POSITIVE DEFINITE FUNCTIONALS IN BANACH SPACES

G. G. HAMEDANI

Abstract

We establish a version of Bochner Theorem due to S. Boylan for Banach spaces with a basis.

1. Introduction

The classical version of Bochner's theorem asserts that a function $\varphi(x)$, $x \in R^n$ is the c.f. (characteristic functional or Fourier transform) of some probability measure on the Borel subsets of R^n if and only if φ is positive definite, $\varphi(0) = 1$, and φ is continuous at $x = 0$. In l_2, and hence in any real separable Hilbert space, it is possible to introduce a topology τ (Gross (1963), Sazanov (1958)) such that a function φ on l_2 is the c.f. of some probability measure on the Borel subsets of l_2 if and only if φ is positive definite, $\varphi(0) = 1$, and φ is continuous at zero (and hence everywhere on l_2) in the τ-topology. Boylan (1975) proved the following version of Bochner's Theorem.

THEOREM B. *Assume $\varphi(x)$ is a bounded continuous functional on l_2 such that $\varphi(0) = 1$. Then $\varphi(x)$ is a c.f. of a positive measure on l_2, if and only if*

(a) $T_t\varphi(y) = E_z^{t\rho}\{\varphi(z)\exp\{i(z, y)\}\} \sqrt{\det(1 + t\rho)} \geqq 0$ *for all $y \in l_2$, for all $t \geqq t_0$, some $t_0 > 0$.*

(b) *Setting $\varphi(\gamma x) = \varphi_\gamma(x)$, we have*

$$\lim_{\gamma \to 0} \lim_{t \to \infty} \int_{l_2} T_t\varphi_\gamma(y)\nu_1(dy) = \varphi(0),$$

where

(i) ν_1 *is a Gauss measure on l_2 with variance parameter 1,*

(ii) ρ *is a linear, symmetric, positive, compact and trace class operator on l_2,*

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(iii) $E_z^{\prime \rho}$ denotes expectation with respect to a Gaussian process $\{z : z \in l_2\}$ with mean zero and covariance ρ.

In proving the above theorem, Boylan characterizes positive definite functionals on l_2 by showing that positive definiteness is equivalent to condition (a). The interesting aspect is that the conditions (a) and (b) use essentially the same entities. The analogue of (a) has been obtained by Boylan (1975) for real separable Banach spaces. However Boylan leaves as an open problem the extension of his version of Bochner's theorem to general spaces. We do this here for Banach spaces with a Schauder basis. We note that the only characterization of positive definiteness we use is through the Hilbert space case, i.e., Theorem B.

2. Terminology and preliminaries

Throughout this paper B will denote a real Banach space with Schauder basis $\{b_n\}$ such that without loss of generality $\|b_n\|_B = 1$ (Day (1962), p. 68). We will write the expansion of x as $\sum_{n=1}^{\infty} \beta_n(x)b_n$ and this emphasizes that the coefficients generate coordinate functionals on B. It is clear that these coordinate functionals are linear and it is well known that they are continuous as well (Day (1962), p. 68). Further it is possible to assume without loss of generality (Day (1962), Theorem 1, p. 67) that

$$(2.1) \qquad \|x\|_B = \sup_k \left\| \sum_{n=1}^{k} \beta_n(x)b_n \right\|_B.$$

We say that $\lambda \in l_\infty^+$ is adequate for B if
(i) for all $x \in B$, $\sum_{i=1}^{\infty} \lambda_i \beta_i^2(x)$ is finite;
(ii) for all x there exists a nonnegative constant C_λ such that

$$\sum_{i=1}^{\infty} \lambda_i \beta_i^2(x) \leq C_\lambda \|x\|_B^2.$$

Here l_∞ denotes the space of bounded sequences. We note at the outset that if B is a Hilbert space then $\lambda = (1, 1, 1, \cdots)$ is adequate and then $C_\lambda = 1$.

2.2. REMARKS. (a) $\lambda \in l_1^+$ is adequate for B, since $\sup_n |\beta_n(x)| \leq k\|x\|_B$ (Day (1962), p. 68) and $C_\lambda = \sum_{i=1}^{\infty} \lambda_i$.

(b) If $B = l_p$ (the set of all sequences $x = (x_1, x_2, \cdots)$, with $\sum_{i=1}^{\infty} |x_i|^p < \infty$) $(p \geq 2)$ then $\lambda \in (l_{p/2}^*)^+$ is adequate with $C_\lambda = \|\lambda\|_{l_{p/2}^*}$.

If $\lambda \in l_\infty^+$ is adequate for B then for that λ we define the inner product

$$(2.3) \qquad (x, y) = \sum_{i=1}^{\infty} \lambda_i \beta_i(x)\beta_i(y) \quad \text{for all} \quad x, y \quad \text{in} \quad B.$$

Let $\|x\|_\lambda = (x, x)^{\frac{1}{2}}$ then $\|x\|_\lambda \leq C_\lambda \|x\|_B$. Let H_λ be the Hilbert space obtained by the completion of B under the norm $\|\cdot\|_\lambda$.

From (2.1) and the fact that β_n is a $\|\cdot\|_\lambda$-continuous function on B, it follows that $\|x\|_B$ is a measurable function in $\|\cdot\|_\lambda$-topology, and hence B is a $\|\cdot\|_\lambda$-measurable subset of H_λ. Therefore if μ is a measure on B, it can be regarded as a measure on H_λ via $\mu(A) = \mu(A \cap B)$ for all $A \in \mathcal{B}(H_\lambda)$, Borel sets of H_λ.

The following ideas are given in Kuelbs and Mandrekar (1972).

If $\lambda \in l_\infty^+$ and $\{\mu_s : s \in A\}$ is a family of probability measures on B such that

$$\mu_s\left\{x \in B : \sum_{n=1}^\infty \lambda_n \beta_n^2(x) < \infty\right\} = 1$$

for each $s \in A$, we say λ is sufficient for the family $\{\mu_s : s \in A\}$.

Every λ adequate for B in l_∞^+ is sufficient for any family of probability measures on B.

A family of probability measures $\{\mu_s : s \in A\}$ on B is a λ-family for some $\lambda \in l_\infty^+$ if λ is sufficient for $\{\mu_s : s \in A\}$ and for every $\varepsilon, \delta > 0$ there is a sequence $\{\varepsilon_N\}$ such that

$$\mu_s\left\{x \in B : \sum_{n=N+1}^\infty \lambda_n \beta_n^2(x) < \delta\right\} > 1 - \varepsilon$$

implies

$$\mu_s\left\{x \in B : \left\|\sum_{n=N+1}^\infty \beta_n(x)b_n\right\|_B < h(\delta)\right\} > 1 - (\varepsilon + \varepsilon_N) \quad \text{for all} \quad s,$$

where $\lim_N \varepsilon_N = 0$ and h is a strictly increasing continuous function on $[0, \infty)$ with $h(0) = 0$.

It is quite clear that any family of probability measures on a real separable Hilbert space is a λ-family with $\lambda = (1, 1, \cdots)$ and $h(\delta) = \delta^{\frac{1}{2}}$.

Finally for $x \in B$, $N = 1, 2, \cdots$, we define

$$P_N x = \sum_{k=1}^N \beta_k(x)b_k.$$

3. Main result

Let Γ be a Gaussian measure with mean zero on $(B, \mathcal{B}(B))$. Then by an argument similar to Lemma 2.2 of (Kuelbs (1970)), Γ is a Gaussian measure on H_λ, and therefore there exists a nonnegative, symmetric, trace class operator (often called an S-operator) ρ on H_λ^* into H_λ such that

$$(\rho x, x) = \int_{H_\lambda} (y, x)^2 d\Gamma(y)$$

for $x \in H_\lambda^*$, and Γ is uniquely determined on H_λ by the operator ρ. We shall assume that ρ has dense range. Let $t > 0$ and let Γ_t be a Gaussian measure on H_λ^* with the covariance operator $tj\rho j^*$ where j is the bijection map on H_λ to H_λ^* given by Riesz Representation Theorem. Let $C_t = (1 + tj\rho j^*)^{\frac{1}{2}}$.

   3.1. THEOREM. *A bounded continuous functional φ on B^* with $\varphi(0) = 1$ is c.f. of a probability measure on B if and only if*
   (i) *the family of measures $\{\mu_N\}$ corresponding to $\varphi(P_N(\cdot))$ is a λ-family for some $\lambda \in l_x^+$ which is adequate for B,*
   (ii) $(\det C_t)\int_B \cdot \varphi(y) \exp\{i\langle x, y\rangle\}\Gamma_t(dy) \geq 0$ *for all $x \in B$, for all $t \geq t_0$, some $t_0 > 0$,*
   (iii) $\lim_{\gamma \to \infty} \lim_{t \to \infty} (\det C_t)\int_B \int_{B^*} \varphi(\gamma y) \exp\{i\langle x, y\rangle\}\Gamma_t(dy)\nu_1(dx) = \varphi(0)$,
   where ν_1 is a Gauss measure on B.

   PROOF. Let μ be a probability measure on B with c.f. φ. Then condition (i) holds for every $\lambda \in l_x^+$ which is adequate for B since $\{\mu_N\}$ is conditionally compact and Lemma 3.1 of Kuelbs and Mandrekar (1972) applies. Now let $\lambda \in l_x^+$ be adequate for B and consider the Hilbert space H_λ. Since B is a measurable subset of H_λ, μ can be regarded as a probability measure on H_λ. Let ψ be a c.f. of μ on H_λ. Then by Theorem 1 of Boylan (1975) we have

(3.2) $(\det C_t)\int_{H_\lambda^*} \psi(y)\exp\{i(x, y)\}\Gamma_t(dy) \geq 0$ for all $x \in H_\lambda$,

for all $t \geq t_0$, some $t_0 > 0$, and

(3.3) $\lim_{\gamma \to 0} \lim_{t \to \infty} (\det C_t)\int_{H_\lambda} \int_{H_\lambda^*} \psi(\gamma y)\exp\{i(x, y)\}\Gamma_t(dy)\nu_1(dx) = \psi(0)$.

Let ν_1 be a Gauss measure on B. Then in view of (3.2) and (3.3) and the fact that H_λ^* is a measurable dense subset of B^* it can be seen that (ii) and (iii) are satisfied.
   Now assume (i), (ii), (iii) hold. Let ψ be restriction of φ over H_λ^*. Then, from (ii) and (iii) and the fact that B is a measurable dense subset of H_λ, it follows that conditions (a) and (b) of Theorem 1 of Boylan (1975) are satisfied for ψ. Hence ψ is c.f. of a probability measure μ on H_λ. Let $\mu_N = \mu \circ P_N^{-1}$, then $\{\mu_N\}$ is compact on H_λ and μ_N converges weakly to μ on H_λ. This together with (i) imply that $\{\mu_N\}$ is compact on B (Hamedani and Mandrekar (1973), Lemma 1.2.11). Hence μ_N converges weakly to μ on B and $\mu(B) = 1$. This implies that the c.f. of μ is $\varphi(x)$ and the theorem is proved.

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Arya–Mehr University of Technology,

Tehran, Iran.

Michigan State University, U.S.A.