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CORRECTION TO 'AUTOMORPHISM AND OUTER AUTOMORPHISM GROUPS OF RIGHT-ANGLED ARTIN GROUPS ARE NOT RELATIVELY HYPERBOLIC'

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It turns out that there is one more case, not covered in [2, Theorem 4.1], where $\mathrm{Out}(A_\Gamma)$ is relatively hyperbolic. This occurs when $\mathrm{Out}(A_\Gamma)$ is a virtually right-angled Artin group (RAAG) whose defining graph consists of at least two components.

1. Main statement

We will prove the following revised statement of [2, Theorem 4.1].

THEOREM 1.1. *If* $\mathrm{Out}(A_\Gamma)$ *is infinite and not virtually a RAAG whose defining graph is either a single vertex or disconnected, then* $\mathrm{Out}(A_\Gamma)$ *is not relatively hyperbolic.*

Following the notation in [2], let S be the set of all transvections and partial conjugations in $\mathrm{Aut}(A_\Gamma)$, and S' the set of all the (nontrivial) images of elements of S in $\mathrm{Out}(A_\Gamma)$. Let $K' = K(\mathrm{Out}^*(A_\Gamma), S')$ be the commutativity graph of $\mathrm{Out}^*(A_\Gamma)$ with respect to S'.

We first consider the case when S' consists of only partial conjugations. In this case, $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to $\mathrm{PSO}(A_\Gamma)$, the pure symmetric outer automorphism group of A_Γ. We complete the proof of Theorem 1.1 by considering the case when S' has a transvection.

2. Pure symmetric (outer) automorphism group

The subgroup $\mathrm{PSA}(A_\Gamma) \le \mathrm{Aut}(A_\Gamma)$ generated by partial conjugations is the *pure symmetric automorphism group* of A_Γ and the subgroup $\mathrm{PSO}(A_\Gamma) \le \mathrm{Out}(A_\Gamma)$ generated by partial conjugations is the *pure symmetric outer automorphism group* of A_Γ. Koban and Piggott showed in [3] that $\mathrm{PSA}(A_\Gamma)$ has a group presentation whose generators are partial conjugations and whose relators are commutators. Moreover, they showed that $\mathrm{PSA}(A_\Gamma)$ is isomorphic to a RAAG if and only if Γ has no SIL-pairs (defined below Lemma 2.1). With a similar flavour, Day and Wade [1] found the criterion for $\mathrm{PSO}(A_\Gamma)$ to be a RAAG.

In the study of $\mathrm{PSA}(A_\Gamma)$ or $\mathrm{PSO}(A_\Gamma)$, the most important thing is to know when two partial conjugations commute, and there is a precise description of this using the following fact.

LEMMA 2.1 [1, Lemma 2.1]. *Let a and b be nonadjacent vertices of Γ. Then the components of $\Gamma - \mathrm{st}(a)$ consist of $A_0, \ldots, A_k, C_1, \ldots, C_l$, and the components of $\Gamma - \mathrm{st}(b)$ consist of $B_0, \ldots, B_m, C_1, \ldots, C_l$, where $b \in A_0$ and $a \in B_0$, $A_1, \ldots, A_k \subset B_0$ and $B_1, \ldots, B_m \subset A_0$.*

In this lemma, A_0 and B_0 are the *dominating components*, the C_i are the *shared components* and the other components are the *subordinate components*. We say (a, b) is an *SIL-pair* if $l \geq 1$. Note that any of k, m or l can be zero; for instance, $l = 0$ implies that there is no shared component.

LEMMA 2.2 [1, Lemma 2.4]. *Let a and b be nonadjacent vertices in Γ such that there are nontrivial partial conjugations P_a^C and P_b^D in $\mathrm{Out}(A_\Gamma)$. Then $[P_a^C, P_b^D] \neq 1$ in $\mathrm{Out}(A_\Gamma)$ if and only if (a, b) is an SIL-pair and one of the following conditions holds:*

- *C and D are the dominating components for the pair (a, b);*
- *one of C or D is dominating and the other is shared;*
- *C and D are identical shared components.*

Now, we examine the nonrelative hyperbolicity of $\mathrm{Out}^*(A_\Gamma)$ when S' consists of only partial conjugations, that is, $\mathrm{Out}^*(A_\Gamma) = \mathrm{PSO}(A_\Gamma)$, by using K'.

PROPOSITION 2.3. *Suppose S' consists of partial conjugations and $|S'| \geq 1$. If there is a vertex $v \in \Gamma$ such that $\Gamma - \mathrm{st}(v)$ has at least three components, then K' is connected. Otherwise, $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to the RAAG whose defining graph is K'.*

PROOF. Obviously, if there is a nontrivial partial conjugation by a vertex $v \in \Gamma$, then any two partial conjugations by v commute.

Suppose there is a vertex v such that $\Gamma - \mathrm{st}(v)$ has at least three components, and there is a nontrivial partial conjugation by w for $w \neq v$. By the first paragraph, it suffices to show that P_v^C and P_w^D commute for some C and D. If (v, w) is not an SIL-pair, by Lemma 2.2, any partial conjugation by w commutes with any partial conjugation by v. Otherwise, there is at least one shared component C_1 for the pair (v, w). If there is one more shared component C_2, by Lemma 2.2, we have $[P_v^{C_1}, P_w^{C_2}] = 1$. Otherwise, there is a subordinate component C' of $\Gamma - \mathrm{st}(v)$, and by Lemma 2.2, $[P_v^{C'}, P_w^{C_1}] = 1$.

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In [1, Theorem B], it is shown that $\mathrm{PSO}(A_\Gamma)$ is isomorphic to a RAAG if and only if the support graph of each vertex $v \in \Gamma$ is a forest, where the support graph is a simplicial graph whose vertices are components of $\Gamma - \mathrm{st}(v)$. If there is no vertex v such that $\Gamma - \mathrm{st}(v)$ has at least three components, then each support graph is a forest (either a single vertex or two vertices with or without an edge). Therefore, $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to a RAAG, and by the results in [1], it can easily be seen that the defining graph of the RAAG is equal to K'. □

Wiedmer showed that any RAAG can be isomorphic to $\mathrm{Out}^*(A_\Gamma)$ for some graph Γ [4]. To completely characterise nonrelative hyperbolicity of $\mathrm{Out}^*(A_\Gamma)$ when $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to $\mathrm{PSO}(A_\Gamma)$, we need the following fundamental fact.

LEMMA 2.4. *A RAAG A_Λ is relatively hyperbolic if and only if its defining graph Λ consists of either a single vertex or at least two components.*

PROOF. If Λ is a single vertex, then A_Λ is isomorphic to $\mathbb{Z}$ and thus (relatively) hyperbolic. If Λ consists of at least two components, then A_Λ is isomorphic to $A_{\Lambda_1} * \cdots * A_{\Lambda_n}$, where the Λ_i are components of Λ; in particular, A_Λ is relatively hyperbolic with respect to $\{A_{\Lambda_1}, \ldots, A_{\Lambda_n}\}$.

If Λ is connected and has at least two vertices, then the commutativity graph is exactly the same as the defining graph by taking the generating set as the usual generators of the RAAG. □

3. Proof of Theorem 1.1

If $|S'| \leq 1$, then $\mathrm{Out}(A_\Gamma)$ is finite or has a finite-index subgroup isomorphic to $\mathbb{Z}$, and thus, it is (relatively) hyperbolic. If Γ has only one vertex, then $\mathrm{Out}(A_\Gamma)$ is obviously finite. If Γ has only two vertices, then $\mathrm{Out}(A_\Gamma)$ is isomorphic to $\mathrm{GL}_2(\mathbb{Z})$, and thus, it is virtually the free group of rank 2. Now we examine the commutativity graph $K' = K(\mathrm{Out}^*(A_\Gamma), S')$ for the case that $|S'| \geq 2$ and Γ has at least three vertices.

If S' does not have any transvection, by Proposition 2.3 and Lemma 2.4, $\mathrm{Out}^*(A_\Gamma)$ is not relatively hyperbolic if and only if $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to a RAAG whose defining graph is connected.

Now, we assume that there is at least one transvection in S'.

Claim A. As long as they exist, any nontrivial partial conjugation and any transvection are joined by a path in K' unless $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to $\mathrm{Aut}^*(\mathbb{F}_2)$.

Let R_{ab} be a transvection and suppose that there is a nontrivial partial conjugation P_c^C. Note that $[R_{ab}, L_{ab}] = 1$ whenever R_{ab} is equal to L_{ab} or not. We will show the existence of a path joining R_{ab} and P_c^C in K'. There are four cases, depending on c and the adjacency of a and b.

(I) If $c = b$, then $[R_{ab}, P_c^C] = 1$ whenever C contains a or not.

(II) Suppose that c is neither a nor b. If a or b is contained in $\mathrm{lk}(c)$, then $a \leq b$ implies that $b \in \mathrm{lk}(c)$ and thus $[P_c^C, R_{ab}] = 1$ for any component C. If a and b

are in the same component C' of $\Gamma - \mathrm{st}(c)$, then we have $[P_c^{C'}, R_{ab}] = 1$, which implies that the claim is true since $[P_c^{C'}, P_c^C] = 1$. If a and b are contained in different components of $\Gamma - \mathrm{st}(c)$, then $a \leq b$ implies that $a \leq c$. Since P_c^C and R_{ac} are joined in K' by an edge by *(I)* and we have $[R_{ab}, L_{ac}] = [L_{ac}, R_{ac}] = 1$, and the claim holds.

(III) Suppose $c = a$, and a and b are adjacent. If there is a nontrivial element P_b^D in $\mathrm{Out}(A_\Gamma)$, then $[P_a^C, P_b^D] = [P_b^D, R_{ab}] = 1$, and thus, P_c^C and R_{ab} are joined by a path. Otherwise, there exists a component C' of $\Gamma - \mathrm{st}(a)$ contained in $\mathrm{lk}(b)$, which implies $[R_{ab}, P_a^{C'}] = 1$ and thus the claim holds.

(IV) Suppose $c = a$ but a and b are nonadjacent. Since $a \leq b$, there is no subordinate component of $\Gamma - \mathrm{st}(a)$ for the pair (a, b). If $\Gamma - \mathrm{st}(a)$ has at least three components, then there are at least two shared components, say C_1 and C_2. Since we have $[P_a^{C_1}, P_b^{C_2}] = [P_b^{C_2}, R_{ab}] = 1$ by Lemma 2.2 and Case (I), the claim holds.

Now, suppose $\Gamma - \mathrm{st}(a)$ has two components, the shared component C' and the dominating component C''. If $\Gamma - \mathrm{st}(b)$ has a subordinate component D, then the claim holds since $[P_a^{C'}, P_b^D] = [P_b^D, R_{ab}] = 1$ by Lemma 2.2 and Case (I). Otherwise, there are two situations depending on the existence of a vertex $x \in \mathrm{lk}(b) - \mathrm{lk}(a)$. If such a vertex x exists, then $x \leq b$ and thus the claim holds since $[R_{ab}, R_{xb}] = [R_{xb}, P_a^{C'}] = 1$. Otherwise, $a \sim b$ (in particular, C'' becomes $\{b\}$) and we have two final cases.

(1) Suppose there is a vertex d in C', which defines a nontrivial partial conjugation in $\mathrm{Out}(A_\Gamma)$. If $\mathrm{lk}(a) \subseteq \mathrm{lk}(d)$, then $a \sim b \leq d$ and thus the claim holds since $[R_{ab}, L_{ad}] = [L_{ad}, R_{bd}] = 1$ and there is a path joining R_{bd} and $P_a^{C'}$ by Case (II). Otherwise, C'' is a subordinate component of $\Gamma - \mathrm{st}(a)$ for the pair (a, d). By Lemma 2.2, any partial conjugation P_d^D by d commute with $P_a^{C''}$. Since there is a path joining P_d^D and R_{ab} by Case (II), the claim holds.

(2) Suppose there is no such vertex d. In this case, only a, b and vertices in $\mathrm{lk}(a)$ $(= \mathrm{lk}(b))$ may be able to define nontrivial partial conjugations in $\mathrm{Out}(A_\Gamma)$. If $x \in \mathrm{lk}(a)$ does, then any nontrivial partial conjugation P_x^X commutes with P_c^C and R_{ab}, and thus the claim holds.

Otherwise, we need to see whether there is a transvection R_{vw} different from R_{ab} or R_{ba}.

Suppose there is such a transvection R_{vw}. If $\{v, w\} \cap \{a, b\} = \emptyset$, then $[R_{ab}, R_{vw}] = 1$. Since there is a path joining R_{vw} and P_a^C by Case (II), the claim holds. If v is either a or b (in particular, $a \sim b \leq w$ and $w \neq a, b$), since w must not define a nontrivial partial conjugation, w is adjacent to both a and b. Since $[R_{ab}, L_{aw}] = [L_{aw}, R_{aw}] = 1$ and there is a path joining R_{aw} and P_a^C by Case (III), the claim holds. If w is either a or b (in particular, $v \leq a \sim b$ and $v \neq a, b$), by Case (I), there is a path joining R_{va} and P_a^C. Since $[R_{va}, L_{vb}] = [L_{vb}, R_{ab}] = 1$, the claim holds.

Finally, if there is no such transvection R_{vw}, then a and b are the only vertices defining nontrivial partial conjugations and each of $\Gamma - \mathrm{st}(a)$ and $\Gamma - \mathrm{st}(b)$ has two components. In particular, S' consists of two partial conjugations and four (two right and two left) transvections. In this case, K' is discrete and $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to $\mathrm{Aut}^*(\mathbb{F}_2)$.

In summary, we checked that if there are a transvection and a partial conjugation in S' which cannot be joined by a path in K'. Thus, $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to $\mathrm{Aut}^*(\mathbb{F}_2)$.

Claim B. Every pair of transvections can be joined by a path in K' unless $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$ or $\mathrm{Aut}^*(\mathbb{F}_2)$.

Since we assumed that Γ has at least three vertices, the existence of a path in $K(\mathrm{Aut}^*(A_\Gamma), S)$ between two transvections (which appeared while proving Claim 1 in [2, the proof of Theorem 3.1]) tells us there is a path in K' between the two transvections, except between R_{ab} and R_{ba}; the path joining them in $K(\mathrm{Aut}^*(A_\Gamma), S)$ may use partial conjugations which have trivial images in $\mathrm{Out}(A_\Gamma)$.

If there is another transvection R_{vw}, by Cases 1, 2, 3 and 4 of the proof of Claim 1 in [2, the proof of Theorem 3.1], there is a path joining R_{vw} to R_{ab} (and R_{ba}) in K', and thus R_{ab} and R_{ba} are joined by a path. If there is no other transvection but there is a nontrivial partial conjugation, by Claim A, there is a path joining R_{ab} to R_{ba} in K' except when $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to $\mathrm{Aut}^*(\mathbb{F}_2)$. Lastly, if $S' = \{R_{ab}, L_{ab}, R_{ba}, L_{ba}\}$, then any partial conjugation by a or b must be the identity in $\mathrm{Out}(A_\Gamma)$, which implies that $R_{ab} = L_{ab}$ and $R_{ba} = L_{ab}$. Thus, $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to $\mathrm{Out}^*(\mathbb{F}_2)$, which is $\mathrm{SL}_2(\mathbb{Z})$.

By these two claims, as long as $|S'| > 1$ and there is at least one transvection in S', K' is connected and [2, Theorem 2.4] yields the nonrelative hyperbolicity of $\mathrm{Out}^*(A_\Gamma)$ unless $\mathrm{Out}^*(A_\Gamma)$ is isomorphic to neither $\mathrm{Aut}^*(\mathbb{F}_2)$ nor $\mathrm{SL}_2(\mathbb{Z})$. Since $\mathrm{Aut}^*(\mathbb{F}_2)$ is nonrelative hyperbolic as explained in the paragraph below the proof of [2, Theorem 3.1], we have completed the proof.

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