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DISPERSED FACTORIZATION STRUCTURES

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0. Introduction. Factorization structures on a category $\mathscr{K}$ form a useful categorical tool. As is known, any $\mathscr{K}$, satisfying suitable completeness—and smallness—conditions, has a sufficient supply of factorization structures; in fact, there is a bijection between the class of all epireflective (full and isomorphism-closed) subcategories of $\mathscr{K}$ and the class of all so called *perfect factorization structures* on $\mathscr{K}$. In this paper, for an arbitrary category $\mathscr{K}$ supplied with a fixed factorization structure (E, M), a similar bijection between the class of all E-reflective (full and isomorphism-closed) subcategories of $\mathscr{K}$ and the class of all (E, M)-*dispersed factorization structures* on $\mathscr{K}$, introduced in this paper, will be established. Moreover (E, M)-dispersed factorization structures (C, D) will be characterized by several sets of conditions, the simplest being that (1) $C \subset E$ and (2) $g \circ f \in C$ and $f \in E$ implies $f \in C$. For these results it is of crucial importance to consider factorizations for arbitrary sources and not for single morphisms only.

In case E is the class of $\mathscr{K}$-epimorphisms, our results extend and simplify corresponding results about perfect factorizations (for which there exists an excellent survey by G. E. Strecker [**24**]) and free the latter from their somewhat unnatural completeness—and smallness—restrictions. In case E is the class of extremal $\mathscr{K}$-epimorphisms, the corresponding so called *dissonant factorization structures* have been recently discovered and investigated independently by G. Salicrup and R. Vázquez [**20**] and by G. Preuss [**16**] in the more restricted setting of topological categories $\mathscr{K}$. At least in this more specialized setting they have a definite topological flavour and a close relation to connectedness properties in $\mathscr{K}$. In particular they contain as special cases the (concordant quotient, dissonant)—factorization of P. Collins [**2**] and the (submonotone quotient, superlight)—factorization of G. E. Strecker [**23**].

In this paper a number of further examples will be provided. In particular it will be shown that the topological category *Rere* of reflexive relations has a proper class of perfect factorization structures and a proper class of dissonant factorization structures, but only one factorization structure, which is simultaneously dissonant and perfect.

In all that follows, $\mathscr{K}$ will denote a category and every subcategory of a category will be assumed to be full and isomorphism-closed.

1. Factorization structures.

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1.1 *Definitions.* (1) A *source* in $\mathscr{K}$ with domain X is a pair $(X, f_i)_I$, where X is a $\mathscr{K}$-object and $(f_i)_I$ is a family of $\mathscr{K}$-morphisms $f_i: X \to Y_i$ indexed by a class I which can be proper, improper or empty. Alternative notations for $(X, f_i)_I$ are $(X, f_i: X \to Y_i)_I$ and $(X, \mathscr{F})$.

(2) (E, M) is said to be a *factorization structure* on $\mathscr{K}$ if and only if E is a class of $\mathscr{K}$-morphisms closed under composition with isomorphisms, M is a conglomerate of sources in $\mathscr{K}$ closed under composition with isomorphisms, each source in $\mathscr{K}$ has an (E, M)-factorization and $\mathscr{K}$ has the (E, M)-diagonalization property, i.e. whenever f and e are morphisms and $(Y, m_i)_I$ and $(Z, f_i)_I$ are sources in $\mathscr{K}$ such that $e \in E$, $(Y, m_i)_I \in M$ and $f_i \circ e = m_i \circ f$ for each $i \in I$, then there exists a unique morphism g in $\mathscr{K}$ such that $g \circ e = f$ and $m_i \circ g = f_i$ for each $i \in I$.

In the following we assemble without proof some properties of factorization structures. For details see e.g. [**10**], [**17**], [**20**], or [**24**].

1.2 PROPOSITION. *If (E, M) is a factorization structure on $\mathscr{K}$ then the following hold:*

(1) *(E, M)-factorizations are unique (up to isomorphisms).*

(2) *$E \cap M$ is the class of isomorphisms in $\mathscr{K}$.*

(3) *E is a class of epimorphisms in $\mathscr{K}$* ([**11**], [**25**]).

(4) *Every extremal monosource, in particular all limits and all sections, belongs to M.*

(5) *If f, g, h are $\mathscr{K}$-morphisms such that $h = g \circ f$ then the following hold:*

(a) *If h belongs to E and f is an epimorphism then g belongs to E.*

(b) *If f and g belong to E then h belongs to E.*

(6) *If $(X, f_i)_I$ is a source in $\mathscr{K}$ and $((X, g_j)_J, ((Z_j, k_{ji})_{I_j})_J)$ is a factorization of $(X, f_i)_I$, i.e. $\bigcup_{j \in J} I_j = I$ and for each $j \in J$ and each $i \in I_j$, $f_i = k_{ji} \circ g_j$, then the following hold:*

(a) *If $(X, f_i)_I$ is in M then $(X, g_j)_J$ is in M.*

(b) *If $(X, g_j)_J$ is in M and for each $j \in J$ $(Z_j, k_{ji})_{I_j}$ is in M then $(X, f_i)_I$ is in M.*

(7) *If $(X, f_i)_I$ is a source in $\mathscr{K}$ and there exists $J \subset I$ such that $(X, f_i)_J$ belongs to M then $(X, f_i)_I$ belongs to M.*

(8) *E and M determine each other through the diagonalization property.*

(9) *Every subcategory $\mathscr{A}$ of $\mathscr{K}$ has an E-reflective hull $\mathscr{B}$. X belongs to $\mathscr{B}$ if and only if there exists a source $(X, m_i: X \to A_i)_I$ in M with A_i in $\mathscr{A}$ for each $i \in I$.*

(10) *In particular, the subcategory $\mathscr{C}$ of $\mathscr{K}$ whose objects are those X such that $(X, \emptyset)$ belongs to M is the smallest E-reflective subcategory of $\mathscr{K}$. The objects of $\mathscr{C}$ can also be described as those X which are E-injective.*

1.3 *Remarks.* (1) Factorization structures can be defined with respect to a certain conglomerate S of sources by requiring the existence of (E, M)-factor-

izations only for sources in S. In particular, S can be the conglomerate of set indexed sources or that of single morphisms (considered as singleton sources).

(2) If $\mathscr{K}$ has products then every factorization structure (E, M) on $\mathscr{K}$ for single morphisms can be uniquely extended to a factorization structure (E, M') for set indexed sources (M' consists of compositions of M-morphisms with products).

(3) If $\mathscr{K}$ is cowell powered and has a factorization structure (E, M) for set indexed sources then the following are equivalent:

(a) (E, M) can be (uniquely) extended to a factorization structure (E, M') on $\mathscr{K}$.
(b) E is a class of epimorphisms in $\mathscr{K}$.
(c) If (X, m) belongs to M then $(X, (m, m))$ belongs to M.
(d) If $(X, \mathscr{F})$ belongs to M, $\mathscr{F} \subset \mathscr{G}$ and $\mathscr{G}$ is set indexed then $(X, \mathscr{G})$ belongs to M.
(e) Every $\mathscr{K}$-section belongs to M.

The above conditions are not automatically satisfied as the factorization structure (maps, products) for set indexed sources in *Set* demonstrates.

(4) If $\mathscr{K}$ has a factorization structure (E, M) then every factorization structure (C, D) for single morphisms with $C \subset E$ can be uniquely extended to a factorization structure (C, D') on $\mathscr{K}$. (D' consists of all compositions of D-morphisms with M-sources).

2. Dispersed factorization structures. From now on let $\mathscr{K}$ be a category supplied with a factorization structure (E, M). The concepts defined in this section will depend on this factorization structure, without making it always explicit in terminology and notation.

2.1 *Definitions and notation.* Let $\mathscr{A}$ be a subcategory of $\mathscr{K}$.
(1) For any object X of $\mathscr{K}$ $(X, \mathscr{F}(X, \mathscr{A}))$ denotes the source of all $\mathscr{K}$-morphisms with domain X and codomain in $\mathscr{A}$.
(2) A $\mathscr{K}$-morphism $f: X \to Y$ is called *$\mathscr{A}$-concentrated* if and only if f belongs to E and is $\mathscr{A}$-extendable; i.e. the source $(X, \mathscr{F}(X, \mathscr{A}))$ can be factorized through f. The class of all $\mathscr{A}$-concentrated morphisms will be denoted by $C\mathscr{A}$.
(3) A source $(X, \mathscr{F})$ in $\mathscr{K}$ is called *$\mathscr{A}$-dispersed* if and only if $(X, \mathscr{F} \cup \mathscr{F}(X, \mathscr{A}))$ belongs to M. The conglomerate of all $\mathscr{A}$-dispersed sources will be denoted by $D\mathscr{A}$.

2.2 *Remark.* If $\mathscr{A}$ is a reflective subcategory of $\mathscr{K}$, X is a $\mathscr{K}$-object and $r: X \to X'$ is an $\mathscr{A}$-reflection then a $\mathscr{K}$-source (X, F) belongs to $D\mathscr{A}$ if and only if $(X, \mathscr{F} \cup \{r\})$ belongs to M and an E-morphism $f: X \to Y$ belongs to $C\mathscr{A}$ if and only if there exists a $\mathscr{K}$-morphism g such that $g \circ f = r$.

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2.3 THEOREM. *If $\mathscr{A}$ is a subcategory of $\mathscr{K}$ then $(C\mathscr{A}, D\mathscr{A})$ is a factorization structure on $\mathscr{K}$.*

Proof. (1) Obviously $C\mathscr{A}$ and $D\mathscr{A}$ are closed under compositions with isomorphisms.

(2) If $(X, f_i)_I$ is a source in $\mathscr{K}$, $(X, \mathscr{F}(X, \mathscr{A})) = (X, f_i)_J$ and

$$(X \xrightarrow{f_i} Y_i)_{I \cup J} = (X \xrightarrow{e} Z \xrightarrow{m_i} Y_i)_{I \cup J}$$

is an (E, M)-factorization of $(X, f_i)_{I \cup J}$, then

$$(X \xrightarrow{f_i} Y_i)_I = (X \xrightarrow{e} Z \xrightarrow{m_i} Y_i)_I$$

is the desired $(C\mathscr{A}, D\mathscr{A})$-factorization of $(X, f_i)_I$.

(3) Let the diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_i} \\ Y & \xrightarrow[m_i]{} & X_i \end{array} \qquad (i \in I)$$

commute with e in $C\mathscr{A}$ and $(Y, m_i)_I$ in $D\mathscr{A}$. If $(Y, \mathscr{F}(Y, \mathscr{A})) = (Y, m_i)_J$, the fact that e belongs to $C\mathscr{A}$ implies that for each $i \in J$ there exists a $\mathscr{K}$-morphism $f_i\colon Z \to X_i$ such that $f_i \circ e = m_i \circ f$. Hence, since $(Y, m_i)_{I \cup J}$ belongs to M, there exists a $\mathscr{K}$-morphism h such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & Z \\ {\scriptstyle f}\downarrow & \overset{h}{\swarrow} & \downarrow{\scriptstyle f_i} \\ Y & \xrightarrow[m_i]{} & X_i \end{array} \qquad (i \in I \cup J)$$

commutes.

2.4 PROPOSITION. *If $\mathscr{A}$ is a subcategory of $\mathscr{K}$ and $\mathscr{B}$ is the E-reflective hull of $\mathscr{A}$ then $C\mathscr{A} = C\mathscr{B}$ and $D\mathscr{A} = D\mathscr{B}$.*

Proof. By 1.2(8) and 2.3, all one has to show is $C\mathscr{A} = C\mathscr{B}$. Since $\mathscr{A} \subset \mathscr{B}$, $C\mathscr{B} \subset C\mathscr{A}$. Let $f\colon X \to Y$ be in $C\mathscr{A}$ and suppose $g\colon X \to B$ is a $\mathscr{K}$-morphism with B in $\mathscr{B}$. By 1.2(9), there exists an M-source $(B, m_i; B \to A_i)_I$ with all A_i in $\mathscr{A}$ and, since f is $\mathscr{A}$-extendable, for each $i \in I$ there exists a $\mathscr{K}$-morphism $g_i\colon Y \to A_i$ such that $m_i \circ g = g_i \circ f$. Hence there exists a $\mathscr{K}$-morphism h

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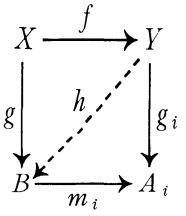
such that the diagram



commutes, and this implies that f belongs to $C\mathscr{B}$.

2.5 PROPOSITION. *If $\mathscr{A}$ is a subcategory of $\mathscr{K}$ then for each $\mathscr{K}$-object X the following are equivalent:*

(a) *X belongs to the E-reflective hull of $\mathscr{A}$.*

(b) *$(X, \emptyset) \in D\mathscr{A}$.*

(c) *X is $C\mathscr{A}$-injective.*

Proof. Since (a) is equivalent to $(X, \mathscr{F}(X, \mathscr{A})) \in M$ the equivalence (a) $\Leftrightarrow$ (b) follows immediately from the definition of $D\mathscr{A}$. The equivalence (b) $\Leftrightarrow$ (c) follows immediately from 1.2(10), since $(C\mathscr{A}, D\mathscr{A})$ is a factorization structure on $\mathscr{K}$.

2.6 COROLLARY. *If $\mathscr{A}$ is a subcategory of $\mathscr{K}$ then the following subcategories of $\mathscr{K}$ coincide:*

(a) *The E-reflective hull of $\mathscr{A}$.*

(b) *The $C\mathscr{A}$-reflective hull of $\mathscr{A}$.*

(c) *The smallest $C\mathscr{A}$-reflective subcategory.*

(d) *The subcategory of $C\mathscr{A}$-injective objects.*

2.7 *Definition.* A factorization structure (C, D) on $\mathscr{K}$ is called *dispersed*, or more precisely *(E, M)-dispersed*, if and only if there exists a subcategory $\mathscr{A}$ of $\mathscr{K}$ such that $C = C\mathscr{A}$ and $D = D\mathscr{A}$.

2.8 *Remark.* 2.3 and 2.4 imply that there exists a bijection between the class of E-reflective subcategories of $\mathscr{K}$ and the class of dispersed factorization structures on $\mathscr{K}$. In particular the factorization structure (E, M) itself is dispersed and corresponds to the smallest E-reflective subcategory of $\mathscr{K}$.

2.9 THEOREM. *If (C, D) is a factorization structure on $\mathscr{K}$ and $C \subset E$ then the following are equivalent:*

(a) *(C, D) is dispersed.*

(b) *If $g \circ f$ belongs to C and f belongs to E then f belongs to C.*

(c) *If $(X, f_i)_I$ is a source in $\mathscr{K}$ which contains a C-morphism then (E, M)- and (C, D)-factorizations of $(X, f_i)_I$ coincide.*

(d) *If $f: X \to Y$ is a $\mathscr{K}$-morphism such that there exists a $\mathscr{K}$-morphism $g: Y \to Z$ and a K-source $(Y, g_i)_I$ with $(X, g_i \circ f)_I$ in D and $g \circ f$ in C, then f belongs to M.*

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(e) *If* $(X, f_i)_I$ *is a D-source and* $f_0: X \to Y_0$ *is a C-morphism then* $(X, f_i)_{I \cup \{0\}}$ *belongs to M.*

Proof. (a) $\Rightarrow$ (b) follows immediately from the definition of $\mathscr{A}$-concentrated morphism.

(b) $\Rightarrow$ (c). Let

$$(X \xrightarrow{f_i} Y_i)_I = (X \xrightarrow{e} Z \xrightarrow{m_i} Y_i)_I$$

be an (E, M)-factorization of $(X, f_i)_I$. Since for some $j \in I$ f_j belongs to C, (b) implies that e belongs to C.

(c) $\Rightarrow$ (d). If $\mathscr{F} = \{g_i \circ f\}_I \cup \{g \circ f\}$ then, by 1.2(7), $(X, \mathscr{F})$ belongs to D. Hence, by (c), $(X, \mathscr{F})$ belongs to M and therefore, according to 1.2(6), f belongs to M.

(d) $\Rightarrow$ (e). Let

$$(X \xrightarrow{f_i} Y_i)_{I \cup \{0\}} = (X \xrightarrow{e} Z \xrightarrow{g_i} Y_i)_{I \cup \{0\}}$$

be an (E, M)-factorization of $(X, f_i)_{I \cup \{0\}}$. By (d) e belongs to M and hence is a $\mathscr{K}$-isomorphism. Consequently $(X, f_i)_{I \cup \{0\}}$ belongs to M.

(e) $\Rightarrow$ (a). Let $\mathscr{A}$ be the C-reflective subcategory of C-injective objects. Obviously each morphism in C is $\mathscr{A}$-concentrated. If $f: X \to Y$ belongs to $C\mathscr{A}$,

$$(X \xrightarrow{f} Y) = (X \xrightarrow{c} Z \xrightarrow{d} Y)$$

is a (C, D)-factorization of f and $s: Z \to Z'$ is an $\mathscr{A}$-reflection then, by (e), $(Z, (d, s))$ belongs to M. Hence (Z, d) belongs to $D\mathscr{A}$ but, since by 1.2(5) d belongs to $C\mathscr{A}$, d is a $\mathscr{K}$-isomorphism and therefore f belongs to C. Hence $C = C\mathscr{A}$ and, by 1.2(8) and 2.3, $D = D\mathscr{A}$.

2.10 *Remark.* If a factorization structure (C, D) is dispersed then $C \subset E$. The converse does not hold (See 2.12 and 3.1).

2.11 THEOREM. *If C is a class of E-morphisms then the following are equivalent:*

(1) $C = C\mathscr{A}$ *for some subcategory* $\mathscr{A}$ *of* $\mathscr{K}$.

(2) *Every (C-injective)-extendable E-morphism belongs to C.*

(3) *The following conditions hold:*

(a) *The subcategory of C-injective objects is C-reflective.*

(b) *If* $g \circ f$ *belongs to C and f belongs to E then f belongs to C.*

(4) *The following conditions hold:*

(a′) *For each* $\mathscr{K}$*-object X there exists a C-morphism* $f: X \to Y$ *with Y C-injective.*

(b) *If* $g \circ f$ *belongs to C and f belongs to E then f belongs to C.*

Proof. (1) $\Rightarrow$ (2). Since $C = C\mathscr{A}$, 1.2(10) and 2.6 imply that the subcategory of C-injective objects is the E-reflective hull of $\mathscr{A}$. Hence, by 2.4, C coincides with the class of $(C$-injective$)$-extendable E-morphisms.

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(2) $\Rightarrow$ (3). If $(X, m_i\colon X \to Y_i)_I$ is an M-source with all Y_i C-injective then, by the (E, M)-diagonalization property of $\mathscr{K}$, X is C-injective. Hence, by 1.2(9), the subcategory of C-injective objects is E-reflective and this, by (2), implies that it is C-reflective.

(b) Follows from (2) and the fact that if $g \circ f$ belongs to C and f belongs to E then f is a (C-injective)-extendable E-morphism.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1). Let $\mathscr{A}$ be the subcategory of C-injective objects. Then $C \subset C\mathscr{A}$. If $f\colon X \to Y$ belongs to $C\mathscr{A}$, by (a′) there exists a C-morphism $g\colon X \to A$ with A C-injective. Hence there exists a $\mathscr{K}$-morphism $k\colon Y \to A$ such that $k \circ f = g$. Therefore, by (b), f belongs to C and this implies that $C = C\mathscr{A}$.

2.12 *Remark.* Obviously (a) implies (a′). To show the independence of (a) and (a′) from (b), consider *Top* supplied with the factorization structure (epimorphisms, extremal monosources). The class C of all quotients satisfies (a) but not (b) and the class C of all epimorphisms with a fixed domain X satisfies (b) but not (a′).

2.13 *Problem.* Is there a nice characterization of those subclasses C of E for which there exists a D such that (C, D) is a factorization structure on $\mathscr{K}$?

2.14 THEOREM. *If D is a conglomerate of sources and $\mathscr{B}$ is the subcategory of $\mathscr{K}$ whose objects are those X in $\mathscr{K}$ for which $(X, \emptyset)$ belongs to D then the following are equivalent:*

(1) $D = D\mathscr{A}$ *for some subcategory* $\mathscr{A}$ *of* $\mathscr{K}$.

(2) *A* $\mathscr{K}$*-source* $(X, \mathscr{F})$ *belongs to* D *if and only if* $(X, \mathscr{F} \cup \mathscr{F}(X, \mathscr{B}))$ *belongs to* M.

Proof. (1) $\Rightarrow$ (2). By 1.2(9) and 2.3 $\mathscr{B}$ is the $C\mathscr{A}$-reflective hull of $\mathscr{A}$ which, by 2.6, coincides with the E-reflective hull of $\mathscr{A}$. Hence, by 2.4, $D = D\mathscr{B}$.

(2) $\Rightarrow$ (1) follows immediately from the definition of $D\mathscr{B}$.

3. Specializations. If $\mathscr{K}$ is supplied with a factorization structure (epimorphisms, extremal monosources) then, for any subcategory $\mathscr{A}$ of $\mathscr{K}$, $\mathscr{A}$-concentrated morphisms coincide with $\mathscr{A}$-extendable epimorphisms, $\mathscr{A}$-dispersed sources coincide with $\mathscr{A}$-*perfect* sources and dispersed factorization structures on $\mathscr{K}$ are called *perfect*. Perfect factorization structures have been well investigated. See for example [**4**], [**5**], [**6**], [**8**], [**9**], [**13**], [**14**], [**21**], [**22**] and in particular the survey paper by G. E. Strecker [**24**].

If $\mathscr{K}$ is supplied with a factorization structure (extremal epimorphisms, monosources) and $\mathscr{A}$ is a subcategory of $\mathscr{K}$ then $\mathscr{A}$-concentrated morphisms will be called $\mathscr{A}$-*concordant*, $\mathscr{A}$-dispersed sources will be called $\mathscr{A}$-*dissonant* and dispersed factorization structures will be called *dissonant*. Dissonant factorizations have been investigated previously by G. Salicrup and R. Vázquez [**20**] and independently by G. Preuss [**16**] in topological categories and even earlier by G. Salicrup and R. Vázquez [**19**] in *Top*. If $\mathscr{K}$ is a topological category

and $\mathscr{A}$ is a quotient reflective subcategory of $\mathscr{K}$ then for each $\mathscr{K}$-object X, the fibres of the $\mathscr{A}$-reflection $r\colon X \to X'$ are called the *$\mathscr{A}$-quasicomponents* of X. In this case a quotient $f\colon X \to Y$ is $\mathscr{A}$-concordant if and only if each fibre of f is contained in an $\mathscr{A}$-quasicomponent of X. G. Preuss [**16**], has shown that this last statement holds if and only if for each $\mathscr{A}$-quasicomponent Y' of Y, $f^{-1}(Y')$ is an $\mathscr{A}$-quasicomponent of X. A map $f\colon X \to Y$ is $\mathscr{A}$-dissonant if and only if each fibre of f intersects each $\mathscr{A}$-quasicomponent of X in at most one point. $\mathscr{A}$-dissonant sources are all the compositions of $\mathscr{A}$-dissonant morphisms with monosources. In particular, if $\mathscr{K} = Top$ and $\mathscr{A}$ is the quotient reflective hull of the discrete two point space then $\mathscr{A}$-quasicomponents are precisely quasicomponents and the factorization structure ($\mathscr{A}$-concordant, $\mathscr{A}$-dissonant) is the (concordant, dissonant)-factorization of P. Collins [**2**]. If $\mathscr{A}$ is the quotient reflective subcategory of Top whose objects are all the totally disconnected spaces, then $\mathscr{A}$-quasicomponents coincide with components and the factorization structure ($\mathscr{A}$-concordant, $\mathscr{A}$-dissonant) is the (submonotone quotient, superlight) factorization of G. E. Strecker [**23**].

If C is a class of epimorphisms in Top, $f\colon X \to Y$ is called *hereditarily-C* if and only if for each open set $B \subset Y$ the restriction

$$f|^{B}_{f^{-1}(B)}\colon f^{-1}(B) \to B$$

belongs to C. P. Collins and R. Dyckhoff [**3**] have shown that for each factorization structure (C, D) in Top, there exists D' such that (hereditarily-C, D') is a factorization structure in Top.

3.1 THEOREM. *Let $\mathscr{A}$ be a quotient reflective subcategory of Top such that $\mathscr{A}$ is different from Top, from T_0 and from the smallest quotient reflective subcategory. Then the factorization structure (C, D) in Top, in which C is the class of hereditarily $\mathscr{A}$-extendable quotients, is neither perfect nor dissonant.*

Proof. Since $\mathscr{A}$ is different from Top and from T_0, $\mathscr{A} \subset T_1$. Let X be the space with three points a, b, c and topology generated by $\{\{a\}, \{c\}\}$, let $f\colon X \to Y$ be the quotient obtained by identifying a and c, and let $g\colon Y \to T$ be the unique map from Y onto a singleton space T. Then f is a quotient and $g \circ f$ is a hereditarily $\mathscr{A}$-extendable quotient. Since $\mathscr{A}$ is different from the smallest quotient reflective subcategory of Top, the restriction $f|^{B}_{f^{-1}(B)}$, where $B = \{f(a)\}$, is not $\mathscr{A}$-extendable. Hence f is not hereditarily $\mathscr{A}$-extendable. By 2.9 this implies that (C, D) is neither perfect nor dissonant.

3.2 THEOREM. *In Top the following hold:*

(1) *There exists a proper class of perfect factorization structures and a proper class of dissonant factorization structures.*

(2) *There exists a non-legitimate collection (in the sense of* [**1**]*) of perfect factorization structures and a non-legitimate collection of dissonant factorization structures.*

(3) *There exists a proper class of factorization structures which are neither perfect nor dissonant.*

(4) *There are exactly two factorization structures which are simultaneously dissonant and perfect.*

(a) *The factorization structure* (*isomorphisms, sources*).

(b) *The factorization structure* (T_0*-extendable epimorphisms,* T_0*-perfect sources*) = (*quotients with indiscrete fibres, sources whose multiple fibres are* T_0).

Proof. (1) G. Salicrup and R. Vázquez [**18**] have shown that *Top* has a proper class of right constant subcategories. Since every right constant subcategory in *Top* is quotient reflective, by 2.8 this implies that there exists a proper class of perfect and a proper class of dissonant factorization structures in *Top*.

(2) V. Kannan and M. Rajagopalan [**12**] have shown that there exists a a proper class K of compact T_2-spaces such that the only non-constant continuous mappings between members of K are the identities (see also [**26**]). Consequently, if for each $S \subset K$ $Q(S)$ denotes the quotient reflective hull of S, $\{Q(S) | S \subset K\}$ is a collection of quotient-reflective subcategories of *Top* that is not even in one-to-one correspondence with any class; hence non-legitimate. Hence, by 2.8, there exists a non-legitimate collection of perfect and a non-legitimate collection of dissonant factorization structures in *Top*.

(3) For every right constant subcategory $\mathscr{A}$ of *Top* let $C(\mathscr{A})$ be the class of all hereditarily $\mathscr{A}$-extendable quotients. By [**3**] there exists a conglomerate $D(\mathscr{A})$ of sources such that $(C(\mathscr{A}), D(\mathscr{A}))$ is a factorization structure in *Top*. By 2.13 of [**19**] every $\mathscr{A}$-reflection map belongs to $C(\mathscr{A})$, which implies that the correspondence $\mathscr{A} \mapsto C(\mathscr{A})$ is one to one. Since, by [**18**], there exists a proper class of right constant subcategories of *Top*, the result follows from 3.1.

(4) It is easy to verify that factorization structures given in (a) and (b) are simultaneously perfect and dissonant. To show that they are the only ones with this property, let $\mathscr{A}$ be a quotient reflective subcategory of *Top* such that $\mathscr{A}$ is different from *Top* and from T_0. Hence $\mathscr{A} \subset T_1$. Let X be the Sierpinski space, Y the two point indiscrete space and Z a one point space. If $f\colon X \to Y$ is an epimorphism and $g\colon Y \to Z$ then $g \circ f$ is $\mathscr{A}$-concordant, but f is not a quotient. Hence f is not $\mathscr{A}$-concordant and consequently, by 2.9, the factorization structure ($\mathscr{A}$-concordant, $\mathscr{A}$-dissonant) is not perfect.

3.3 *Remark.* As opposed to the above, the category *Set* has only two factorization structures at all [**11**].

4. The category of reflexive relations.

4.1 *Definitions and notation.* (1) *Rere* is the category whose objects are pairs (X, ρ) with X a set and ρ a reflexive relation on X and whose morphisms $f\colon (X, \rho) \to (Y, \sigma)$ are relation preserving maps $f\colon X \to Y$.

(2) For each natural number $n \geqq 2$, S_n denotes the object (X, ρ) in *Rere* such that $X = \{1, \ldots, n\}$ and ρ is the reflexive relation generated by

$$\{(i, i+1) | 1 \leqq i \leqq n-1\} \cup \{(n, 1)\}.$$

(3) An *n-cycle* on an object (X, ρ) in *Rere* is a monomorphism $m: S_n \to (X, \rho)$, and will be usually denoted by its image $(x_1, \ldots, x_n)$.

(4) An object (X, ρ) in *Rere* is *cycle-free* if and only if there are no n-cycles on (X, ρ) for $n \geqq 2$.

(5) B_∞ denotes the subcategory of *Rere* whose objects are all those (X, ρ) which are cycle-free. For each $n \geqq 3$, B_n denotes the subcategory of *Rere* whose objects are all those (X, ρ) which have no m-cycles for $2 \leqq m < n$. In particular, B_3 consists of the antisymmetric reflexive relations.

4.2 *Remarks.* (1) *Rere* is a properly fibred topological category and hence is endowed with a factorization structure (quotients, monosources).

(2) For any object (X, ρ) in *Rere*, ρ is discrete if and only if ρ is the diagonal on $X \times X$ and ρ is indiscrete if and only if $\rho = X \times X$.

4.3 *Notation.* For each natural number $n \geqq 3$ and each cardinal $k \geqq 2$, $X_{n,k}$ denotes the set

$$(\{\alpha | \alpha \text{ ordinal}, 0 < \alpha < k\} \times \{1, 2\}) \cup \{1, \ldots, n-2\},$$

$\rho_{n,k}$ denotes the reflexive relation on $X_{n,k}$ generated by

$$\{((\alpha, i), (\alpha', i') | \alpha < \alpha' \quad \text{or} \quad (\alpha = \alpha' \quad \text{and} \quad i < i')\} \cup \{(i, i+1)) | 1 \leqq i \leqq n-3\} \cup \{((\alpha, 2), 1) | \alpha < k\} \cup \{(n-2, (\alpha, 1)) | \alpha < k\}$$

and $T_{n,k}$ denotes the object $(X_{n,k}, \rho_{n,k})$ in *Rere*.

4.4 Proposition. *For any natural $n \geqq 3$ and each cardinal $k \geqq 2$, if $f: T_{n,k} \to B$ is a morphism in Rere and B belongs to B_n then f is either injective or constant.*

Proof. Consider first the case $n > 3$. Since any $y \in X_{n,k}$ belongs to some n-cycle $((\alpha, 1), (\alpha, 2), 1, \ldots, n-2)$, it suffices to prove that if f is not injective then $f|\{1, \ldots, n-2\}$ is constant. Let x, x' be two different points of $X_{n,k}$ such that $f(x) = f(x')$. If $\{x, x'\} = \{(\alpha, 2), (\alpha', 2)\}$ with $\alpha < \alpha'$ then $f|\{(\alpha, 2), (\alpha', 1), (\alpha', 2)\}$ has to be constant, hence $f|\{(\alpha', 1), (\alpha', 2), 1, \ldots, n-2\}$ is constant. If $\{x, x'\} = \{(\alpha, 1), (\alpha', 1)\}$ with $\alpha < \alpha'$, then analogously $f|\{(\alpha, 1), (\alpha, 2), (\alpha', 1)\}$ and hence $f|\{(\alpha, 1), (\alpha, 2), 1, \ldots, n-2\}$ have to be constant. If $\{x, x'\} = \{(\alpha, 2), (\alpha', 1)\}$ with $\alpha < \alpha'$ then $f(\{(\alpha, 2), (\alpha', 1), 1, \ldots, n-2\})$ has to be a point because otherwise it would contain an m-cycle for some $2 \leqq m < n$. If $\{x, x'\}$ has neither of the forms considered above, then $\{x, x'\}$ has to be contained in an n-cycle $((\alpha, 1), (\alpha', 2), 1, \ldots, n-2)$, hence $f|((\alpha, 1), (\alpha', 2), 1, \ldots, n-2)$ is constant. In any of the three cases, $f|\{1, \ldots, n-2\}$ is constant. The proof for $n = 3$ is similar.

4.5 *Notation.* (1) For any subcategory $\mathscr{A}$ of *Rere*, $Q(\mathscr{A})$ denotes the quotient reflective hull of $\mathscr{A}$. If $|\mathscr{A}| = \{X\}$ then $Q(X)$ denotes $Q(\mathscr{A})$.

(2) $(1, 2)$ denotes the object (X, ρ) in *Rere* with $X = \{1, 2\}$ and ρ discrete.

(3) $(1 \to 2)$ denotes the object (X, ρ) in *Rere* with $X = \{1, 2\}$ and $\rho = \{(1, 1), (2, 2), (1, 2)\}$.

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(4) $\mathscr{D}$ denotes the subcategory of all discrete objects in *Rere*.

(5) $\mathscr{S}$ = {subterminal} denotes the subcategory of all those (X, ρ) in *Rere* with $|X| \leqq 1$.

(6) A quotient reflective subcategory of *Rere* is *trivial* if and only if it coincides with $\mathscr{S}$, $\mathscr{D}$ or *Rere*.

4.6 THEOREM. *The following hold in Rere:*

(1) $Rere = Q(S_2) \supsetneq Q(S_3) \supsetneq \ldots \supsetneq \bigcap_2^\infty Q(S_n) = B_\infty = Q((1 \to 2)) \supsetneq \mathscr{D} = Q((1, 2)) \supsetneq \mathscr{S} = Q(\emptyset)$.

(2) *For each* $n \geqq 3$ B_n *is quotient reflective,* $Q(S_n) \subset B_n$ *and* {*antisymmetric*} $= B_3 \supsetneq B_4 \supsetneq \ldots \supsetneq \bigcap_3^\infty B_n = B_\infty$.

(3) *If* $\mathscr{B}$ *is a non-trivial quotient reflective subcategory of Rere then* $B_\infty \subset \mathscr{B}$, *and if* $B_\infty \subsetneq \mathscr{B}$ *and* n *is the least natural for which there exists an* n*-cycle* $m: S_n \to B$ *with* $B \in \mathscr{B}$, *then* $n \geqq 3$ *and* $Q(S_n) \subset \mathscr{B} \subset B_n$.

(4) *If* $n \geqq 3$ *then* $\{Q(T_{n,k}) | k$ *cardinal,* $k > 2\}$ *is a proper, linearly ordered class such that, if* $k < k'$ *then*

$$Q(S_n) = Q(T_{n,2}) \subsetneq Q(T_{n,k}) \subsetneq Q(T_{n,k'}) \subsetneq B_n.$$

(5) *For any* $n \geqq 3$, $B_n \neq Q(B)$ *for each* $B \in B_n$.

Proof. (1) Since S_2 is indiscrete $Q(S_2) = Rere$. For each $n \geqq 2$ the map $f: S_{n+1} \to S_n \times S_n$ such that $f(i) = (i, i)$ for $1 \leqq i \leqq n-1$, $f(n) = (n, n-1)$ and $f(n+1) = (n, n)$ is a monomorphism in *Rere*, which implies $S_{n+1} \in Q(S_n)$ and hence $Q(S_{n+1}) \subset Q(S_n)$. Since any morphism $f: S_n \to S_{n+1}$ is constant, $S_n \in Q(S_n) \backslash Q(S_{n+1})$. Hence $Q(S_n) \supsetneq Q(S_{n+1})$ and $\bigcap_2^\infty Q(S_n) \subset B_\infty$. If (X, ρ) belongs to B_∞ then there exists a monosource $((X, \rho), m_i: (X, \rho) \to (1 \to 2))_I$ and for each $n \geqq 2$ the inclusion $(1 \to 2) \to S_n$ is a morphism in *Rere*. Therefore $\bigcap_2^\infty Q(S_n) = B_\infty = Q((1 \to 2))$.

(2) If $((X, \rho), m_i: (X, \rho) \to (Y_i, \beta_i))_I$ is a monosource with all (Y_i, β_i) in B_n then (X, ρ) belongs to B_n, hence B_n is quotient reflective. For each $n \geqq 3$, S_n belongs to $B_n \backslash B_{n+1}$, therefore $Q(S_n) \subset B_n \supsetneq B_{n+1}$.

(3) If $(X, \rho) \in \mathscr{B} \backslash \mathscr{D}$ then there exists a monomorphism $m: (1 \to 2) \to (X, \rho)$. Therefore $Q(1 \to 2) = B_\infty \subset \mathscr{B}$. If (X, ρ) belongs to $\mathscr{B}$ and $m: S_n \to (X, \rho)$ is a monomorphism then $Q(S_n) \subset \mathscr{B}$ and, since $\mathscr{B}$ is not trivial, this implies that $n \geqq 3$. Obviously $\mathscr{B} \subset B_n$.

(4) S_n is isomorphic to $T_{n,2}$ and obviously $T_{n,k} \in B_n$ for each $n \geqq 3$. If $2 \leqq k < k'$ then the inclusion $X_{n,k} \to X_{n,k'}$ defines a monomorphism $m: T_{n,k} \to T_{n,k'}$ in *Rere*. Hence $Q(T_{n,k}) \subset Q(T_{n,k'})$. According to 4.4, each morphism $f: T_{n,k'} \to T_{n,k}$ is constant, hence

$$T_{n,k'} \in Q(T_{n,k'}) \backslash Q(T_{n,k}).$$

(5) If $B \in B_n$ let k be a cardinal such that $k > |B|$. Hence, by 4.4, any morphism $f: T_{n,k} \to B$ is constant and this implies that $T_{n,k} \in B_n \backslash Q(B)$.

The following diagram depicts the results of 4.6.

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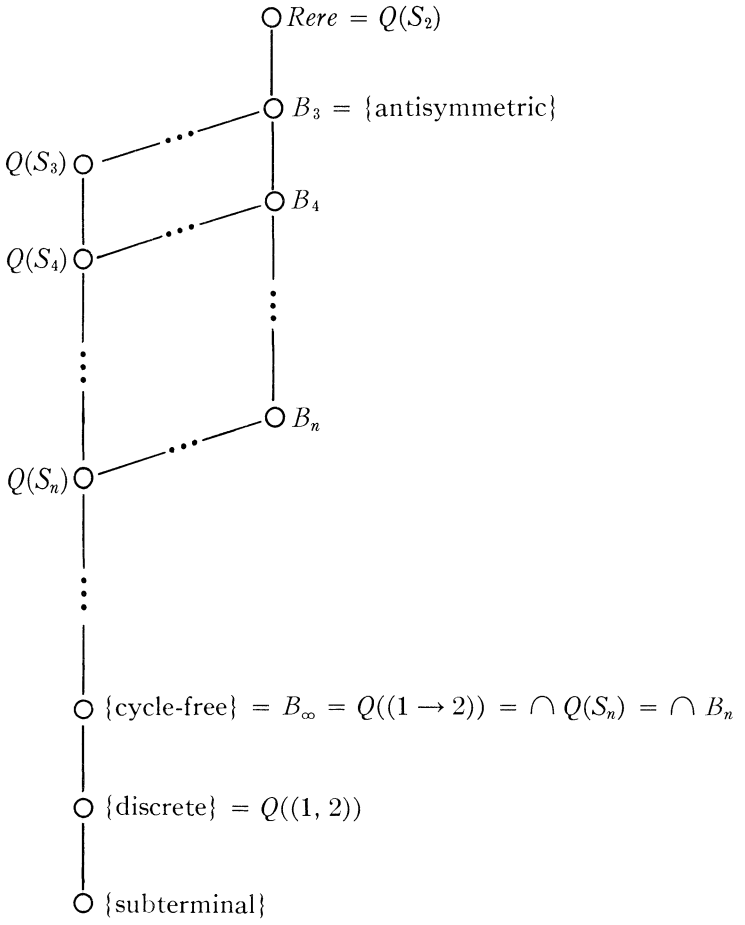

4.7 THEOREM. *In Rere the following hold:*

(1) *Rere has a proper class of dissonant factorization structures.*

(2) *Rere has a proper class of perfect factorization structures.*

(3) *(isomorphisms, sources) is the only factorization structure on Rere which is simultaneously dissonant and perfect.*

Proof. (1) and (2) follow immediately from 2.8 and 4.6(4).

(3) Let $\mathscr{A}$ be a quotient reflective subcategory of *Rere* different from *Rere* itself. Then, by 4.6(1), $\mathscr{A} \subset \{\text{antisymmetric}\}$. If X is the three point set $\{1, 2, 3\}$, ρ is the reflexive relation on X generated by $(1, 2)$, $(2, 1)$, $(2, 3)$, $(3, 2)$, σ is the indiscrete reflexive relation on X and (A, α) is a one point object in *Rere*, then the composition

$$(X, \rho) \xrightarrow{1_X} (X, \sigma) \xrightarrow{g} (A, \alpha)$$

is $\mathscr{A}$-concordant but 1_X is not a quotient. Hence, by 2.9, the factorization structure ($\mathscr{A}$-concordant, $\mathscr{A}$-dissonant) in *Rere* is not perfect.

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