'_{f,h})(x, s) = \begin{cases} \theta H_{1-2s}(x), & 0 \leqq s \leqq \frac{1}{2}, \quad x \in L, \\ G_{2s-1}(f(x)), & \frac{1}{2} \leqq s \leqq 1, \quad x \in L. \end{cases}$$

Our assettion follows from these.

11.2. Let f, h, θ be as above and let $\phi : E \to Z$ be given; then we set $\theta' = \phi \circ (I\theta) : \Omega Y \to Z$. Note that $\theta' \circ \Omega\theta \simeq 0$ by the exactness of $(\mathfrak{M}\theta)_*$.

Following Peterson [10] we shall define the *secondary θ-operation* Φ_θ determined by ϕ as follows:

$$\Phi_\theta(h) = [\phi \circ \xi_{h,\theta}],$$

which is regarded as an element of $\pi(K, Z)$ classified by ϕ-image of left operation of $\pi(K, \Omega Y)$. Thus, in order to describe Φ_θ completely, we need more explicit information about $\phi \circ \mu$, where $\mu : \Omega Y \times E_\theta \to E_\theta$ is left operation.

From the above definitions we see that

$$\theta' \circ \eta_{h,\theta} \circ \xi_{f,h} = \phi \circ (I\theta) \circ \eta_{h,\theta} \circ \xi_{f,h} = \phi \circ \xi_{h,\theta} \circ f.$$

Therefore we have proved

PROPOSITION 8. (cf. Th. 7.1 in [10]) $\theta'(\bar{\theta}_f(h)) = f^* \Phi_\theta(h) \bmod \theta' f^* \pi(K, \Omega Y) + f^* \phi$ (*left operation of* $\pi(K, \Omega Y)$).

11.3. Next, let $f : L \to K$ be a fibering with fibre F, and let $i : F \to L$ be the inclusion. Suppose $h : K \to X$ is such that $h \circ f \simeq 0$ by a homotopy H_t $(0 \leqq t \leqq 1)$, and let $\phi : X \to Y$ be a map with $\phi \circ h \simeq 0$. Then we have the diagram

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$$\begin{array}{ccccccccc}
F & \xrightarrow{i} & L & & & & & & \\
\Phi\downarrow & & \| & & & & & & \\
E_f & \xrightarrow{Pf} & L & \xrightarrow{f} & K & \xrightarrow{P'f} & C_f & & \\
\eta_{f,h}\downarrow & \xi_{f,h}\downarrow & & & \| & & \downarrow v & & \\
\Omega X & \xrightarrow{Ih} & E_h & \xrightarrow{Ph} & K & \xrightarrow{h} & X & & \\
\| & \eta_{h,\psi}\downarrow & & & \downarrow & \xi_{h,\psi} & \| & & \\
\Omega X & \xrightarrow[\Omega-\phi]{} & \Omega Y & \xrightarrow[I\phi]{} & E_\psi & \xrightarrow[P\phi]{} & X & \xrightarrow[\phi]{} & Y,
\end{array}$$

where Φ is a natural equivalence (cf. Th. 1) and v is defined by

$$v(k) = h(k) \qquad \text{for } k \in K,$$
$$v(l, s) = H_{1-s}(l) \qquad \text{for } l \in L,\ 0 \leqq s \leqq 1.$$

Since $\bar{\sigma}^*([v]) = [\eta_{f,h} \circ \Phi]$ in the fibering f (cf. 8.1, Remark), it follows from the homotopy commutativity of the diagram

PROPOSITION 9. (cf. [10], Lemma 6.2) $i^*\bar{\psi}_f(h) = -(\Omega\psi)_*\bar{\sigma}^*([v])$ mod $(\Omega\psi)_*\pi(L, \Omega X)$.

11.4. Let $L \xrightarrow{f} K \xrightarrow{h} X \xrightarrow{\theta} Y \xrightarrow{\phi} Z$ be maps such that there exist homotopies $H_t : \theta \circ h \circ f \simeq 0$, $G_t : \phi \circ \theta \simeq 0 (0 \leqq t \leqq 1)$. Then we can easily verify that both $\eta_{\theta,\psi} \circ \xi_{h\circ f,\theta}$ and $\eta_{\theta\circ h,\psi} \circ \xi_{f,\theta\circ h}$ are given by

$$(x, s) \to \begin{cases} G_{1-2s}(hf(x)), & 0 \leqq s \leqq \frac{1}{2}, \quad x \in L, \\ \psi H_{2s-1}(x), & \frac{1}{2} \leqq s \leqq 1, \quad x \in L. \end{cases}$$

Hence we have shown

PROPOSITION 10. $\bar{\psi}_{h\circ f}(\theta) = \bar{\psi}_f(\theta \circ h)$.

In case $\bar{\sigma}^* : \pi(C_0, Z) \to \pi(E_0, \Omega Z)$ is onto, then there exists a map $\phi : Y \to Z$ such that $[\eta_{\theta,\psi}] = [\phi]$ for any $\phi : E_\theta \to \Omega Z$, and then we have $(I\theta)^*[\phi] = -[\Omega\psi]$. We deduce from Prop. 10.

COROLLARY. $\Phi_\theta(h \circ f) = \bar{\psi}_f(\theta \circ h)$ *mod* $(\Omega\psi)_*\pi(L, \Omega Y) + f^*\pi(K, \Omega Z)$ (cf. [10], Th. 6.3).

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