ON A CONJECTURE OF GUINAND FOR THE PLANE PARTITION FUNCTION

by GEORGE E. ANDREWS†

(Received 26th June 1970)

In (**1**; p. 38), A. P. Guinand discusses the plane partition function $q(n)$. He observes that $q(3)$, $q(6)$, $q(9)$, $q(15)$, $q(18)$, $q(21)$, and $q(24)$ are respectively 6, 48, 282, 1479, 6879, 29601, 118794, and 451194. As all these are multiples of 3 he suggests the conjecture that $q(3n) \equiv 0 \pmod 3$ for all positive integers n.

Unfortunately, $q(27) = 1632658 \equiv 1 \pmod 3$; however, we may prove that the phenomenon observed by Guinand has even more striking counterparts for higher-dimensional partition functions. An n-dimensional partition of N is a representation of N as an n-fold sum:

$$N = \sum_{x_1, x_2, \ldots, x_n \geqq 0} y(x_1, \ldots, x_n)$$

where the $y(x_1, \ldots, x_n)$ are all positive integers with $y(x_1, \ldots, x_n) \geqq y(x_1', \ldots, x_n')$ whenever $x_i \leqq x_i'$ for all i.

Theorem. *If $p_n(N)$ denotes the number of n-dimensional partitions of N and if $n+1$ is a prime, then*

$$p_n((n+1)N) \equiv 0 \pmod{n+1}, \quad 0 < N < (n+1)^n, \tag{1}$$

$$p_n((n+1)^{n+1}) \equiv 1 \pmod{n+1}. \tag{2}$$

Proof. We represent each n-dimensional partition as a solid $(n+1)$-dimensional graph (cf. (**2**; p. 174) for the case $n = 2$) where the nodes lie on lattice points with non-negative integer coordinates. The $n+1$ successive transformations which send

$$(x_1, x_2, \ldots, x_{n+1}) \to (x_{n+1}, x_1, \ldots, x_n) \to \ldots \to (x_2, \ldots, x_{n+1}, x_1) \to (x_1, x_2, \ldots, x_{n+1})$$

always produce from a given partition either $n+1$ distinct partitions or else leave the partition fixed. This is because these transformations form a cyclic group of order $n+1$, a prime; consequently any element of the group other than the identity generates the full group. If however, the partition is left fixed then since the only fixed points of our transformations lie on the diagonal, we see that the first partition which is a multiple of $n+1$ and invariant under the given rotations is just the $(n+1)$-dimensional hypercube with $n+1$ dots on an edge. Consequently, the partitions of $(n+1)N$ can be divided into disjoint sets of $n+1$ elements whenever $0 < N < (n+1)^n$. Hence (1) follows.

† Partially supported by NSF Grant GP-9660.

On the other hand, the only partition of $(n+1)^{n+1}$ which has $n+1$ dots on the diagonal is the above described hypercube. Thus since all the other n-dimensional partitions of $(n+1)^{n+1}$ may be separated into sets of $n+1$ elements, we deduce that

$$p_n((n+1)^{n+1}) \equiv 1 \pmod{n+1}.$$

The procedure used here is derived from Wright's work in (3) for 1-restricted $(n+1)$-dimensional partitions.

REFERENCES

(1) A. P. Guinand, *Report of the Research Committee on the Summer Research Institutes* (Canadian Math. Congress, 1969).

(2) P. A. MacMahon, *Combinatory Analysis*, vol. 2 (Cambridge University Press, Cambridge, 1916).

(3) E. M. Wright, Rotatable partitions, *J. London Math. Soc.*, 43 (1968), 501-505.

Pennsylvania State University
University Park, Pennsylvania 16802