Can. J. Math. Vol. **50** (4), 1998 pp. 739–755

EIGENPOLYTOPES OF DISTANCE REGULAR GRAPHS

C. D. GODSIL

ABSTRACT. Let X be a graph with vertex set V and let A be its adjacency matrix. If E is the matrix representing orthogonal projection onto an eigenspace of A with dimension m, then E is positive semi-definite. Hence it is the Gram matrix of a set of $|V|$ vectors in R^m. We call the convex hull of a such a set of vectors an eigenpolytope of X. The connection between the properties of this polytope and the graph is strongest when X is distance regular and, in this case, it is most natural to consider the eigenpolytope associated to the second largest eigenvalue of A. The main result of this paper is the characterisation of those distance regular graphs X for which the 1-skeleton of this eigenpolytope is isomorphic to X.

1. **Introduction.** Let X be a graph with vertex set V and adjacency matrix A. Let θ be an eigenvalue of A with multiplicity m and let U_θ be a matrix whose columns form an orthonormal basis for the eigenspace of A belonging to θ. If $u \in V$, define $u(\theta)$ to be the row of U_θ indexed by u. The *eigenpolytope* of X belonging to θ is defined to be the convex hull of the vectors $u(\theta)$, where u ranges over the vertices of X. This definition is dependent on the orthonormal basis chosen for the eigenspace but the inner product $\langle u(\theta), v(\theta)\rangle$ is independent of this choice, and this is all that matters for us.

If E_θ denotes the matrix representing orthogonal projection onto the eigenspace belonging to θ then $E_\theta = U_\theta U_\theta^T$, hence

$$\langle u(\theta), v(\theta)\rangle = (E_\theta)_{u,v}$$

If $u \in V$, let e_u be the vector in $\mathbf{R}^V$ which is 1 on u and 0 elsewhere. We have

$$\langle u(\theta), v(\theta)\rangle = e_u^T E_\theta^T E_\theta e_v = e_u^T E_\theta^2 v(\theta) = u(\theta) E_\theta v(\theta) = (E_\theta)_{u,v}.$$

Since E_θ is a polynomial in A, this implies that $\langle u(\theta), v(\theta)\rangle$ is determined by θ and the numbers $(A^r)_{u,v}$ for $r \geq 0$. In other words, it is determined by the number of walks in X from u to v with length r, for all non-negative integers r. Therefore the geometry of an eigenpolytope of X is related to the structure of X.

An eigenpolytope has at least one property not shared by polytopes in general. If u and v are vertices of X then we write $u \sim v$ to denote that u and v are adjacent. Because A is a 01-matrix and $\theta U_\theta = AU_\theta$, we easily derive the following condition.

Received by the editors September 28, 1995; revised April 16, 1998.

Support from a National Sciences and Engineering Council of Canada operating grant is gratefully acknowledged.

AMS subject classification: 05E30, 05C50.

https://doi.org/10.4153/CJM-1998-040-8 Published online by Cambridge University Press

LEMMA 1.1. *Let X be a graph and let θ be an eigenvalue of its adjacency matrix. Then, for each vertex u of X,*

$$\theta u(\theta) = \sum_{v \sim u} v(\theta). \qquad \blacksquare$$

Powers and Licata [11] call a polytope *self-reproducing* if it has the property expressed by this lemma.

In this paper we will only consider eigenpolytopes of distance regular graphs. For the basic notation and theory of the latter, see [3] (or even [7]). We do recall that if X is distance-regular with diameter d then for $i = 0, \ldots, d$ there are constants c_i, a_i and b_i such that if u and v are vertices in X at distance i then the number of neighbours of v at distances $i-1$, i and $i+1$ from u is c_i, a_i and b_i respectively. This implies that if X is distance-regular then it is regular with valency b_0. Following tradition we will usually denote b_0 by k. We observe also that $a_0 = b_d = 0$ and $c_1 = 1$. (In practice we will be most concerned with a_1, b_1 and c_2.)

If X is distance-regular with vertex set V and diameter d, let X_i be the graph with vertex set V, with two vertices adjacent if and only if there are at distance i in X. Thus $X_1 = X$ and the edge sets of the graphs X_i partition the edge set of the complete graph on V. If we define A_0 to be the identity matrix I and $A_i := A(X_i)$ then

$$\sum_i A_i = J.$$

(Here, as usual, J is the matrix with all entries equal to one.) It can be shown that A_i can be written as a polynomial of degree i in A, and that every polynomial in A is a linear combination of the matrices $A_0, \ldots, A_d$. Consequently, if θ is an eigenvalue of A and u and v are vertices of X then $(E_\theta)_{u,v}$ is determined by the distance between u and v in X. For eigenpolytopes, this has the following consequence.

LEMMA 1.2. *Let X be a distance-regular graph with vertex set V and adjacency matrix A, and let θ be an eigenvalue of A. If u and v are vertices of X then $\langle u(\theta), v(\theta)\rangle$ is determined by the distance between u and v in X.* $\blacksquare$

This implies that the length of $u(\theta)$ is independent of the choice of the vertex u, and hence that its length is $m/|V|$, where m is the multiplicity of θ. Thus the vertices of an eigenpolytope of a distance regular graph all lie on a sphere in $\mathbf{R}^m$ centred at the origin.

If X is distance regular with diameter d then it has exactly $d+1$ distinct eigenvalues, which we will denote by $\theta_0, \ldots, \theta_d$, in decreasing order. If X has valency k then $\theta_0 = k$ and θ_0 is simple (because X is connected). For reasons which we present later, the eigenpolytope belonging to θ_1 is particularly interesting. One of the main results of this paper is a characterisation of the distance regular graphs X such that the 1-skeleton of the θ_1-eigenpolytope is isomorphic to X. Before we can do this we need to establish some of the basic theory of convex polytopes (and explain words such as '1-skeleton'). This is the task of the next section.

2. **Polytopes.** Brøndsted's book [2] is a convenient reference for most of our polytopal needs. We do assume some familiarity with the elements of the theory of convex sets.

A convex polytope is defined to be the convex hull of a finite set of points. The *dimension* of a polytope is the dimension of the smallest affine space which contains all its points; we will often refer to a polytope of dimension m as an *m-polytope*. A 0-polytope is a complicated name for a point.

An affine hyperplane H is a *supporting hyperplane* for a polytope $\mathcal{P}$ if it contains at least one point of $\mathcal{P}$ and all points of $\mathcal{P}$ not on H lie on the same side of H. A *face* of $\mathcal{P}$ is any set of points $\mathcal{P} \cap H$, where H is a supporting hyperplane. Any face is itself a convex polytope, and a face of a face of $\mathcal{P}$ is a face of $\mathcal{P}$. (These facts may seem entirely obvious, but they require proof.) There is an alternative definition of faces which will be useful. Suppose that $\mathcal{P}$ is a polytope in $\mathbf{R}^m$. The set of points in $\mathcal{P}$ at which a linear functional on $\mathbf{R}^m$ takes its maximum value is a face, and all faces can be obtained in this way. Less formally, if $h \in \mathbf{R}^m$ then the points x in $\mathcal{P}$ such that $h^T x$ is maximal form a face of $\mathcal{P}$. It is not too hard to see that these two definitions of faces are equivalent. An r-face is a face which has dimension r. A 0-face is usually called a *vertex* and a 1-face is called an *edge*. An $(m-1)$-face of an m-polytope is a *facet*. The vertices and edges of a polytope form a graph, which is the 1-*skeleton* of the polytope.

THEOREM 2.1. *If X is the* 1*-skeleton of an m-polytope $\mathcal{P}$ and C is a cutset in X then the vertices in C span an affine hyperplane, and hence $|C| \geq m$.*

PROOF. Let C be a subset of the vertices of X which does not span $\mathbf{R}^m$. If C is contained in a face of P then $X \setminus C$ is connected by [2: Theorem 15.5]. Otherwise there is a hyperplane containing C and at least one other vertex of P. The proof of Theorem 15.6 from [2] now yields that $X \setminus C$ is connected. ∎

Balinski [1] proved that the 1-skeleton of an m-polytope is m-connected; this is proved in [2] as Theorem 15.6. Thus Theorem 2.1 is essentially a reformulation of this result, and we will also make use of it in this form. In either form this result implies that the 1-skeleton of an m-polytope has minimum valency at least m. We note one simple consequence of this.

LEMMA 2.2. *Suppose X is distance regular with valency k, let θ is an eigenvalue with multiplicity m and let $\mathcal{P}$ be the associated eigenpolytope. If $k < m$ then X is not isomorphic to the* 1*-skeleton of $\mathcal{P}$.*

PROOF. Theorem 2.1 implies that the 1-skeleton of $\mathcal{P}$ is m-connected, and therefore its minimum valency is at least m. ∎

A polytope is *simplicial* if every face is a simplex. An m-polytope is *simple* if every k-face lies in exactly $m-k$ facets. There is a more intuitive characterisation, given as Theorem 12.12 in [2].

THEOREM 2.3. *An m-polytope is simple if and only if its 1-skeleton is regular of valency m.* ∎

THEOREM 2.4. *Let $\mathcal{P}$ be a simple polytope. Then:*
(a) Every face of $\mathcal{P}$ is simple.
(b) Suppose u and $v_1, \ldots, v_k$ are vertices of $\mathcal{P}$ such that uv_i is an edge of $\mathcal{P}$ for $i = 1, \ldots, k$, and let F be the smallest face of $\mathcal{P}$ containing u and the vertices v_i. Then F has dimension k and the edges uv_i are the only edges in F on u.

PROOF. See Theorem 12.15 and 12.17 respectively from [2]. ∎

Suppose that A is the adjacency matrix of the graph X and θ is an eigenvalue of A with multiplicity m. Let $\mathcal{P}$ be the eigenpolytope of X belonging to θ and let h be a vector in $\mathbf{R}^m$. Then the function which maps u in V onto $\langle h, u(\theta)\rangle$ is an eigenvector of A with eigenvalue θ and each eigenvector of A with eigenvalue θ can be obtained in this way. As noted by Powers [14], the vertices on which an eigenvector assumes its maximum value form a face of $\mathcal{P}$. In particular if there is an eigenvector equal to 1 on u and v and less than one on all other vertices of X then uv is an edge in $\mathcal{P}$. We will make use of this later.

As also noted by Powers [14], equitable partitions can be used to derive information about the faces of eigenpolytopes. We explain this. If V is the vertex set of X and π is a partition of V, let $F(\pi)$ denote the vector space of all functions on V which are constant on the cells of π. Call π *equitable* if $F(\pi)$ is A-invariant. If π is an equitable partition of X then $F(\pi)$ contains eigenvectors for A, each of which must be constant on the cells of π. Therefore at least two cells of π are faces of some eigenpolytope of X. (This will of course still be true if we assume only that $F(\pi)$ contains an eigenvector of A, but I have found no use for this generality yet.) If $S \subseteq V$ then the *distance partition* of X relative to S is the partition with cells C_i, $i = 0, \ldots, r$ say, where C_i is the set of vertices of X at distance i from S. (So $C_0 = S$.) A subset is *completely regular* if its distance partition is equitable. Any vertex in a distance regular graph is a completely regular subset.

For an introduction to equitable partitions see [7: Section 5.1] and [8, 9]. Completely regular subsets are discussed in [3: Section 11.1] and [7: Section 11.7].

3. **Cosines.**

Let X be a distance regular graph with diameter d and let θ be an eigenvalue of its adjacency matrix. If u and v are vertices of X at distance i, let w_i be the cosine of the angle between the vectors $u(\theta)$ and $v(\theta)$. The existence of the cosines w_i for $i = 0, \ldots, d$ is a consequence of Lemma 1.2. In this section we summarise some of the properties of these cosines, and their geometric consequences. Not surprisingly, the treatment in this section follows [7: Chapter 13], and most of what we discuss will also be found in [3: Chapters 3 and 4].

First, however, there is a point that we have glossed over. The mapping

$$u \in V \longmapsto u(\theta)$$

need not be injective, even when X is distance regular. Note that this mapping is injective if and only $w_i = 1$ implies $i = 0$. An extreme example arises if we take θ to be the

https://doi.org/10.4153/CJM-1998-040-8 Published online by Cambridge University Press

valency, k say, of X. Then θ is simple, the all-ones vector $\mathbf{1}$ is an eigenvector and $w_i = 1$ for all i. If X is bipartite then $-k$ is an eigenvalue and, in this case, $w_i = (-1)^i$. These difficulties can be avoided by not using the eigenpolytopes belonging to the eigenvalues k and $-k$, but this is not enough. However it can be shown that if $|\theta| \neq k$ and $w_i = 1$ then i must be the diameter of X, and X must be antipodal. A distance regular graph X which is antipodal of diameter two is a complete multipartite graph. Here $\theta_1 = 0$, the corresponding eigenvectors are constant on the colour-classes of X and the eigenpolytope is a simplex.

In this paper we will concentrate on the eigenpolytopes associated to the second-largest eigenvalue θ_1 of a distance regular graph. For this eigenvalue there are no difficulties with injectivity. To see this, define a *sign-change* in a sequence $w_0, \ldots, w_d$ of non-zero real numbers to be an index i such that $w_{i-1}w_i < 0$. The number of sign-changes in a sequence with terms equal to 0 will be defined to be the number of sign-changes in the sequence obtained by deleting all terms that are 0. For one proof of the next result, see [7: Lemma 13.2.1].

LEMMA 3.1. *Let X be a distance regular graph with eigenvalues $\theta_0, \ldots, \theta_d$. Then the sequence of cosines $w_0, \ldots, w_d$ for θ_i has exactly i sign-changes and, if $i > 0$, the sequence of differences $w_0 - w_1, \ldots, w_{d-1} - w_d$ has exactly $i - 1$ sign-changes.* ∎

COROLLARY 3.2. *Let X be a distance regular graph with eigenvalues $\theta_0, \ldots, \theta_d$. Then the cosine sequence for θ_1 is non-increasing.* ∎

This shows that the Euclidean distance between two vertices of the θ_1-eigenpolytope is a non-decreasing function of the graphical distance between the corresponding vertices of X.

Suppose now that u and y are vertices at distance i in X. Taking the inner product with $u(\theta)$ of the equation

$$\theta y(\theta) = \sum_{z \sim y} z(\theta)$$

and dividing by $\|u(\theta)\|^2$, we obtain

$$\theta w_i = c_i w_{i-1} + a_i w_i + b_i w_{i+1}. \tag{3.1}$$

We set $w_{-1} = w_{d+1} = 0$, so this identity holds for $i = 0, \ldots, d$. One consequence of this is a three-term recurrence for w_i:

$$w_{i+1} = \frac{1}{b_i}[(\theta - a_i)w_i - c_i w_{i-1}]. \tag{3.2}$$

This implies that if $w_i = 0$ then $w_{i-1}w_{i+1} < 0$, and that $w_d \neq 0$.

Our next task is to present some more specific information about the cosines w_1 and w_2. The recurrence (3.2) yields immediately that

$$w_1 = \frac{\theta}{k}, \quad w_2 = \frac{1}{kb_1}(\theta^2 - a_1\theta - k). \tag{3.3}$$

https://doi.org/10.4153/CJM-1998-040-8 Published online by Cambridge University Press

As w_1 is a cosine, $|w_1| \leq 1$ and hence we deduce from the above expression for w_1 that $|\theta| \leq k$. (This is, of course, well known.) Recalling that $b_1 = k - a_1 - 1$, we obtain from these identities that

$$1 - w_2 = (1 - w_1)\frac{b_1 + \theta + 1}{b_1}. \tag{3.4}$$

As w_2 is a cosine, $|w_2| \leq 1$, and therefore (3.4) implies that $\theta \geq a_1 - k$. (Although we will not need this bound.) We also find that

$$w_1 - w_2 = (1 - w_1)\frac{\theta + 1}{b_1}, \tag{3.5}$$

with the consequence that $w_1 > w_2$ when $\theta > -1$. Together the last two equations imply that

$$1 - 2w_1 + w_2 = (1 - w_1)\frac{b_1 - \theta - 1}{b_1}. \tag{3.6}$$

To complete this section, we derive some information about the 1-skeletons of eigenpolytopes. Let θ be an eigenvalue of the distance regular graph X with diameter d and let $w_0, \ldots, w_d$ be its sequence of cosines. If $u \in V(X)$ we define the *standard eigenvector* for θ relative to u to be the vector with v-entry equal to the cosine of the angle between $u(\theta)$ and $v(\theta)$. Part (a) of the next result is taken from [3: Theorem 4.4.9]. If $u \in V(X)$ then $X(u)$ denotes the set of vertices in X adjacent to u.

THEOREM 3.3. *Let X be a distance regular graph with diameter d, let θ be an eigenvalue of X with cosine sequence $w_0, \ldots, w_d$ and assume that $w_1 \geq w_i$ if $i > 1$. Let $\mathcal{P}$ be the eigenpolytope for θ.*

(a) If X contains an induced C_4 then $1 - 2w_1 + w_2 \geq 0$.
(b) If u and v are adjacent vertices in X then uv is an edge of $\mathcal{P}$.
(c) If $1 - 2w_1 + w_2 > 0$ and u and v are at distance two in X then uv is an edge of $\mathcal{P}$.
(d) If $1 - 2w_1 + w_2 = 0$ and u and v are at distance two in X then each vertex in $X(u) \cap X(v)$ is not adjacent to at most one other vertex in $X(u) \cap X(v)$. Further, uv is an edge of $\mathcal{P}$ if and only if $X(u) \cap X(v)$ is a clique.

PROOF. Suppose u, a, v and b induce a copy of C_4, with u not adjacent to v and a not adjacent to b. Then the squared length of the vector $u(\theta) + v(\theta) - a(\theta) - b(\theta)$ is $4\langle u(\theta), u(\theta)\rangle(1 - 2w_1 + w_2)$, whence (a) follows.

Note that, if equality holds, then $u(\theta) + v(\theta) = a(\theta) + b(\theta)$ and so the line through u and v has a point in common with the line through a and b. Hence these lines are coplanar and therefore any supporting hyperplane of $\mathcal{P}$ that contains u and v must contain a and b. Accordingly uv cannot be an edge of $\mathcal{P}$. We will use this in proving (d).

Let z_u and z_v be the standard eigenvectors for θ relative to u and v respectively.

If $u \sim v$ then the u and v entries of $z_u + z_v$ are both equal to $1 + w_1$, and any other entry is at most $2w_1$. Therefore $\{u, v\}$ is the set of vertices on which $z_u + z_v$ takes its maximum value, and so uv is an edge in $\mathcal{P}$.

If u and v are at distance two in X, the u and v entries of $z_u + z_v$ are equal to $1 + w_2$, while any other entry is at most $2w_1$. It follows that uv is an edge of $\mathcal{P}$ when $1 - 2w_1 + w_2 > 0$.

Assume now that $1 - 2w_1 + w_2 = 0$ and u and v are at distance two. Then

$$\{u, v, X(u) \cap X(v)\}$$

is the set of vertices on which $z_u + z_v$ takes its maximum value, hence this set is a face of $\mathcal{P}$. If a and b are common neighbours of u and v which are not adjacent then u, a, v and b induce a copy of C_4 and from the proof of (a) it follows that uv is not an edge of $\mathcal{P}$ and that $b(\theta) = u(\theta) + v(\theta) - a(\theta)$. The latter shows that there can be at most one vertex in $X(u) \cap X(v)$ not adjacent to a.

Finally, suppose that $X(u) \cap X(v)$ is a clique and let α be defined by

$$\alpha = \sum_{x \in X(u) \cap X(v)} x(\theta).$$

If $x \in X(u) \cap X(v)$ then

$$\langle \alpha, x(\theta) \rangle = \big(1 + (c_2 - 1)w_1\big) \langle x(\theta), x(\theta) \rangle$$

while

$$\langle \alpha, u(\theta) \rangle = \langle \alpha, v(\theta) \rangle = c_2 w_1 \langle u(\theta), u(\theta) \rangle.$$

It follows that uv is a face of a face of $\mathcal{P}$, hence it is an edge in $\mathcal{P}$. ∎

COROLLARY 3.4. *Let X be a distance regular graph with diameter d, let θ be an eigenvalue of X with cosine sequence $w_0, \ldots, w_d$ and assume that $w_1 \geq w_i$ if $i > 1$. Then X is a spanning subgraph of the 1-skeleton of the eigenpolytope belonging to θ.* ∎

The reader might object that Theorem 3.3 is not needed to prove this corollary. If $w_1 \geq w_i$ when $i > 1$ and $u \sim v$ then no vertex of $\mathcal{P}$ is closer to $u(\theta)$ than $v(\theta)$ is. Hence it seems obvious that uv is an edge of $\mathcal{P}$. Even in the plane this is false—any rhombus which is not a square provides a counterexample. As noted in [6], Corollary 3.4 implies that X is planar when θ_1 has multiplicity three.

4. Polytopal distance-regular graphs.

We wish to determine the distance regular graphs X which are isomorphic to the 1-skeleton of the eigenpolytope associated to their second-largest eigenvalue. Our first tool is a version of a result from Terwilliger [16].

LEMMA 4.1. *Let X be a distance regular graph of diameter d with an eigenvalue θ and let $w_0, \ldots, w_d$ be the corresponding cosine sequence. Let u be a vertex in X, let N be the adjacency matrix of $X(u)$ and let τ be an eigenvalue of N. Then $(1 - w_2) + (w_1 - w_2)\tau \geq 0$. If $\theta \neq 0$ then equality holds if and only the vectors $v(\theta)$ with $v \in X(u)$ form a linearly dependent set.*

PROOF. Define the matrix M by

$$M = I + w_1 N + w_2 (J - I - N).$$

https://doi.org/10.4153/CJM-1998-040-8 Published online by Cambridge University Press

Then $\frac{m}{|V(X)|}M$ is the principal submatrix of E_θ with rows and columns indexed by the neighbours of u, consequently M is positive semi-definite and therefore its eigenvalues are non-negative. As the neighbourhood of u is regular (with valency a_1) the vector $\mathbf{1}$ is an eigenvector of N, and hence is eigenvector of M with eigenvalue

$$1 + a_1 w_1 + (k - a_1 - 1) w_2 = 1 + a_1 w_1 + b_1 w_2.$$

Using (3.3), we find that this equals θ^2/k.

Suppose that z is an eigenvector for N that is orthogonal to $\mathbf{1}$ and has eigenvalue τ. Then z is an eigenvector for M with eigenvalue $(1 - w_2) + (w_1 - w_2)\tau$, and therefore

$$(1 - w_2) + (w_1 - w_2)\tau \geq 0.$$

Equality will hold if and only if there is an eigenvector of M that is orthogonal to $\mathbf{1}$ and has eigenvalue 0. This proves the last claim. ∎

If $k > m$ then the vectors $v(\theta)$ for v in $X(u)$ will be linearly dependent; this will be very helpful when we come to prove our main result, Theorem 4.3.

The following is a combination of important results from Brouwer, Cohen and Neumaier [3], that are based in part on earlier work of Neumaier [13] and Terwilliger [17].

THEOREM 4.2. *Let X be a distance regular graph with diameter d and valency k, let $w_0, \ldots, w_d$ be the cosine sequence for θ_1 and let m be the multiplicity of θ_1. If $c_2 > 1$, $1 - 2w_1 + w_2 = 0$ and $k \geq m$ then X is:*

(a) a Johnson graph $J(v, k)$,

(b) a Hamming graph $H(n, q)$ or a Doob graph,

(c) a halved n-cube,

(d) the Schläfli graph or one of the three Chang graphs,

(e) the Gosset graph.

PROOF. If $1 - 2w_1 + w_2 = 0$ then (3.6) implies that $b_1 = \theta_1 + 1$. Graphs satisfying this condition are classified by Theorem 4.4.11 and Theorem 3.12.4 of [3], as follows.

If X has diameter two then, by [3: Theorem 4.4.11(i)], its least eigenvalue is -2, whence [3: Theorem 3.12.4] yields that X is $J(v, 2)$, $H(2, n)$, the Shrikande graph (which is a Doob graph) or one of the graphs listed in (d).

If the diameter of X is greater than two then, since we have $c_2 \geq 2$, Theorem 4.4.11 of [3] yields that $c_2 \in \{2, 4, 6, 10\}$ and determines the graphs that can arise for each of the four possible values of c_2. We consider these in turn.

If $c_2 = 2$ then X is a Hamming graph, a Doob graph or is one of two locally Petersen graphs described in [3: Theorem 1.16.5(ii) and (iii)]. From the tables at the end of [3] we see that $k < m$ for both of these graphs.

If $c_2 = 4$ or 6 then X must be a Johnson graph or a halved-cube respectively. If $c_2 = 10$ then X is the Gosset graph. ∎

A Doob graph is the Cartesian product of $H(n, 4)$ with some number of copies of the Shrikande graph. The latter is a strongly regular graph on 16 vertices with the same

https://doi.org/10.4153/CJM-1998-040-8 Published online by Cambridge University Press

parameters as $H(2,4)$. The Schläfli graph is a strongly regular graph on 27 vertices with valency 16; the Chang graphs are strongly regular graphs on 28 vertices with the same parameters as $J(8,2)$. The halved 5-cube is also known as the Clebsch graph, and occurs in this guise in [3: Theorem 3.12.4]. The Gosset graph is an antipodal distance-regular graph with diameter 3 on 56 vertices with valency 7, it is locally a Schläfli graph. For further information about these graphs, we refer the reader to [3].

We now come to our main result.

THEOREM 4.3. *Let X be distance regular and let $\mathcal{P}$ be the eigenpolytope associated to the second-largest eigenvalue of X. Then X is the* 1*-skeleton of $\mathcal{P}$ if and only if it is one of the following:*

(a) a Johnson graph $J(v,k)$,
(b) a Hamming graph $H(n,q)$,
(c) a halved n-cube,
(d) the Schläfli graph,
(e) the Gosset graph,
(e) the icosahedron,
(f) the dodecahedron,
(g) the complement of r copies of K_2, or
(h) a cycle.

PROOF. Assume X is distance regular with diameter d and valency k, let θ denote its second-largest eigenvalue and let $\mathcal{P}$ be the corresponding eigenpolytope. Let m be the multiplicity of θ. We assume that X is the 1-skeleton of $\mathcal{P}$, whence $k \geq m$. We prove that X is either a Chang graph, a Doob graph, or one of the graphs listed in the statement of the theorem. We may assume $k \geq 3$.

If $\theta < 0$ then $\bar{X}$ has least eigenvalue greater than -1, and therefore X is complete. If $\theta = 0$ then $\bar{X}$ has least eigenvalue -1. It follows that each component of $\bar{X}$ is a complete graph, and hence that X is a regular complete multipartite graph. Assume that $\bar{X}$ consists of r disjoint copies of K_m. Because $\theta = 0$ we have $w_1 = 0$ and, from (3.5), we find that $w_2 = -1/b_1 = -1/(m-1)$. Consequently $1 - 2w_1 + w_2 > 0$ unless $m = 2$. When $m = 2$, vertices of $\mathcal{P}$ corresponding to vertices at distance two in X are antipodal and so X is isomorphic to the 1-skeleton of $\mathcal{P}$. Consequently we may assume that $\theta > 0$.

Let u be a vertex of X and let N be the adjacency matrix of $X(u)$. Assume henceforth that τ denotes the least eigenvalue of N. We have

$$1 - 2w_1 + w_2 - [1 - w_2 + (w_1 - w_2)\tau] = -(w_1 - w_2)(\tau + 2).$$

By Lemma 4.1, this implies that $1 - 2w_1 + w_2 > 0$ when $\tau < -2$. Therefore $\tau \geq -2$. If $\tau = -2$ then Lemma 4.1 yields that $1 - 2w_1 + w_2 \geq 0$, therefore $1 - 2w_1 + w_2 = 0$. If $c_2 > 1$ we appeal to Theorem 4.2, if $c_2 = 1$ then $X(u)$ cannot contain an induced path of length two, therefore it is a disjoint union of complete graphs and $\tau \geq -1$.

Thus we may assume that $\tau > -2$, and hence that each component of $X(u)$ is a clique or an odd cycle.

https://doi.org/10.4153/CJM-1998-040-8 Published online by Cambridge University Press

Suppose $c_2 > 1$. By Theorem 4.2 we may assume that $1-2w_1+w_2 < 0$, whence Theorem 3.3 implies that X contains no induced 4-cycle, and therefore the common neighbours of any two vertices at distance two must be a clique. Suppose a and b are neighbours of u that are distance at least three in $X(u)$. The common neighbours of a and b form a clique containing u, implying that u is the only common neighbour of a and b. As $c_2 > 1$, this is impossible. Consequently the neighbourhood of each vertex in X is a pentagon; given this, it is not hard to show that X is the icosahedron.

If $k > m$ then equality holds in the bound of Lemma 4.1; together with the identities from Section 3 this yields

$$\tau = -1 - \frac{b_1}{\theta + 1}.$$

If $a_1 = 0$ then $\tau = 0$, but then $b_1 < 0$ which is impossible. If $a_1 = 1$ then $\tau = -1$ but then we find that $b_1 = 0$ and X is complete. Suppose $\tau < -1$. Then $a_1 \geq 2$ and, as the least eigenvalue of an even cycle is -2, this means that $X(u)$ is a disjoint union of odd cycles and at least one of these cycles has length at least five. This implies that $c_2 > 1$.

Hence we are left with the case where $c_2 = 1$ and $k = m$. As X is the 1-skeleton of $\mathcal{P}$, Theorem 2.3 implies that $\mathcal{P}$ is simple and Theorem 2.4(b) that every path of length two in X lies in a 2-face, necessarily unique. Since $c_2 = 1$, no face is a 4-gon.

If every 2-face is a triangle then there is no induced copy of P_3 and X is complete. Suppose then that a, b and c induce a copy of P_3, and that the 2-face which contains this is an n-gon. The angle between the vectors $a(\theta) - b(\theta)$ and $c(\theta) - b(\theta)$ is $\pi - \frac{2\pi}{n}$. A straightforward computation yields that

$$\cos\Big(\pi - \frac{2\pi}{n}\Big) = \frac{1 - 2w_1 + w_2}{2(1 - w_1)} = \frac{b_1 - 1 - \theta}{2b_1}$$

and therefore

$$\cos\Big(\frac{2\pi}{n}\Big) = \frac{\theta + 1 - b_1}{2b_1}. \tag{4.1}$$

Thus each 2-face of $\mathcal{P}$ is a triangle or an n-gon, where n is determined by θ and b_1, and is at least five.

Assume now that $a_1 = 0$. As $\theta < k$, equation (4.1) implies that

$$\cos\Big(\frac{2\pi}{n}\Big) < \frac{1}{k-1}.$$

If $k \geq 5$, this implies $n \geq 6$. However any 3-face of $\mathcal{P}$ is a cubic planar graph and, from Euler's relation, it follows that such a graph must have a face of size at most five. Therefore $k \leq 4$, and so $m \leq 4$. If $k = 3$ or 4 then $n = 5$. Hence if $k = 3$ then X is the dodecahedron. If $k = 4$ then Euler's formula yields that each 3-face of $\mathcal{P}$ is a dodecahedron. Therefore $\mathcal{P}$ is a regular 4-polytope and the only regular 4-polytopes with distance-regular 1-skeletons are the simplex, the 4-cube and its dual. (This follows for example, from Zhu [18], where the distance-regular graphs with an eigenvalue of multiplicity four are determined.)

Assume now that $a_1 > 0$. As $c_2 = 1$, the neighbourhood of a vertex in X is the disjoint union of t cliques of size $a_1 + 1$, where $t = k/(a_1 + 1)$. Assume first that $t \geq 3$.

Let u be a fixed vertex in X and let p, v and w be neighbours of u such that $v \sim w$ but neither v nor w is adjacent to p. Let $\mathcal{F}$ denote the smallest face of $\mathcal{P}$ that contains u and these three neighbours. By Theorem 2.4(b), it follows that $\mathcal{F}$ is a simple 3-polytope with all faces triangles or n-gons. The 1-skeleton of $\mathcal{F}$ is cubic, hence there is a vertex, v' say, adjacent to p but not u and lying in an n-gon that contains u and v. Similarly there is a vertex w' adjacent to p but not u and lying in an n-gon that contains w. These two n-gons lie in affine planes that intersect in the line through u and p. The configuration formed by these two n-gons is symmetric about the plane through u, p and $(v+w)/2$. It follows that

$$\|v'(\theta) - w'(\theta)\| = \|v(\theta) - w(\theta)\|,$$

which implies that v' and w' must be adjacent. From this we conclude, by a trivial induction argument, that each vertex in $\mathcal{F}$ lies in exactly one triangle. Consequently n must be even—every other edge in a 2-face of size n lies in a triangle—and so $n \geq 6$.

Now let $\mathcal{F}$ denote the simple 3-polytope formed by the smallest face of $\mathcal{P}$ that contains u and three pairwise non-adjacent neighbours of u. (Such neighbours exist, because $t \geq 3$.) Then the 1-skeleton of $\mathcal{F}$ is a cubic graph that, by the argument of the last paragraph, is triangle-free. As it is a planar graph, Euler's formula implies that some face of $\mathcal{F}$ has length at most five. This implies that $n = 5$, a contradiction.

Thus we are left with the cases when $t \leq 2$. If $t = 1$ then X is complete and no more need be said. If $t = 2$ then X is the line graph of a triangle-free graph. By [3: Theorem 4.2.16] it follows that if X is not listed in the statement of theorem then it is the line graph of either

(a) the incidence graph of a regular generalised d-gon with $d \in \{3, 4, 6\}$, or

(b) a Moore graph (with diameter two and valency 3, 7 or 57).

The incidence graph of a regular generalised d-gon is bipartite with girth $2d$ and therefore the shortest cycle other than a triangle in its line graph has length at least $2d$. Since X contains pentagons, this case cannot arise. The eigenvalues and their multiplicities for line graphs of Moore graphs appear in [3: p. 149]; in each case $m > k$.

We prove next that a Doob graph is never the 1-skeleton of its θ_1-eigenpolytope. A Doob graph is defined to be the Cartesian product of $H(n, 4)$ with some positive number of copies of the Shrikande graph. We quote the information we need from [3: pp. 103–104]. The Shrikande graph is a strongly regular graph on 16 vertices with the same parameters as $H(2, 4)$; hence it has $a_1 = c_2 = 2$ and $\theta_1 = 2$ with multiplicity six. The neighbourhood of any vertex is isomorphic to C_6, whereas in $H(2, 4)$ all neighbourhoods are isomorphic to $2K_3$. The least eigenvalue τ of C_6 is -2, whence Lemma 4.1 yields that the image in $\mathcal{P}$ of the neighbourhood of a vertex is linearly dependent, and therefore lies in a 4-dimensional affine space. Since $\mathcal{P}$ is 6-dimensional this contradicts Theorem 2.1. Therefore the Shrikande graph is not the 1-skeleton of its θ_1-eigenpolytope.

The same argument works in general. The neighbourhood of a vertex in a Doob graph X must be a disjoint union of copies of K_3 and C_6, with at least one C_6. Hence $\tau = -2$ and therefore the image of a C_6 is linearly dependent.

For the Chang graphs $c_2 = 4$. A computer search carried out by M. Conder (using the package CAYLEY) yielded that in each of these three graphs there is a pair of vertices at distance two whose four common neighbours induce a clique. Hence the Chang graphs are not isomorphic to the 1-skeletons of their θ_1-eigenpolytopes.

Now we must show that the graphs listed in the statement of the theorem are isomorphic to the 1-skeletons of their θ_1-eigenpolytopes. For the cycle no proof should be required. To treat the Hamming and Johnson graphs we prove that if $1 - 2w_1 + w_2 = 0$ and every induced path of length two lies in an induced 4-cycle then X is the 1-skeleton of $\mathcal{P}$. If $a \in V(X)$, let $C(a)$ denote the convex cone generated by the vectors $u(\theta) - a(\theta)$ as u ranges over the vertices of X and let $C_1(a)$ denote the cone generated by the vectors $u(\theta) - a(\theta)$ as u ranges over the neighbours of a. Clearly $C_1(a) \subseteq C(a)$. We show that every vector in $C(a)$ is a non-negative linear combination of vectors $u(\theta) - a(\theta)$, where $u \sim a$. This implies that X is the 1-skeleton of $\mathcal{P}$.

Let v be a vertex at distance i from a, and let b be a neighbour of v at distance $i - 1$ from a. We prove by induction on i that there is a neighbour, u say, of a such that

$$u(\theta) - a(\theta) = v(\theta) - b(\theta). \tag{4.2}$$

If $i = 1$ there is nothing to prove. If $i \geq 2$, let b' be a neighbour of b at distance $i-2$ from a. Then b', b and v lie in an induced 4-cycle with a fourth vertex v'. Because $1-2w_1+w_2 = 0$ this implies that

$$v'(\theta) - b'(\theta) = v(\theta) - b(\theta)$$

and by induction on i there is a neighbour u of a such that

$$u(\theta) - a(\theta) = v'(\theta) - b'(\theta).$$

It follows that if $v \in V(X)$ then $v(\theta) - a(\theta)$ is equal to a sum of vectors of the form $u(\theta) - a(\theta)$, where $u \in X(a)$, and hence $C_1(a) = C(a)$.

The common neighbours of two distinct non-adjacent vertices in the Schläfli graph induce a subgraph isomorphic to K_8 with a perfect matching deleted. (This follows, for example, from the description of this graph as the complement of the point graph of the generalised quadrangle of order $(2, 4)$—see page 103 of [3].) Hence Theorem 3.3 yields that it is isomorphic to the 1-skeleton of its θ_1-eigenpolytope.

The Gosset graph is locally a Schläfli graph and has $c_2 = 10$. Let u and v be two vertices in it at distance two, and let a be one of their common neighbours. As $X(a)$ is a Schläfli graph, there are exactly eight common neighbours of u and v adjacent to a, hence their tenth common neighbour is not adjacent to a and so $X(u) \cap X(v)$ is isomorphic to the complement of $5K_2$. Thus Theorem 3.3 yields that the Gosset graph is isomorphic to the 1-skeleton of its θ_1-eigenpolytope.

https://doi.org/10.4153/CJM-1998-040-8 Published online by Cambridge University Press

The dodecahedron and icosahedron remain. Let $u_1, \ldots, u_{20}$ be a set of vectors forming the vertices of a regular dodecahedron in $\mathbf{R}^3$. By symmetry, the sum of the vectors u_j corresponding to the vertices adjacent to u_i is equal to cu_i for some real number c and, by symmetry, c is independent of i. Given this it is not too hard to see that $u_1, \ldots, u_{20}$ can be taken to be the vertices of the θ_1-eigenpolytope of the dodecahedron. A similar argument works for the icosahedron. ∎

5. **Faces.** For the Johnson graphs, Hamming graphs and the halved cubes we can describe all faces of the θ_1-eigenpolytope.

First a definition. Let $\operatorname{dist}_X(u,v)$ denote the distance between vertices u and v in a graph. A subgraph Y of a graph X is *convex* if, given any two vertices u and v from Y, any vertex x of X such that

$$\operatorname{dist}_X(u,x) + \operatorname{dist}_X(x,v) = \operatorname{dist}_X(u,v)$$

also lies in Y. (In other words, any vertex in X on a shortest path in X that joins u to v must lie in Y.)

LEMMA 5.1. *Let X be a distance regular graph with diameter d and let θ be an eigenvalue of X with cosine sequence $w_0, \ldots, w_d$. If $1-2w_1+w_2 = 0$ and every induced path of length two lies in a 4-cycle then every face of the θ-eigenpolytope is a convex subgraph of X.*

PROOF. Assume θ has multiplicity m, let $\mathcal{P}$ be the θ-eigenpolytope of X and let F be a face of $\mathcal{P}$. Then there is a vector h in $\mathbf{R}^m$ such that $h^T z = 1$ when $z \in F$ and $h^T z < 1$ when $z \in \mathcal{P} \setminus F$.

Suppose the lemma is false for F. Then there are vertices u and v in F and a shortest uv-path P such that

$$P \cap F = \{u, v\}.$$

If b is the neighbour of v on P then $h^T b(\theta) < 1$ and thus $h^T\big(v(\theta) - b(\theta)\big) > 0$. From the last part of the proof of Theorem 4.3, there is a neighbour, a say, of u such that

$$a(\theta) - u(\theta) = v(\theta) - b(\theta)$$

and therefore $h^T a(\theta) > 1$. ∎

Let $J(v,k)$ have as vertices all k-subsets of the v-set V. Let S and T be disjoint subsets of V. Then the k-subsets of V which contain S and intersect T in the empty set form a convex subset, and Lambeck [10: Chapter 5] has shown that all convex subsets of $J(v,k)$ are of this form. It is easy to verify that all these subsets are faces of the θ_1-eigenpolytope of $J(v,k)$.

View $H(n,q)$ as the Cartesian product of n copies of K_q. If $r_1, \ldots, r_n$ are integers between 1 and q, the Cartesian product of the graphs K_{r_i} is a convex subgraph, and all convex subgraphs have this form. (This is probably folklore, and an easy exercise. A proof does appear in [10: Chapter 5].) Again, it is easy to verify that all these subsets are faces of the θ_1-eigenpolytope of $H(n,q)$.

For the halved cubes we can use the information we have obtained for Hamming graphs. Let X be the n-cube $H(n,2)$ and let X_2 be the graph with the same vertex set as X, but with two vertices adjacent if and only if they are at distance two in X. Then X_2 has two connected components with vertex sets the colour classes of X, each of which is isomorphic to a halved n-cube. Let W denote one of the colour classes of X. Let θ_1 be the second-largest eigenvalue of X and let $\mathcal{P}$ be its eigenpolytope. Note that the multiplicity of θ_1 is $n-1$. Let $\mathcal{Q}$ be the convex hull of the vectors $u(\theta_1)$, for u in W. Then $\mathcal{Q}$ is the θ_1-eigenpolytope of the halved n-cube. We prove that a face of $\mathcal{Q}$ is either:

(a) A set of vertices in W with a common neighbour not in W, or

(b) The intersection of W with a face of X.

The subgraphs corresponding to the faces in (a) are cliques, those in (b) are halved m-cubes. We take it as given that these are faces, and show that there are no others. Assume $h \in \mathbf{R}^{n-1}$ and that the vectors x such that $h^T x$ is maximal form a face F of $\mathcal{Q}$. We may assume that $h^T x = 1$ for all x in F. If $h^T x \leq 1$ for all x in $\mathcal{P}$ then F is the intersection of $\mathcal{Q}$ with a face of $\mathcal{P}$. Suppose then that there is a vertex y in $\mathcal{P}$ such that $h^T y > 1$, and choose y so that $h^T y$ is maximal. Then y lies in the face, F' say, of $\mathcal{P}$ determined by h. No vertex in F' can be the image of a vertex in W, but the 1-skeleton of F' is a connected subgraph of X. Hence F' must be the vertex y, and therefore F must be a subset of the neighbours of y. The convex hull of the neighbours of y is a simplex—this follows from [2: Theorem 12.13] and the fact that the n-cube is a simple polytope—and therefore any subset of it is a face.

Meyerowitz [12] has determined the completely regular designs of strength zero in the Johnson and Hamming graphs. As any such design must form a face in the θ_1-polytope, the results in this section provide another approach to his work.

6. **Other eigenvalues, other graphs.** We collect some observations about the polytopes of $J(v,k)$ associated with eigenvalues other than θ_1. Delsarte [4: Theorem 4.6] determines the principal idempotents for the Johnson scheme, from which we find that if α and β are k-sets with $|\alpha \cap \beta| = i$ then

$$\binom{v-4}{k-2}(E_2)_{\alpha,\beta} = \binom{i}{2} - \frac{(k-1)^2}{v-2}i + \frac{\binom{k}{2}^2}{\binom{v-1}{2}}.$$

From this we find immediately that $1 - 2w_1 + w_2 > 0$, and therefore the 1-skeleton contains at least $X_1 \cup X_2$.

We also have

$$E_k = \frac{v-2k+1}{v-k+1}\sum_{r=0}^{k}(-1)^r \binom{v-k}{r}^{-1} A_r,$$

from which it follows that the terms in the cosine sequence for θ_k alternate in sign and decrease strictly in absolute value. So in this case vertices at even distance in $J(v,k)$ are closer in the polytope than vertices at odd distance. Could the 1-skeleton be X_2? The

https://doi.org/10.4153/CJM-1998-040-8 Published online by Cambridge University Press

multiplicity of $-k$ as an eigenvalue is

$$\binom{v}{k} - \binom{v}{k-1}$$

which, in general, is greater than $\binom{k}{2}\binom{v-k}{2}$, the valency of X_2. Hence X_2 cannot be the 1-skeleton of this polytope.

Our next result indicates that we should expect difficulties in identifying the facets of an eigenpolytope.

LEMMA 6.1. *The edges in a regular subgraph Y of K_v are the vertices of a face of the θ_2-polytope of $J(v,2)$. This face is a facet if and only Y is connected and not bipartite.*

PROOF. Any regular subgraph Y of K_v, with its complement, determines an equitable partition of the vertices of $J(v,2)$ with two cells. By our remarks at the end of Section 2, this gives rise to two parallel faces of $J(v,2)$. The face determined by Y will be a facet if and only if the vertex-edge incidence matrix of Y has rank v, which happens if and only Y is connected and not bipartite. ∎

For $J(v,3)$ we note that any Steiner triple system on v points forms a completely regular design of strength two, hence determines a face in the θ_3-polytope. This again provides us with vast numbers of faces in general.

LEMMA 6.2. *Let X be a distance regular graph with valency k and diameter d and let θ be an eigenvalue of X with cosine sequence $w_0, \dots, w_d$. Assume that $k > \theta > -1$ and $w_2 \geq w_i$ when $i \geq 2$. If $b_1 > 2(\theta + 1)$ then each set of three vertices of X with any two at distance at most two forms a 2-face of the θ-polytope of X.*

PROOF. The condition $k > \theta$ implies that $w_1 < 1$ and the condition $\theta > -1$ implies that $w_1 > w_2$ (by (3.3)). Let a, b and c be distinct vertices of X, and let z_1, z_2 and z_3 be the corresponding standard eigenvectors. As $1 > w_1$, it follows that if z_1, z_2 and z_3 form a triangle, then $z_1 + z_2 + z_3$ takes its maximum value on precisely the set $\{a, b, c\}$.

Suppose a is adjacent to b and c, and that b and c are at distance two. Define

$$\gamma := \frac{1 - 2w_1 + w_2}{w_1 - w_2}, \quad z := \gamma z_1 + z_2 + z_3.$$

Then z takes the same value, $\gamma + 2w_1$, on a, b and c. As

$$\frac{1 - 2w_1 + w_2}{1 - w_1} = 1 - \frac{\theta + 1}{b_1},$$

we see that $\gamma \geq 0$; consequently the value of z on any vertex other than a, b or c is at most $(\gamma + 2)w_1$. Because

$$\gamma + 2w_1 - (\gamma + 2)w_1 = \gamma(1 - w_1) > 0$$

it follows that $\{a, b, c\}$ is the set of vertices on which z is maximal.

Suppose next that a is at distance two from b and c, which are adjacent. If

$$\gamma := (1 + w_1 - 2w_2)/(1 - w_2), \quad z = \gamma z_1 + z_2 + z_3,$$

then z takes the value $\gamma + 2w_2$ on a, b and c. As

$$\gamma - 1 = (w_1 - w_2)/(1 - w_2) > 0,$$

the value of z on any vertex other than a, b or c is at most $(\gamma + 2)w_1$. After some work, we find that

$$\begin{aligned}\gamma + 2w_2 - (\gamma + 2)w_1 &= \gamma(1 - w_1) - 2(w_1 - w_2)\\ &= (1 - w_1)^2 - 2(w_1 - w_2)^2,\end{aligned}$$

whence it follows that $\{a, b, c\}$ is the set of vertices on which z is maximal when

$$\left(\frac{1 - w_1}{w_1 - w_2}\right)^2 > 2.$$

This is equivalent to the condition $b_1 > \sqrt{2}(\theta + 1)$.

Now assume that a, b and c are pairwise at distance two and

$$z := z_1 + z_2 + z_3.$$

Then z is equal to $1 + 2w_2$ on a, b and c, and these three vertices form a face if

$$0 < 1 + 2w_2 - 3w_1 = (1 - w_1) - 2(w_1 - w_2).$$

This holds if and only if $b_1 > 2(\theta + 1)$. ∎

If no three vertices pairwise at distance two in X have a common neighbour, the condition of the lemma can be relaxed to $b_1 > \sqrt{2}(\theta + 1)$. Combining this lemma with Theorem 3.3(b) yields the following.

COROLLARY 6.3. *Let X be a strongly regular graph, let θ be an eigenvalue of X, not the valency, and let $\mathcal{P}$ be the associated polytope. If $b_1 > \theta + 1$ then the 1-skeleton of $\mathcal{P}$ is complete. If $b_1 > 2(\theta + 1)$, every triple of vertices in X forms a face of $\mathcal{P}$.* ∎

A polytope is *k-neighbourly* if every set of k vertices forms a face. The above result thus asserts that $\mathcal{P}$ is 3-neighbourly when $b_1 > 2(\theta + 1)$. For information about neighbourly polytopes, see [2: Section 14].

We provide some examples where the conditions of the corollary hold. If X is the complement of the line graph of K_n then $b_1 = 2n-8$ and $\theta_1 = 1$. Here θ_1 has multiplicity $n-1$, and the θ_1-polytope is 3-neighbourly when $n \geq 7$. If X is the block graph of a Steiner triple system on v points then $\theta_1 = (v - 9)/2$ and $b_1 = v - 5$. The multiplicity of θ_1 is $v - 1$, and the polytope is 3-neighbourly when $v \geq 7$. (The vertices of the block graph are the triples of the Steiner triple system, two triples are adjacent if they have a vertex in common.) A Paley graph on n vertices is self-complementary, with $b_1 = (n - 1)/4$ and $\theta_1 = (-1 + \sqrt{n})/2$. The multiplicity of θ_1 is $(n - 1)/2$, which is also the valency. The polytope is 3-neighbourly when $n > 25$; for the graph to exist n must be a prime power congruent to 1, modulo 4.

7. **Questions.** An m-polytope can be at most $\lfloor m/2 \rfloor$-neighbourly (see [2: Corollary 14.5]); can this bound can be realised by eigenpolytopes of distance-regular graphs more than finitely often when $m \geq 4$?

It appears that the 1-skeleton of an eigenpolytope of a distance regular graph is often complete; hence it would be interesting to find more examples where it is not.

Finally it would be good to have more examples of eigenpolytopes of distance regular graphs where we can explicitly describe the facets.

ACKNOWLEDGMENT. I wish to thank Marston Conder for his help with the computations on the Chang graphs.

REFERENCES

1. M. Balinski, *On the graph structure of convex polyhedra in n-space*. Pacific J. Math. **11**(1961), 431–434.
2. A. Brøndsted, *An Introduction to Convex Polytopes*. Springer, New York, 1983.
3. A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*. Springer, Berlin, 1989.
4. P. Delsarte, *An algebraic approach to the association schemes of coding theory*. Philips Research Reports Supplements **10**, 1973.
5. C. D. Godsil, *Graphs, groups and polytopes*. In: Combinatorial Mathematics. (eds. by D. A. Holton and Jennifer Seberry), Lecture Notes in Math. **686**, Springer, Berlin, 1978, 157–164.
6. ______, *Bounding the diameter of distance regular graphs*. Combinatorica **8**(1988), 333–343.
7. ______, *Algebraic Combinatorics*. Chapman and Hall, New York, 1993.
8. ______, *Equitable partitions*. In: Combinatorics, Paul Erdős is Eighty, Vol. I. (eds. D. Miklós, V. T. Sós, T. Szőnyi), János Bolyai Mathematical Society, Budapest, 1993, 173–192.
9. C. D. Godsil and W. J. Martin, *Quotients of association schemes*. J. Combin. Theory Ser. A **69**(1995), 185–199.
10. E. W. Lambeck, *Contributions to the Theory of Distance Regular Graphs*. Ph. D. Thesis, Technical University Eindhoven, 1990.
11. C. Licata and D. L. Powers, *A surprising property of some regular polytopes*. Scientia **1**(1988), 73–80.
12. A. Meyerowitz, *Cycle-balanced partitions in distance-regular graphs*. submitted.
13. A. Neumaier, *Characterization of a class of distance-regular graphs*. J. Reine Angew. Math. **357**(1985), 182–192.
14. D. L. Powers, *The Petersen polytopes*. Technical Report, Clarkson University, 1986.
15. ______, *Eigenvectors of distance-regular graphs*. SIAM J. Matrix Anal. Appl. **9**(1988), 399–407.
16. P. Terwilliger, *A new feasibility condition for distance-regular graphs*. Discrete Math. **61**(1986), 311–315.
17. ______, *Root systems and the Johnson and Hamming graphs*. European J. Combin. **8**(1987), 73–102.
18. R. Zhu, *Distance-regular graphs with an eigenvalue of multiplicity four*. J. Combin. Theory Ser. B **57**(1993), 157–182.

Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
N2L 3G1
e-mail: cgodsil@math.uwaterloo.ca