



A Lift of the Schur and Hall–Littlewood Bases to Non-commutative Symmetric Functions

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Abstract. We introduce a new basis of the algebra of non-commutative symmetric functions whose images under the forgetful map are Schur functions when indexed by a partition. Dually, we build a basis of the quasi-symmetric functions that expand positively in the fundamental quasi-symmetric functions. We then use the basis to construct a non-commutative lift of the Hall–Littlewood symmetric functions with similar properties to their commutative counterparts.

1 Introduction: Yet Another Schur-like Basis of NSym

The algebras of non-commutative symmetric functions NSym and quasi-symmetric functions QSym are dual Hopf algebras. They have been of great importance to algebraic combinatorics. As seen in [1], they are universal in the category of combinatorial Hopf algebras. They also represent the Grothendieck rings for the finitely generated projective representations and the finite dimensional representation theory of the 0-Hecke algebra. We will not attempt to summarize these notions in great detail; the interested reader should see [21].

This paper is the result of an exploration of what a “Schur” analogue in the non-commutative setting should look like. We focused on a minimal set of axioms that define a basis of NSym whose image under the forgetful map is a Schur function when the basis element is indexed by a partition.

The primary goal in this paper is to build a new basis, the “*immaculate basis*,” of NSym and to develop its theory. This basis has many of the same properties as the classical basis of Schur functions of the symmetric function algebra. The immaculate basis has a positive right-Pieri rule (Theorem 3.5), a simple Jacobi–Trudi formula (Theorem 3.27), and a creation operator construction (Definition 3.2). Furthermore, under the forgetful map χ from the non-commutative symmetric functions to the symmetric functions (Equation (2.1)) their image is a Jacobi–Trudi expression for a Schur function (Corollary 3.30). Thus, immaculate functions map to Schur functions

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using a *signed sorting* action (Proposition 2.2). By duality, these functions give rise to a basis of the quasi-symmetric function algebra and expand positively into the monomial and fundamental bases of quasi-symmetric functions (Propositions 3.36 and 3.37). We also give a combinatorial expansion of the Schur functions (Theorem 3.38) in this basis.

In the commutative setting there are many characterizing properties of the Schur functions (e.g., triangularity with other bases, a Pieri rule, uniquely determined by an orthonormalization process). We immediately focused on the concept of a Jacobi–Trudi identity defining a basis of NSym. The reader interested only in the definition of our basis may wish to take the Jacobi–Trudi rule (Theorem 3.27) as the definition of the immaculate basis. However, to simplify the proofs and provide a coherent story, we started with the functions being defined through creation operators. Since they are defined through the process of creation (applying certain creation operators to the identity), we decided to name them the “immaculately conceived basis,” which we have shortened to the immaculate basis.

Recently, many of NSym’s enthusiasts have developed bases for the algebra that have various properties in common with different classical bases of Sym. The “immaculate” term within this paper is intended to be humorous; our basis shares many of the properties of the Schur basis of Sym, but is in no way a perfect analogue of the Schur basis (for instance, products of immaculate functions do not expand positively in the immaculate basis). We have no supportive evidence to believe that any basis of NSym will ever be a perfect analogue of the Schur basis.

It should be noted that this basis is not the non-commutative Schur basis (dual to the quasi-symmetric Schur basis of [7, 13, 14]), even though they share several similar properties. The non-commutative Schur basis has the property that the image under the map χ of an element indexed by a composition is a Schur function indexed by the parts of the composition sorted in decreasing order. The basis studied here has the property that an element indexed by a composition is sent to a Jacobi–Trudi determinant expression (Theorem 3.27 and Corollary 3.30). We have not developed any connections at this point between the non-commutative Schur basis and our immaculate basis.

Starting with the immaculate basis, we construct lifts of the Hall–Littlewood symmetric functions in the non-commutative symmetric function algebra. It should be noted that many different versions of a non-commutative Hall–Littlewood symmetric function already exist [5, 16, 23, 28, 35], but as far as we are aware, none of these project for all partitions to the classical Hall–Littlewood functions (ours do!). Having a basis of NSym that projects to the Hall–Littlewood basis could prove to be a powerful tool towards their study; fundamental problems in the classical theory of Hall–Littlewood symmetric functions, such as a combinatorial understanding of their structure coefficients, remain open.

The results of this paper are mostly combinatorial and algebraic. In a forthcoming paper we turn our attention to the representation theoretic interpretation of our basis [2]. We construct indecomposable modules of the 0-Hecke algebra whose characteristics, under the identification with elements of QSym, form the dual basis to the immaculate basis.

2 Background

2.1 Compositions and Combinatorics

A *partition* of a non-negative integer n is a tuple $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_m]$ of positive integers satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ that sum to n ; it is denoted $\lambda \vdash n$. Partitions are of particular importance to algebraic combinatorics; among other things, partitions of n index a basis for the symmetric functions of degree n , Sym_n , and the character ring for the representations of the symmetric group. These concepts are intimately connected; we assume the reader is well versed in this area (see for instance [29] for background details).

A *composition* of a non-negative integer n is a tuple $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_m]$ of positive integers that sum to n , often written $\alpha \models n$. The entries α_i of the composition are referred to as the parts of the composition. The size of the composition is the sum of the parts and will be denoted $|\alpha| := n$. The length of the composition is the number of parts and will be denoted $\ell(\alpha) := m$. In this paper we study dual graded Hopf algebras whose bases at level n are indexed by compositions of n .

Compositions of n are in bijection with subsets of $\{1, 2, \dots, n - 1\}$. We will follow the convention of identifying $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_m]$ with the subset $D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots, \alpha_1 + \alpha_2 + \dots + \alpha_{m-1}\}$.

If α and β are both compositions of n , we say that $\alpha \leq \beta$ in *refinement order* if $D(\beta) \subseteq D(\alpha)$. For instance, $[1, 1, 2, 1, 3, 2, 1, 4, 2] \leq [4, 4, 2, 7]$, since

$$D([1, 1, 2, 1, 3, 2, 1, 4, 2]) = \{1, 2, 4, 5, 8, 10, 11, 15\}$$

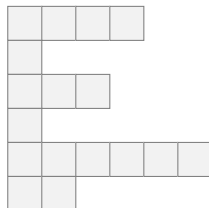
$$\text{and } D([4, 4, 2, 7]) = \{4, 8, 10\}.$$

We introduce a new notion that will arise in our Pieri rule (Theorem 3.5); we say that $\alpha \subset_i \beta$ if:

- (a) $|\beta| = |\alpha| + i$,
- (b) $\alpha_j \leq \beta_j$ for all $1 \leq j \leq \ell(\alpha)$,
- (c) $\ell(\beta) \leq \ell(\alpha) + 1$.

For a composition $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_\ell]$ and a positive integer m , we let $[m, \alpha]$ stand for the composition $[m, \alpha_1, \alpha_2, \dots, \alpha_\ell]$.

In this presentation, compositions will be represented as diagrams of left aligned rows of cells. The combinatorics of the elements that we introduce will lead us to represent our diagrams in this way rather than as a ribbon (as is the usual method for representing compositions when working with the ribbon Schur and fundamental bases). We will also use the matrix convention (“English” notation) that the first row of the diagram is at the top and the last row is at the bottom. For example, the composition $[4, 1, 3, 1, 6, 2]$ is represented as



Note that in examples and in a few formulas, compositions used as subscripts indexing elements of an algebra will often be written without enclosing brackets.

2.2 Schur Functions and Creation Operators

We use the standard notation for the common bases of Sym: h_λ for complete homogeneous; e_λ for elementary; m_λ for monomial; p_λ for power sums; s_λ for Schur. For simplicity, we let $h_i, e_i, m_i, p_i,$ and s_i denote the corresponding generators indexed by the partition $[i]$.

We next define a Schur function indexed by an arbitrary tuple of integers. The family of symmetric functions indexed by partitions λ are the usual Schur basis of the symmetric functions.

Definition 2.1 For an arbitrary integer tuple $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_\ell] \in \mathbb{Z}^\ell$, we define

$$s_\alpha := \det \begin{bmatrix} h_{\alpha_1} & h_{\alpha_1+1} & \cdots & h_{\alpha_1+\ell-1} \\ h_{\alpha_2-1} & h_{\alpha_2} & \cdots & h_{\alpha_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\alpha_\ell-\ell+1} & h_{\alpha_\ell-\ell+2} & \cdots & h_{\alpha_\ell} \end{bmatrix} = \det[h_{\alpha_i+j-i}]_{1 \leq i, j \leq \ell},$$

where we use the convention that $h_0 = 1$ and $h_{-m} = 0$ for $m > 0$.

With this definition, we notice that switching two adjacent rows of the defining matrix has the effect of changing the sign of the determinant. Switching rows of the matrix implies that we have the following equality:

$$s_{\alpha_1, \alpha_2, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_\ell} = -s_{\alpha_1, \alpha_2, \dots, \alpha_{r+1}-1, \alpha_r+1, \dots, \alpha_\ell}.$$

Two rows of the matrix are equal if $\alpha_i - i = \alpha_j - j$. This implies part of the following well-known result.

Proposition 2.2 *If α is a composition of n with length equal to k , then $s_\alpha = 0$ if and only if there exists $i, j \in \{1, 2, \dots, k\}$ with $i \neq j$ such that $\alpha_i - i = \alpha_j - j$. If this is not the case, then there is a unique permutation σ such that*

$$[\alpha_{\sigma_1} + 1 - \sigma_1, \alpha_{\sigma_2} + 2 - \sigma_2, \dots, \alpha_{\sigma_k} + k - \sigma_k]$$

is a partition. In this case,

$$s_\alpha = (-1)^\sigma s_{\alpha_{\sigma_1}+1-\sigma_1, \alpha_{\sigma_2}+2-\sigma_2, \dots, \alpha_{\sigma_k}+k-\sigma_k}.$$

Sym is a self-dual Hopf algebra. It has a pairing (the Hall scalar product) defined by

$$\langle h_\lambda, m_\mu \rangle = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}.$$

An element $f \in \text{Sym}$ gives rise to an operator $f^\perp: \text{Sym} \rightarrow \text{Sym}$ according to the relation

$$\langle fg, h \rangle = \langle g, f^\perp h \rangle \quad \text{for all } g, h \in \text{Sym}.$$

Using this as a definition, the action of the operator f^\perp can be calculated on another symmetric function by the formula

$$f^\perp(g) = \sum_\lambda \langle g, fa_\lambda \rangle b_\lambda,$$

where $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$ are any two bases that are dual with respect to the pairing $\langle \cdot, \cdot \rangle$.

We define a “creation” operator $\mathbf{B}_m: \text{Sym}_n \rightarrow \text{Sym}_{m+n}$ by

$$\mathbf{B}_m := \sum_{i \geq 0} (-1)^i h_{m+i} e_i^\perp.$$

The following theorem, which states that creation operators construct Schur functions, will become one of the motivations for our new basis of NSym (see Definition 3.2).

Theorem 2.3 (Bernstein [37, pp. 69–70]) *For all tuples $\alpha \in \mathbb{Z}^m$,*

$$s_\alpha = \mathbf{B}_{\alpha_1} \mathbf{B}_{\alpha_2} \cdots \mathbf{B}_{\alpha_m}(1).$$

Because of this result we shall refer to the \mathbf{B}_m operators as either creation or Bernstein operators.

2.3 Non-commutative Symmetric Functions

The algebra of non-commutative symmetric functions NSym is a non-commutative analogue of Sym that arises by considering an algebra with one non-commutative generator at each positive degree. In addition to the relationship with the symmetric functions, this algebra has links to Solomon’s descent algebra in type A [27], the algebra of quasi-symmetric functions [27], and representation theory of the type A Hecke algebra at $q = 0$ [21]. It is an example of a combinatorial Hopf algebra [1]. While we will follow the foundational results and definitions from references such as [11, 27], we have chosen to use notation here that is suggestive of analogous results in Sym.

We define NSym as the algebra with generators $\{H_1, H_2, \dots\}$ and no relations. Each generator H_i is defined to be of degree i , giving NSym the structure of a graded algebra. We let NSym_n denote the graded component of NSym of degree n . A basis for NSym_n are the complete homogeneous functions $\{H_\alpha := H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_m}\}_{\alpha \vdash n}$ indexed by compositions of n . To make this convention consistent, some formulas will use expressions that have H indexed by tuples of integers and we use the convention that $H_0 = 1$ and $H_{-r} = 0$ for $r > 0$.

There exists a map (sometimes referred to as the forgetful map) that we shall also denote $\chi: \text{NSym} \rightarrow \text{Sym}$, defined by sending the basis element H_α to the complete homogeneous symmetric function

$$(2.1) \quad \chi(H_\alpha) := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_{\ell(\alpha)}} \in \text{Sym}$$

and extended linearly to all of NSym. This map is a surjection onto Sym.

Similarly to the study of Sym and the ring of characters for the symmetric groups, the ring of non-commutative symmetric functions is isomorphic to the Grothendieck ring of finitely generated indecomposable projective representations of the 0-Hecke algebra. We state this fact only as an analogy to Sym; we will not use it in this paper. We refer the reader to [21] for details.

The element of NSym that corresponds to the indecomposable projective representation indexed by α is here denoted R_α . The collection of R_α are a basis of NSym, usually called the *ribbon basis* of NSym. They are defined through their expansion in the complete homogeneous basis:

$$(2.2) \quad R_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} H_\beta \quad \text{or equivalently} \quad H_\alpha = \sum_{\beta \geq \alpha} R_\beta.$$

The product expansion follows easily from the non-commutative product on the generators

$$H_\alpha H_\beta = H_{\alpha_1, \dots, \alpha_{\ell(\alpha)}, \beta_1, \dots, \beta_{\ell(\beta)}}.$$

NSym has a coalgebra structure, which is defined on the generators by

$$\Delta(H_j) = \sum_{i=0}^j H_i \otimes H_{j-i}.$$

This determines the action of the coproduct on the basis H_α , since the coproduct is an algebra morphism with respect to the product. Explicitly we have

$$\Delta(H_\alpha) = \Delta(H_{\alpha_1}) \Delta(H_{\alpha_2}) \cdots \Delta(H_{\alpha_{\ell(\alpha)}}).$$

2.4 Quasi-symmetric Functions

The algebra of quasi-symmetric functions, QSym, was introduced in [10] (see also subsequent references such as [12, 34]). This algebra has become a useful tool for algebraic combinatorics, since it is an algebra that is dual to NSym and contains Sym as a subalgebra.

As with the algebra NSym, the graded component QSym_n is indexed by compositions of n . The algebra is most readily realized within the ring of power series of bounded degree $\mathbb{Q}[[x_1, x_2, \dots]]$. The monomial quasi-symmetric function indexed by a composition α is defined as

$$M_\alpha = \sum_{i_1 < i_2 < \dots < i_m} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_m}^{\alpha_m}.$$

The algebra of quasi-symmetric functions, QSym, can then be defined as the algebra with the monomial quasi-symmetric functions as a basis, whose multiplication is inherited as a subalgebra of $\mathbb{Q}[[x_1, x_2, \dots]]$. We define the coproduct on this basis as

$$\Delta(M_\alpha) = \sum_{S \subset \{1, 2, \dots, \ell(\alpha)\}} M_{\alpha_S} \otimes M_{\alpha_{S^c}},$$

where if $S = \{i_1 < i_2 < \dots < i_{|S|}\}$, then $\alpha_S = [\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{|S|}}]$.

We view Sym as a subalgebra of QSym . In fact, the usual monomial symmetric functions $m_\lambda \in \text{Sym}$ expand positively in the quasi-symmetric monomial functions

$$m_\lambda = \sum_{\text{sort}(\alpha)=\lambda} M_\alpha,$$

where $\text{sort}(\alpha)$ is the partition obtained by organizing the parts of α from the largest to the smallest.

Similarly to NSym , the algebra QSym is isomorphic to the Grothendieck ring of finite-dimensional representations of the 0-Hecke algebra. The irreducible representations of the 0-Hecke algebra form a basis for this ring, and under this isomorphism the irreducible representation indexed by α is identified with an element of QSym , the *fundamental quasi-symmetric function*, denoted F_α . The F_α , for $\alpha \models n$, form a basis of QSym_n and are defined by their expansion in the monomial quasi-symmetric basis:

$$F_\alpha = \sum_{\beta \leq \alpha} M_\beta.$$

2.5 Identities Relating Non-commutative and Quasi-symmetric Functions

The algebras QSym and NSym form graded dual Hopf algebras. The monomial basis of QSym is dual in this context to the complete homogeneous basis of NSym , and the fundamental basis of QSym is dual to the ribbon basis of NSym . NSym and QSym have a pairing $\langle \cdot, \cdot \rangle : \text{NSym} \times \text{QSym} \rightarrow \mathbb{Q}$, defined under this duality as either $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$ or $\langle R_\alpha, F_\beta \rangle = \delta_{\alpha,\beta}$.

We will generalize the operation that is dual to multiplication by a quasi-symmetric function using this pairing. For $F \in \text{QSym}$, let F^\perp be the operator that acts on elements $H \in \text{NSym}$ according to the relation $\langle H, FG \rangle = \langle F^\perp H, G \rangle$. To expand $F^\perp(H)$, we take a basis $\{A_\alpha\}_\alpha$ of QSym and $\{B_\alpha\}_\alpha$ a basis of NSym such that $\langle B_\alpha, A_\beta \rangle = \delta_{\alpha,\beta}$; then

$$F^\perp(H) = \sum_\alpha \langle H, FA_\alpha \rangle B_\alpha.$$

By the duality of the product and the coproduct structure of NSym and QSym , we have for $F, G \in \text{QSym}$ and $H, K \in \text{NSym}$, that the pairing satisfies $\langle HK, G \rangle = \langle H \otimes K, \Delta(G) \rangle$ and $\langle H, FG \rangle = \langle \Delta(H), F \otimes G \rangle$. As a consequence of the first of these two identities, we have the following lemma.

Lemma 2.4 *If $G \in \text{QSym}$ and $\Delta(G) = \sum_i G^{(i)} \otimes G_{(i)}$, then for $H, K \in \text{NSym}$,*

$$G^\perp(HK) = \sum_i G^{(i)\perp}(H)G_{(i)}^\perp(K).$$

Proof Because of the duality of the Hopf algebra structure between NSym and QSym,

$$\begin{aligned}
 G^\perp(HK) &= \sum_{\alpha} \langle G^\perp(HK), A_\alpha \rangle B_\alpha = \sum_{\alpha} \langle HK, GA_\alpha \rangle B_\alpha \\
 &= \sum_{\alpha} \langle H \otimes K, \Delta(GA_\alpha) \rangle B_\alpha = \sum_{\alpha} \langle H \otimes K, \Delta(G)\Delta(A_\alpha) \rangle B_\alpha \\
 &= \sum_{\alpha} \sum_i \langle H \otimes K, (G^{(i)} \otimes G_{(i)})\Delta(A_\alpha) \rangle B_\alpha \\
 &= \sum_{\alpha} \sum_i \langle G^{(i)\perp}(H)G_{(i)}^\perp(K), A_\alpha \rangle B_\alpha. \quad \blacksquare
 \end{aligned}$$

Another way we can compute the action of M_α^\perp is by using the following lemma.

Lemma 2.5 For $G \in \text{NSym}$, if the coproduct on G has the expansion $\Delta(G) = \sum_{\gamma} H_\gamma \otimes G^{(\gamma)}$, then $M_\alpha^\perp(G) = G^{(\alpha)}$.

Proof Let G be an element of NSym such that $\Delta(G)$ has the expansion $\sum_{\gamma} H_\gamma \otimes G^{(\gamma)}$ with the H -basis in the left tensor. A direct computation shows

$$\begin{aligned}
 M_\alpha^\perp(G) &= \sum_{\beta} \langle G, M_\alpha M_\beta \rangle H_\beta = \sum_{\beta} \langle \Delta(G), M_\alpha \otimes M_\beta \rangle H_\beta \\
 &= \sum_{\beta} \sum_{\gamma} \langle H_\gamma \otimes G^{(\gamma)}, M_\alpha \otimes M_\beta \rangle H_\beta = \sum_{\beta} \langle G^{(\alpha)}, M_\beta \rangle H_\beta = G^{(\alpha)}. \quad \blacksquare
 \end{aligned}$$

To develop some of the formulas for the immaculate basis we will need some algebraic identities on NSym and QSym. These are standard results that are analogous to the corresponding formulas in Sym, but require some development of the algebra to verify their correctness.

As a consequence of Lemma 2.4 we have the following relations.

Lemma 2.6 For $i, j > 0$ and for $f \in \text{NSym}$,

$$(2.3) \quad F_{1^i}^\perp(fH_j) = F_{1^i}^\perp(f)H_j + F_{1^{i-1}}^\perp(f)H_{j-1}$$

$$(2.4) \quad F_i^\perp(fH_j) = \sum_{k=0}^{\min(i,j)} F_{i-k}^\perp(f)H_{j-k}.$$

Proof Since F_i and F_{1^i} are, respectively, h_i and e_i in Sym, we know the coproduct rule $\Delta(F_{1^i}) = \sum_{k=0}^i F_{1^{i-k}} \otimes F_{1^k}$, and $\Delta(F_i) = \sum_{k=0}^i F_{i-k} \otimes F_k$. From Lemma 2.5 and the fact that $F_r^\perp = \sum_{\alpha \models r} M_\alpha^\perp$, we calculate that $F_r^\perp(H_j) = \sum_{\alpha \models r} M_\alpha^\perp(H_j) = H_{j-r}$ for $1 \leq r \leq j$. Another application of Lemma 2.5 with $F_{1^r}^\perp = M_{1^r}^\perp$ shows that for $s > 1$, $F_{1^s}^\perp(H_j) = M_{1^s}^\perp(H_j) = 0$. Equations (2.3) and (2.4) are a consequence of Lemma 2.4. ■

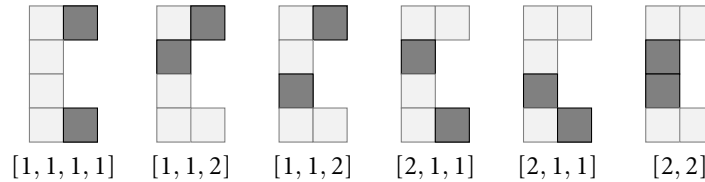
By applying the formulas in the previous lemma, the following expansions may be shown by induction on the length of the composition.

Corollary 2.7 For all $i \geq 0$ and all compositions $\alpha \models n$ such that $\ell(\alpha) = m$,

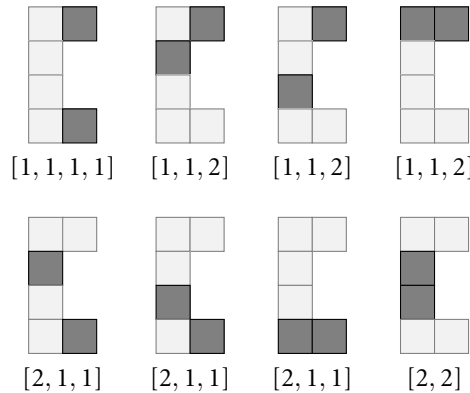
$$F_i^\perp(H_\alpha) = \sum_{\substack{\beta \in \mathbb{N}^m \\ |\beta| = |\alpha| - i \\ \alpha_j - 1 \leq \beta_j \leq \alpha_j}} H_\beta, \quad F_i^\perp(H_\alpha) = \sum_{\substack{\gamma \in \mathbb{N}^m \\ |\gamma| = |\alpha| - i \\ 0 \leq \gamma_j \leq \alpha_j}} H_\gamma.$$

In both sums we use the convention that parts of size 0 are deleted from the tuple when it is re-expressed in the H -basis because $H_0 = 1$.

Example 2.8 We compute $F_{11}^\perp H_{2112} = H_{22} + 2H_{211} + 2H_{112} + H_{1111}$. This comes from removing two boxes from $[2, 1, 1, 2]$ in the following ways and not considering parts of size 0:



Example 2.9 We compute $F_2^\perp H_{2112} = H_{1111} + 3H_{112} + 3H_{211} + H_{22}$. This comes from removing two boxes from $[2, 1, 1, 2]$ in the following ways:



We call a linear ordering of variables (y_1, y_2, \dots) an *alphabet*. Note that an alphabet could be finite or countable.

For an alphabet Y , we define

$$\tilde{\Omega}_Y = \sum_{\alpha} M_{\alpha}[Y] H_{\alpha},$$

where $M_{\alpha}[Y]$ is the monomial quasi-symmetric function indexed by α expanded over an alphabet Y , and the monomial quasi-symmetric functions are allowed to

commute with the non-commutative symmetric functions. In particular when Y is an alphabet with a single variable,

$$\tilde{\Omega}_z = \sum_{d \geq 0} z^d H_d.$$

If $Y = (y_1, y_2, \dots)$ is an alphabet and z is another variable not in Y , we let z, Y denote the alphabet (z, y_1, y_2, \dots) . We notice that $\tilde{\Omega}_z \tilde{\Omega}_Y = \tilde{\Omega}_{z,Y}$ by calculating

$$\begin{aligned} \tilde{\Omega}_{z,Y} &= \sum_{\alpha} M_{\alpha}[z, Y] H_{\alpha} = \sum_{\alpha} (M_{\alpha}[Y] + z^{\alpha_1} M_{\alpha_2, \dots, \alpha_m}[Y]) H_{\alpha} \\ &= \sum_{\alpha} M_{\alpha}[Y] H_{\alpha} + \sum_{\alpha} z^{\alpha_1} H_{\alpha_1} M_{\alpha_2, \dots, \alpha_m}[Y] H_{\alpha_2, \dots, \alpha_m} \\ &= \sum_{\alpha} M_{\alpha}[Y] H_{\alpha} + \sum_{d > 0} z^d H_d \sum_{\beta} M_{\beta}[Y] H_{\beta} \\ &= \left(\sum_{d \geq 0} z^d H_d \right) \left(\sum_{\gamma} M_{\gamma}[Y] H_{\gamma} \right) = \tilde{\Omega}_z \tilde{\Omega}_Y. \end{aligned}$$

We can then determine by induction that $\tilde{\Omega}_Z \tilde{\Omega}_Y = \tilde{\Omega}_{Z,Y}$ for a finite alphabet Z .

For $F \in \text{QSym}$, F^{\perp} acts on the non-commutative symmetric functions and does not affect the monomial quasi-symmetric function coefficients

$$\begin{aligned} F^{\perp} \tilde{\Omega}_Y &= \sum_{\alpha} M_{\alpha}[Y] F^{\perp}(H_{\alpha}) = \sum_{\alpha} M_{\alpha}[Y] \sum_{\beta} \langle H_{\alpha}, FM_{\beta} \rangle H_{\beta} \\ &= \sum_{\beta} \sum_{\alpha} \langle H_{\alpha}, FM_{\beta} \rangle M_{\alpha}[Y] H_{\beta} = \sum_{\beta} F[Y] M_{\beta}[Y] H_{\beta} = F[Y] \tilde{\Omega}_Y. \end{aligned}$$

We also define the two operators with the parameter z as

$$\mathcal{E}_z^{\perp} = \sum_{i \geq 0} z^i F_{1^i}^{\perp}, \text{ and } \mathcal{H}_z^{\perp} = \sum_{i \geq 0} z^i F_i^{\perp},$$

then

$$\begin{aligned} \mathcal{E}_z^{\perp} \tilde{\Omega}_Y &= \sum_{i \geq 0} z^i F_{1^i}[Y] \tilde{\Omega}_Y = \tilde{\Omega}_Y \prod_{y \in Y} (1 + zy), \\ \mathcal{H}_z^{\perp} \tilde{\Omega}_Y &= \sum_{i \geq 0} z^i F_i[Y] \tilde{\Omega}_Y = \tilde{\Omega}_Y / \prod_{y \in Y} (1 - zy). \end{aligned}$$

3 A New Basis for NSym

We are now ready to introduce our new basis of NSym. These functions were discovered while playing with a non-commutative analogue of the Jacobi–Trudi identity (Theorem 3.27). They may also be defined as the unique functions in NSym that satisfy a right-Pieri rule (Theorem 3.5 and Proposition 3.32). In order to streamline our proofs and extend our definitions to a Hall–Littlewood analogue, we start by building our new basis using a non-commutative version of the Bernstein operators (Theorem 2.3).

3.1 Non-commutative Immaculate Functions

We continue with the notation of the previous section: H_i is the complete homogeneous non-commutative symmetric function; F_α is the fundamental quasi-symmetric function indexed by the composition α ; and F_α^\perp is the linear transformation of NSym that is adjoint to multiplication by F_α in QSym.

Definition 3.1 We define the *non-commutative Bernstein operators* \mathbb{B}_m as

$$\mathbb{B}_m = \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^i}^\perp.$$

Using the non-commutative Bernstein operators, we can inductively build functions using creation operators similar to Bernstein’s formula (Theorem 2.3) for the Schur functions.

Definition 3.2 For any $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_m] \in \mathbb{Z}^m$, the *immaculate function* $\mathfrak{S}_\alpha \in \text{NSym}$ is defined as the composition of the operators applied to 1 in the expression

$$\mathfrak{S}_\alpha := \mathbb{B}_{\alpha_1} \mathbb{B}_{\alpha_2} \cdots \mathbb{B}_{\alpha_m}(1).$$

Calculations in the next subsection will show that the elements $\{\mathfrak{S}_\alpha\}_{|\alpha|=n}$ form a basis for NSym_n .

Example 3.3 If $\alpha = (a)$ has only one part, then \mathfrak{S}_a is just the complete homogeneous generator H_a . If $\alpha = [a, b]$ consists of two parts, then $\mathfrak{S}_{ab} = \mathbb{B}_a(H_b) = H_a H_b - H_{a+1} H_{b-1}$.

3.2 The Right-Pieri Rule for Immaculate Functions

Lemma 3.4 For $s \geq 0$ and $m \in \mathbb{Z}$, and for f an element of NSym,

$$\mathbb{B}_m(f)H_s = \mathbb{B}_{m+1}(f)H_{s-1} + \mathbb{B}_m(fH_s).$$

Proof By definition,

$$\mathbb{B}_m(fH_s) = \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^i}^\perp(fH_s).$$

Using Lemma 2.6, one obtains

$$\mathbb{B}_m(fH_s) = \sum_{i \geq 0} (-1)^i H_{m+i} (F_{i-1}^\perp(f)H_{s-1} + F_i^\perp(f)H_s),$$

which by associativity and reindexing gives

$$\mathbb{B}_m(fH_s) = -\mathbb{B}_{m+1}(f)H_{s-1} + \mathbb{B}_m(f)H_s. \quad \blacksquare$$

Theorem 3.5 For a composition α , the \mathfrak{S}_α satisfy a multiplicity free right-Pieri rule for multiplication by H_s ,

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta,$$

where the notation \subset_s is introduced in Section 2.1.

Proof Let $m = \alpha_1$ and let $\bar{\alpha} = [\alpha_2, \alpha_3, \dots, \alpha_k]$ denote the composition with first part removed. The proof will be by induction on $s + \ell(\alpha)$, the base case being trivial. By definition, $\mathbb{B}_m \mathfrak{S}_{\bar{\alpha}} = \mathfrak{S}_\alpha$, so

$$\begin{aligned} \mathfrak{S}_\alpha H_s &= \mathbb{B}_m(\mathfrak{S}_{\bar{\alpha}})H_s \\ &= \mathbb{B}_{m+1}(\mathfrak{S}_{\bar{\alpha}})H_{s-1} + \mathbb{B}_m(\mathfrak{S}_{\bar{\alpha}})H_s \quad \text{by Lemma 3.4,} \\ &= \mathfrak{S}_{[m+1, \bar{\alpha}]}H_{s-1} + \mathbb{B}_m\left(\sum_{\bar{\alpha} \subset_s \eta} \mathfrak{S}_\eta\right) \quad \text{by the Pieri rule on } \bar{\alpha} \text{ and } s, \\ &= \sum_{[m+1, \bar{\alpha}] \subset_{s-1} \gamma} \mathfrak{S}_\gamma + \sum_{\bar{\alpha} \subset_s \eta} \mathfrak{S}_{[m, \eta]} \quad \text{by the Pieri rule on } [m+1, \bar{\alpha}] \text{ and } s-1. \end{aligned}$$

The first sum counts all γ that arise from adding boxes to α , adding at least one to the first part of the composition, which are bounded in length by $\ell(\alpha) + 1$. The second sum counts all η that arise from adding boxes to α , without adding to the first part, which are bounded in length by $\ell(\alpha) + 1$. The statement now follows from combining the two sums. ■

Remark 3.6 Products of the form $H_m \mathfrak{S}_\alpha$ do not have as nice an expression as $\mathfrak{S}_\alpha H_m$, since they generally have negative signs in their expansion and there is no obvious containment of resulting compositions. For example,

$$H_1 \mathfrak{S}_{13} = \mathfrak{S}_{113} - \mathfrak{S}_{221} - \mathfrak{S}_{32}.$$

The reason for this is that left multiplication by H_m can be re-expressed as

$$H_m = \sum_{i \geq 0} \mathbb{B}_{m+i} F_i^\perp.$$

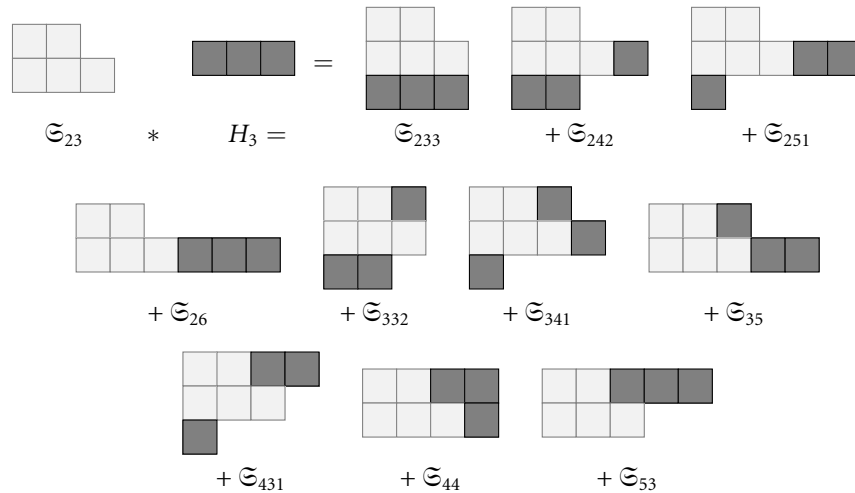
We will develop the algebra required to understand where the negative signs potentially arise, but we will not give a satisfactory left Pieri rule. We conjecture that the left Pieri rule is multiplicity free, up to sign.

Conjecture 3.7

$$H_m \mathfrak{S}_\alpha = \sum_{\beta} (-1)^{\text{sign}(\alpha, \beta)} \mathfrak{S}_\beta,$$

where the sum is over some collection of compositions β of size $|\alpha| + m$ and sign is some statistic that depends on α and β .

Example 3.8 The expansion of \mathfrak{S}_{23} multiplied on the right by H_3 is done below, along with corresponding pictures.



3.3 Relationship with the Classical Bases of NSym

We will now develop some relations between the classical bases of NSym and the immaculate basis. In particular, the first result will establish the fact that the immaculate functions indexed by compositions do in fact form a graded basis of NSym. First, we need the notion of an immaculate tableau.

3.3.1 Immaculate Tableaux

Definition 3.9 Let α and β be compositions. An *immaculate tableau* of shape α and content β is a labelling of the boxes of the diagram of α by positive integers in such a way that:

- (a) the number of boxes labelled by i is β_i ;
- (b) the sequence of entries in each row, from left to right, is weakly increasing;
- (c) the sequence of entries in the *first* column, from top to bottom, is increasing.

An immaculate tableau is said to be *standard* if it has content $1^{|\alpha|}$.

Let $K_{\alpha, \beta}$ denote the number of immaculate tableaux of shape α and content β .

We reiterate that, aside from the elements in the first column, there is no relation among the elements in the columns of an immaculate tableau.

Example 3.10 There are five immaculate tableaux of shape $[4, 2, 3]$ and content $[3, 1, 2, 3]$:

1	1	1	3
2	3		
4	4	4	

1	1	1	3
2	4		
3	4	4	

1	1	1	4
2	3		
3	4	4	

1	1	1	4
2	4		
3	3	4	

1	1	1	2
3	3		
4	4	4	

Standard immaculate tableaux of size n can be identified with set partitions of $\{1, 2, \dots, n\}$ by ordering the parts in the partition by minimal elements. This remark¹ allows us to state a surprising enumeration formula for standard immaculate tableaux that is analogous to the hook length formula for standard tableaux. In order to state and prove this formula, we need to define the standardization of immaculate tableaux and hooks of cells.

Definition 3.11 Given an immaculate tableau T of shape α and content β , we form a standard immaculate tableau $\text{std}(T) = S$ of shape α and content (1^n) as follows. We order the entries of the tableau T , reading first all entries valued 1, then 2, etc. Among all entries with the same value, we first read all entries in the lowest row, starting at the leftmost position and read first to the right and then up rows. The order of the entries forms a standard immaculate tableau that we call the standardization of T .

Example 3.12 The following tableau has shape $[6, 5, 7]$ and content $[2, 3, 5, 1, 4, 3]$.

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 & 4 \\ \hline 2 & 3 & 3 & 3 & 3 & \\ \hline 5 & 5 & 5 & 5 & 6 & 6 & 6 \\ \hline \end{array} .$$

The standardization of T is:

$$S = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 10 & 11 \\ \hline 3 & 6 & 7 & 8 & 9 & \\ \hline 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ \hline \end{array} .$$

Let $c = (i, j)$ be a cell in row i and column j of the diagram for a composition α (that is, $1 \leq i \leq \ell(\alpha)$ and $1 \leq j \leq \alpha_i$). If $c = (i, 1)$, define the hook of c in α to be $h_\alpha(c) = \alpha_i + \alpha_{i+1} + \dots + \alpha_{\ell(\alpha)}$ (the number of cells below and to the right in the diagram). If $j > 1$, then the hook of c in α is $h_\alpha(c) = \alpha_i - j + 1$ (the number of cells weakly to the right in the same row).

Proposition 3.13 If $\alpha \models n$, the number of standard immaculate tableaux of shape α is equal to

$$(3.1) \quad K_{\alpha, 1^n} = \frac{n!}{\prod_{c \in \alpha} h_\alpha(c)},$$

where $c \in \alpha$ indicates $c = (i, j)$ with $1 \leq i \leq \ell(\alpha)$ and $1 \leq j \leq \alpha_i$.

¹Pointed out to us in a discussion with Martha Yip.

Proof Consider a standard immaculate tableau of shape $[k, \alpha] \models n + k$. The first row contains 1 and a subset S of $k - 1$ other integers from $\{2, 3, \dots, n + k\}$. Moreover, if we standardize rows 2 through $\ell(\alpha) + 1$ of this tableau, then we have a standard immaculate tableau of shape α . This gives us a bijection between standard immaculate tableaux of shape $[k, \alpha]$ and the set of pairs (S, T) , where S is a subset of $\{2, 3, \dots, n + k\}$ of size $k - 1$ and T is a standard immaculate tableau of shape α . This is a bijective proof of the recursion

$$K_{[k, \alpha], 1^{n+k}} = \binom{n+k-1}{k-1} K_{\alpha, 1^n}.$$

Now the hook length formula follows by an induction argument on the length of the composition α . Assume that we know equation (3.1) holds for compositions α of length ℓ , then the hooks of the cells in the first row of $[k, \alpha]$ are (respectively) $n + k, k - 1, k - 2, \dots, 2, 1$:

$$\begin{aligned} K_{[k, \alpha], 1^{n+k}} &= \binom{n+k-1}{k-1} K_{\alpha, 1^n} = \binom{n+k-1}{k-1} \frac{n!}{\prod_{c \in \alpha} h_\alpha(c)} \\ &= \frac{(n+k-1)(n+k-2) \cdots (n+1)}{(k-1)!} \frac{n!}{\prod_{c \in \alpha} h_\alpha(c)} \\ &= \frac{(n+k)(n+k-1)(n+k-2) \cdots (n+1)}{(n+k)(k-1)!} \frac{n!}{\prod_{c \in \alpha} h_\alpha(c)} \\ &= \frac{(n+k)!}{\prod_{c \in [k, \alpha]} h_\alpha(c)}. \end{aligned}$$

This shows that (3.1) holds for compositions of length $\ell + 1$. The base case for $\ell = 0$ holds trivially. Therefore the hook length formula (3.1) holds for all compositions. ■

Example 3.14 The hook length formula (3.1) says that since the hooks of $[4, 2, 3]$ are given by the entries in the diagram

9	3	2	1	,
5	1			
3	2	1		

The number of standard immaculate tableaux of shape $[4, 2, 3]$ is equal to

$$\frac{9!}{9 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 224.$$

Proposition 3.15 Recall that $K_{\alpha,\beta}$ denote the number of immaculate tableaux of shape α and content β . Then

- (i) $K_{\alpha,\alpha} = 1$,
- (ii) $K_{\alpha,\beta} = 0$ unless $\alpha \geq_{\ell} \beta$,

where \geq_{ℓ} represents the lexicographic order on compositions.

Proof The first fact follows from definition: there is only one tableau with shape and content α ; it consists of α_i many i 's in row i . For the second statement, we argue by contradiction. Suppose there is such a tableau and let i be the first integer such that $\alpha_i < \beta_i$ (so that $\alpha_j = \beta_j$ for $j < i$). As above, one must place β_j many j 's in row j , for $j < i$, filling all positions in the rows above row i . Then one must place β_i many i 's into row i , which contains only α_i spaces, a contradiction. ■

3.3.2 Expansion of the Homogeneous Basis

Proposition 3.16 The complete homogeneous basis H_{α} has a positive, uni-triangular expansion in the immaculate functions indexed by compositions. Specifically,

$$H_{\beta} = \sum_{\alpha \geq_{\ell} \beta} K_{\alpha,\beta} \mathfrak{S}_{\alpha}.$$

Proof Follows from repeated application of the Pieri rule (Theorem 3.5). To see this, start with $H_{\beta_1} = \mathfrak{S}_{\beta_1}$, which is clearly a sum over immaculate tableaux of shape and content β_1 . Each time one multiplies by H_i , by the Pieri rule one adds to the immaculate tableaux in the index of summation, a set of entries labelled i that satisfy the conditions of the definition of an immaculate tableau. The reason that we need only sum over $\alpha \geq_{\ell} \beta$ is explained in Proposition 3.15. ■

Example 3.17 Continuing from Example 3.10, we see that

$$H_{3123} = \cdots + 5\mathfrak{S}_{423} + \cdots.$$

Corollary 3.18 The $\{\mathfrak{S}_{\alpha} : \alpha \vDash n\}$ form a basis of NSym_n .

Proof These elements span NSym_n by Proposition 3.16 and we have the correct number of elements, so they form a basis. ■

Remark 3.19 We will now use the term “immaculate basis” to mean those immaculate functions that are indexed by compositions.

3.3.3 Expansion of the Ribbon Basis

Next we will expand the ribbon functions in the immaculate basis. To do this, we first need the notions of standardization and descent.

Definition 3.20 We say that a standard immaculate tableau T has a descent in position i if $(i+1)$ is in a row strictly lower than i in T . The symbol $D(T)$ will represent the set of descents in T and $\alpha(D(T))$ will represent the corresponding composition.

Example 3.21 The standard immaculate tableau S of Example 3.12 has descents in positions $\{2, 5, 11\}$. The descent composition of S is then $[2, 3, 6, 7]$.

Remark 3.22 The content of an immaculate tableau T must be a refinement of the descent composition $\alpha(D(\text{std}(T)))$. This follows from the definition of standardization; the only places descents can happen in $\text{std}(T)$ are the final occurrence of a given content in T .

We let $L_{\alpha,\beta}$ denote the number of standard immaculate tableaux of shape α and descent composition β .

Lemma 3.23

$$K_{\alpha,\gamma} = \sum_{\beta \geq \gamma} L_{\alpha,\beta}.$$

Proof Standardization provides a bijection between the set of immaculate tableaux of shape α and content γ and the set of standard immaculate tableaux of shape α and descent composition $\beta \geq \gamma$. The result follows from this bijection. ■

Example 3.24 The five immaculate tableaux having shape $[4, 2, 3]$ and content $[3, 1, 2, 3]$

1	1	1	3	1	1	1	3	1	1	1	4	1	1	1	4	1	1	1	2
2	3			2	4			2	3			2	4			3	3		
4	4	4		3	4	4		3	4	4		3	3	4		4	4	4	

are bijected, under standardization, with the five standard immaculate tableaux of shape $[4, 2, 3]$ and descent composition $\beta \geq [3, 1, 2, 3]$ below:

1	2	3	6	1	2	3	6	1	2	3	9	1	2	3	9	1	2	3	4
4	5			4	9			4	6			4	8			5	6		
7	8	9		5	7	8		5	7	8		5	6	7		7	8	9	
[3, 3, 3]				[3, 1, 2, 3]				[3, 1, 2, 3]				[3, 1, 5]				[4, 2, 3].			

Theorem 3.25 The ribbon function R_β has a positive expansion in the immaculate basis. Specifically,

$$R_\beta = \sum_{\alpha \geq \beta} L_{\alpha,\beta} \mathfrak{S}_\alpha.$$

Proof By Proposition 3.16, we know that $H_\gamma = \sum_{\alpha} K_{\alpha,\gamma} \mathfrak{S}_\alpha$, and we can substitute Lemma 3.23 and obtain

$$H_\gamma = \sum_{\alpha} K_{\alpha,\gamma} \mathfrak{S}_\alpha = \sum_{\alpha} \sum_{\beta \geq \gamma} L_{\alpha,\beta} \mathfrak{S}_\alpha = \sum_{\beta \geq \gamma} \left(\sum_{\alpha} L_{\alpha,\beta} \mathfrak{S}_\alpha \right),$$

which is the defining relation for the ribbon basis (see equation (2.2)). We know that $L_{\alpha,\beta} \leq K_{\alpha,\beta}$, hence we need only sum over $\alpha \geq \beta$. ■

Example 3.26 There are eight standard immaculate tableaux with descent composition $[2, 2, 2]$, giving the expansion of R_{222} into the immaculate basis:

$$\begin{aligned}
 R_{222} = & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} \mathfrak{S}_{222} & + \begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline 3 & 4 & 6 \\ \hline 5 & & \\ \hline \end{array} \mathfrak{S}_{231} & + \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline 5 & 6 & \\ \hline \end{array} \mathfrak{S}_{312} & + \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \hline 5 & & \\ \hline \end{array} \mathfrak{S}_{321} & + 2 \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & \\ \hline 5 & & \\ \hline \end{array} \mathfrak{S}_{321} \\
 & + \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline & & \\ \hline \end{array} \mathfrak{S}_{33} & + \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & & & \\ \hline 5 & & & \\ \hline \end{array} \mathfrak{S}_{411} & + \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & & \\ \hline & & & \\ \hline \end{array} \mathfrak{S}_{42}
 \end{aligned}$$

3.4 Jacobi–Trudi Rule for NSym

Another compelling reason to study the immaculate functions is that they have an expansion in the H_α basis that makes them a clear analogue of the Jacobi–Trudi rule of Definition 2.1.

Theorem 3.27 For $\alpha \in \mathbb{Z}^m$,

$$(3.2) \quad \mathfrak{S}_\alpha = \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1 + \sigma_1 - 1, \alpha_2 + \sigma_2 - 2, \dots, \alpha_m + \sigma_m - m},$$

where we have used the convention that $H_0 = 1$ and $H_{-m} = 0$ for $m > 0$.

Remark 3.28 This sum is a non-commutative analogue of the determinant of the following matrix:

$$\begin{bmatrix} H_{\alpha_1} & H_{\alpha_1+1} & \cdots & H_{\alpha_1+\ell-1} \\ H_{\alpha_2-1} & H_{\alpha_2} & \cdots & H_{\alpha_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\alpha_\ell-\ell+1} & H_{\alpha_\ell-\ell+2} & \cdots & H_{\alpha_\ell} \end{bmatrix}.$$

The non-commutative analogue of the determinant corresponds to expanding the determinant of this matrix about the first row and multiplying those elements on the left.

Remark 3.29 One might ask why one would naturally expand about the first row rather than, say, the first column or the last row. What we considered to be the natural analogue of expanding about the first column however is not a basis, since, for instance, the matrix corresponding to $\alpha = [1, 2]$ would be 0 under this analogue.

Before we begin with the proof we introduce some notation that will prove useful in our development of this identity. Let z be a single variable and define the operator $\mathbb{B}(z) = \sum_{r \in \mathbb{Z}} z^r \mathbb{B}_r$. We will show that the coefficient of $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ in $\mathbb{B}(z_1)\mathbb{B}(z_2) \cdots \mathbb{B}(z_m)(1)$ is the right-hand side of (3.2).

Using the notation developed in Section 2.5,

$$\begin{aligned} \mathbb{B}(z) &= \sum_{m \in \mathbb{Z}} z^m \mathbb{B}_m = \sum_{m \in \mathbb{Z}} z^m \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^i}^\perp \\ &= \sum_{d \geq 0} \sum_{i \geq 0} z^{d-i} (-1)^i H_d F_{1^i}^\perp \\ &= \left(\sum_{d \geq 0} z^d H_d \right) \left(\sum_{i \geq 0} (-1/z)^i F_{1^i}^\perp \right) = \tilde{\Omega}_z \mathcal{E}_{-1/z}^\perp. \end{aligned}$$

Hence, for every indeterminate z_r and every alphabet Y of commuting indeterminates that commute with z_r but are distinct from z_r , we have

$$(3.3) \quad \mathbb{B}(z_r) \tilde{\Omega}_Y = \tilde{\Omega}_{z_r} \mathcal{E}_{-1/z_r}^\perp \tilde{\Omega}_Y = \tilde{\Omega}_{z_r} \tilde{\Omega}_Y \prod_{y \in Y} (1 - y/z_r) = \tilde{\Omega}_{z_r, Y} \prod_{y \in Y} (1 - y/z_r).$$

Repeated application of (3.3) yields that for any alphabet Y of commuting indeterminates that commute with z_1, z_2, \dots, z_r but are distinct from each of them, we have

$$(3.4) \quad \mathbb{B}(z_1) \mathbb{B}(z_2) \cdots \mathbb{B}(z_m) \tilde{\Omega}_Y = \tilde{\Omega}_{z_1, z_2, \dots, z_m, Y} \prod_{r=1}^m \left(\prod_{y \in \{z_{r+1}, \dots, z_m\} \cup Y} (1 - y/z_r) \right).$$

Proof of Theorem 3.27 Taking $Y = \emptyset$ in (3.4), we obtain the identity

$$(3.5) \quad \mathbb{B}(z_1) \mathbb{B}(z_2) \cdots \mathbb{B}(z_m) 1 = \tilde{\Omega}_{z_1, z_2, \dots, z_m} \prod_{1 \leq i < j \leq m} (1 - z_j/z_i).$$

Using the Vandermonde determinant identity,

$$\prod_{1 \leq i < j \leq m} (z_i - z_j) = \sum_{\sigma \in S_m} (-1)^\sigma z_1^{m-\sigma_1} z_2^{m-\sigma_2} \cdots z_m^{m-\sigma_m},$$

we can rewrite (3.5) as

$$\begin{aligned} \mathbb{B}(z_1) \mathbb{B}(z_2) \cdots \mathbb{B}(z_m) 1 &= \tilde{\Omega}_{z_1, z_2, \dots, z_m} \prod_{1 \leq i < j \leq m} (1 - z_j/z_i) \\ &= \tilde{\Omega}_{z_1, z_2, \dots, z_m} z_1^{1-m} z_2^{2-m} \cdots z_m^{m-m} \prod_{1 \leq i < j \leq m} (z_i - z_j) \\ &= \tilde{\Omega}_{z_1, z_2, \dots, z_m} \sum_{\sigma \in S_m} (-1)^\sigma z_1^{1-\sigma_1} z_2^{2-\sigma_2} \cdots z_m^{m-\sigma_m}. \end{aligned}$$

So the coefficient of $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ in $\mathbb{B}(z_1) \cdots \mathbb{B}(z_m)(1)$ is equal to

$$\begin{aligned} \mathfrak{S}_\alpha &= \mathbb{B}_{\alpha_1} \mathbb{B}_{\alpha_2} \cdots \mathbb{B}_{\alpha_m} 1 = \mathbb{B}(z_1) \mathbb{B}(z_2) \cdots \mathbb{B}(z_m) 1 \Big|_{z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m}} \\ &= \tilde{\Omega}_{z_1, z_2, \dots, z_m} \sum_{\sigma \in S_m} (-1)^\sigma z_1^{1-\sigma_1} z_2^{2-\sigma_2} \cdots z_m^{m-\sigma_m} \Big|_{z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m}} \\ &= \sum_{\sigma \in S_m} (-1)^\sigma \tilde{\Omega}_{z_1, z_2, \dots, z_m} \Big|_{z_1^{\alpha_1+\sigma_1-1} z_2^{\alpha_2+\sigma_2-2} \cdots z_m^{\alpha_m+\sigma_m-m}} \\ &= \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1, \alpha_2+\sigma_2-2, \dots, \alpha_m+\sigma_m-m}. \quad \blacksquare \end{aligned}$$

Of course, the original reason for considering this definition is the property that they are a lift of the symmetric functions corresponding to the Jacobi–Trudi matrix.

Corollary 3.30 For any composition α , we have $\chi(\mathfrak{S}_\alpha) = s_\alpha$.

Proof This follows from Definition 2.1 of the Jacobi–Trudi rule and the fact that $\chi(H_i) = h_i$. ■

3.5 A Pieri Rule for the Elementary Basis

Theorem 3.27 shows that in particular for the case of $\alpha = [1^n]$ that \mathfrak{S}_{1^n} is the usual analogue of the elementary generators of Sym . The elementary generators of NSym are the elements E_i that satisfy that $E_0 = 1$, and for $n \geq 1$,

$$E_n := \sum_{i=1}^n (-1)^{i-1} H_i E_{n-i}.$$

The antipode map is both an algebra antimorphism and a coalgebra morphism, so that $S(H_n) = (-1)^n E_n$.

Corollary 3.31 For $n \geq 0$,

$$\mathfrak{S}_{1^n} = \sum_{\alpha \models n} (-1)^{n-\ell(\alpha)} H_\alpha$$

and as a consequence, $F_{1^r}^\perp(\mathfrak{S}_{1^n}) = \mathfrak{S}_{1^{n-r}}$ and for $s > 1$, $F_s^\perp(\mathfrak{S}_{1^n}) = 0$.

Proof The expansion of \mathfrak{S}_{1^n} in terms of the complete homogeneous basis is a direct consequence of Theorem 3.27. For $n > 1$, expand \mathfrak{S}_{1^n} in the leftmost occurrence of H and group the rest of the terms together as

$$\mathfrak{S}_{1^n} = \sum_{i=1}^n \sum_{\beta \models n-i} (-1)^{n-\ell(\beta)-1} H_i H_\beta = \sum_{i=1}^n (-1)^{i-1} H_i \mathfrak{S}_{1^{n-i}}.$$

This implies that $\mathfrak{S}_{1^n} = E_n = (-1)^n S(H_n)$ is the usual analogue of the elementary generators of NSym. Because the antipode is a coalgebra morphism,

$$\Delta(\mathfrak{S}_{1^n}) = (-1)^n \Delta(S(H_n)) = (-1)^n \sum_{i=0}^n S(H_{n-i}) \otimes S(H_i) = \sum_{i=0}^n \mathfrak{S}_{1^{n-i}} \otimes \mathfrak{S}_{1^i}.$$

By Lemma 2.5 we know that $F_{1^r}^\perp(\mathfrak{S}_{1^n}) = M_{1^r}^\perp(\mathfrak{S}_{1^n}) = \mathfrak{S}_{1^{n-r}}$. Also by Lemma 2.5, for $s > 1$,

$$F_s^\perp(\mathfrak{S}_{1^n}) = \sum_{\alpha \models s} M_\alpha^\perp(\mathfrak{S}_{1^n}) = \sum_{\alpha \models s} (-1)^{s-\ell(\alpha)} \mathfrak{S}_{1^{n-s}} = 0. \quad \blacksquare$$

This allows us to give a right Pieri rule for \mathfrak{S}_{1^r} .

Proposition 3.32 For α a composition and $s \geq 0$,

$$\mathfrak{S}_\alpha \mathfrak{S}_{1^s} = \sum_{\substack{\beta \models |\alpha|+s \\ \alpha_i \leq \beta_i \leq \alpha_i+1}} \mathfrak{S}_\beta,$$

where we use the convention that $\alpha_i = 0$ for $i > \ell(\alpha)$.

Proof By combining results in Lemma 2.4 and Corollary 3.31, we know that

$$\begin{aligned} F_{1^i}^\perp(F\mathfrak{S}_{1^s}) - F_{1^{i-1}}^\perp(F\mathfrak{S}_{1^{s-1}}) &= \sum_{j=0}^i F_{1^j}^\perp(F)F_{1^{i-j}}^\perp(\mathfrak{S}_{1^s}) - \sum_{j=0}^{i-1} F_{1^j}^\perp(F)F_{1^{i-j-1}}^\perp(\mathfrak{S}_{1^{s-1}}) \\ &= F_{1^i}^\perp(F)\mathfrak{S}_{1^s}. \end{aligned}$$

Hence, we calculate directly that

$$\begin{aligned} \mathbb{B}_m(F)\mathfrak{S}_{1^s} &= \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^i}^\perp(F)\mathfrak{S}_{1^s} \\ &= \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^i}^\perp(F\mathfrak{S}_{1^s}) - \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^{i-1}}^\perp(F\mathfrak{S}_{1^{s-1}}) \\ &= \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^i}^\perp(F\mathfrak{S}_{1^s}) - \sum_{i \geq 0} (-1)^{i+1} H_{m+i+1} F_{1^i}^\perp(F\mathfrak{S}_{1^{s-1}}) \\ &= \mathbb{B}_m(F\mathfrak{S}_{1^s}) + \mathbb{B}_{m+1}(F\mathfrak{S}_{1^{s-1}}). \end{aligned}$$

From this identity a proof of the right-Pieri rule follows by a straightforward induction on the length of the composition α , which we leave to the reader. \blacksquare

3.6 Pieri Rules for Skew Operators

The following development contains expressions that include integer tuples (as opposed to compositions). To ensure that the sets of integer tuples that we consider are finite, we note that it is not necessary to consider more than a finite set once the degree of the element is fixed.

Lemma 3.33 For $\alpha \in \mathbb{Z}^m$, if $\alpha_i < i - m$ for some $1 \leq i \leq m$, then $\mathfrak{S}_\alpha = 0$. Also, if $\sum_{i=1}^m \alpha_i < 0$, then $\mathfrak{S}_\alpha = 0$.

Proof By Theorem 3.27,

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1 + \sigma_1 - 1, \alpha_2 + \sigma_2 - 2, \dots, \alpha_m + \sigma_m - m}.$$

Thus, if $\alpha_i < i - m$ for some i , then $\alpha_i + \sigma_i - i < 0$ for all $\sigma \in S_m$, and hence $H_{\alpha_i + \sigma_i - i} = 0$. Note that \mathfrak{S}_α is an element of homogeneous degree $\sum_{i=1}^m \alpha_i$, so that if $\sum_{i=1}^m \alpha_i < 0$, then $\mathfrak{S}_\alpha = 0$. ■

From the expressions that we have thus far developed, we are able to describe the action of F_{1r}^\perp and F_r^\perp on the immaculate basis.

Proposition 3.34 For $r \geq 0$, and for $\alpha \in \mathbb{Z}^m$,

$$(3.6) \quad F_{1r}^\perp \mathfrak{S}_\alpha = \sum_{\substack{\beta \in \mathbb{Z}^m \\ \alpha_i - \beta_i \in \{0,1\} \\ |\beta| = |\alpha| - r}} \mathfrak{S}_\beta.$$

If $r > \ell(\alpha)$, then $F_{1r}^\perp \mathfrak{S}_\alpha = 0$.

Proof Using the identities developed in Section 2.5, we calculate

$$\begin{aligned} \mathcal{E}_z^\perp \mathbb{B}(z_1) \mathbb{B}(z_2) \cdots \mathbb{B}(z_m) 1 &= \mathcal{E}_z^\perp \tilde{\Omega}_{z_1, z_2, \dots, z_m} \prod_{1 \leq i < j \leq m} (1 - z_j / z_i) \\ &= \tilde{\Omega}_{z_1, z_2, \dots, z_m} \prod_{1 \leq i < j \leq m} (1 - z_j / z_i) \prod_{i=1}^m (1 + z z_i) \\ &= \mathbb{B}(z_1) \mathbb{B}(z_2) \cdots \mathbb{B}(z_m) 1 \prod_{i=1}^m (1 + z z_i) \\ &= \mathbb{B}(z_1) \mathbb{B}(z_2) \cdots \mathbb{B}(z_m) 1 \sum_{S \subseteq \{1, 2, \dots, m\}} z^{|S|} \prod_{i \in S} z_i. \end{aligned}$$

Now by taking the coefficient of $z^r z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ in both sides of the equation, on the left we have $F_{1r}^\perp \mathfrak{S}_\alpha$, and on the right we have

$$\sum_{\substack{S \subseteq \{1, 2, \dots, m\} \\ |S|=r}} \mathfrak{S}_{\alpha_1 - \delta_{1 \in S}, \alpha_2 - \delta_{2 \in S}, \dots, \alpha_m - \delta_{m \in S}}$$

(where $\delta_{\text{true}} = 1$ and $\delta_{\text{false}} = 0$), which is equivalent to the right-hand side of equation (3.6). ■

Proposition 3.35 For $r \geq 0$ and for $\alpha \in \mathbb{Z}^m$,

$$F_r^\perp \mathfrak{S}_\alpha = \sum_{\substack{\beta \in \mathbb{Z}^m \\ i - m \leq \beta_i \leq \alpha_i \\ |\beta| = |\alpha| - r}} \mathfrak{S}_\beta.$$

Proof Again, using identities from Section 2.5,

$$\begin{aligned} \mathcal{H}_z^\perp \mathbb{B}(z_1)\mathbb{B}(z_2) \cdots \mathbb{B}(z_m) 1 &= \mathcal{H}_z^\perp \tilde{\Omega}_{z_1, z_2, \dots, z_m} \prod_{1 \leq i < j \leq m} (1 - z_j/z_i) \\ &= \mathbb{B}(z_1)\mathbb{B}(z_2) \cdots \mathbb{B}(z_m) 1 \prod_{i=1}^m 1/(1 - zz_i) \\ &= \sum_{r \geq 0} z^r \sum_{\substack{\gamma \in \mathbb{N}^m \\ |\gamma|=r}} z_1^{\gamma_1} z_2^{\gamma_2} \cdots z_m^{\gamma_m} \mathbb{B}(z_1)\mathbb{B}(z_2) \cdots \mathbb{B}(z_m) 1. \end{aligned}$$

If we take the coefficient of $z^r z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ and let $\beta_i = \alpha_i - \gamma_i$, then we have that

$$F_r^\perp \mathfrak{S}_\alpha = \sum_{\substack{\beta \in \mathbb{Z}^m \\ \beta_i \leq \alpha_i \\ |\beta|=|\alpha|-r}} \mathfrak{S}_\beta.$$

By Lemma 3.33 we may restrict our attention to the $\beta \in \mathbb{Z}^m$ such that $\beta_i \geq i - m$. ■

3.7 The Dual Immaculate Basis

Every basis X_α of NSym_n gives rise to a basis Y_β of QSym_n defined by duality; Y_β is the unique basis satisfying $\langle X_\alpha, Y_\beta \rangle = \delta_{\alpha, \beta}$. The dual basis to the immaculate basis of NSym , denoted \mathfrak{S}_α^* , have positive expansions in the monomial and fundamental bases of QSym . Furthermore, Corollary 3.30 allows us to give an expansion of the usual Schur functions of Sym (Theorem 3.38) in terms of these elements. All of these results are dual statements to the statements earlier in this section and follow since for a quasi-symmetric function $G \in \text{QSym}$,

$$G = \sum_{\beta} \langle H_\beta, G \rangle M_\beta = \sum_{\beta} \langle R_\beta, G \rangle F_\beta.$$

Proposition 3.36 *The dual immaculate functions \mathfrak{S}_α^* are monomial positive. Specifically, they expand as*

$$\mathfrak{S}_\alpha^* = \sum_{\beta \leq_t \alpha} K_{\alpha, \beta} M_\beta.$$

Proof This statement will follow from Proposition 3.16 and duality. Specifically, the coefficient of M_β in \mathfrak{S}_α^* is equal to

$$\langle H_\beta, \mathfrak{S}_\alpha^* \rangle = \left\langle \sum_{\gamma} K_{\gamma, \beta} \mathfrak{S}_\gamma, \mathfrak{S}_\alpha^* \right\rangle = K_{\alpha, \beta}. \quad \blacksquare$$

Proposition 3.37 *The dual immaculate functions \mathfrak{S}_α^* are fundamental positive. Specifically, they expand as*

$$\mathfrak{S}_\alpha^* = \sum_{\beta \leq_t \alpha} L_{\alpha, \beta} F_\beta.$$

Proof From Theorem 3.25, we have $\langle R_\beta, \mathfrak{S}_\alpha^* \rangle = L_{\alpha\beta}$. By Lemma 3.23, $L_{\alpha\beta} = 0$ unless $\alpha \geq_\ell \beta$. ■

Duality will also yield an explicit expansion of the Schur functions into the dual immaculate basis.

Theorem 3.38 *The Schur function s_λ with $\ell(\lambda) = k$ expands into the dual immaculate basis as follows:*

$$s_\lambda = \sum_{\sigma \in \mathcal{S}_k} (-1)^\sigma \mathfrak{S}_{\lambda_{\sigma_1+1-\sigma_1}, \lambda_{\sigma_2+2-\sigma_2}, \dots, \lambda_{\sigma_k+k-\sigma_k}}^*$$

where the sum is over permutations σ such that $\lambda_{\sigma_i+i-\sigma_i} > 0$ for all $i \in \{1, 2, \dots, k\}$.

Proof Suppose $s_\lambda = \sum_\alpha c_\alpha \mathfrak{S}_\alpha^*$. Then

$$c_\beta = \left\langle \mathfrak{S}_\beta, \sum_\alpha c_\alpha \mathfrak{S}_\alpha^* \right\rangle = \langle \mathfrak{S}_\beta, s_\lambda \rangle = \langle \chi(\mathfrak{S}_\beta), s_\lambda \rangle = \langle s_\beta, s_\lambda \rangle = \mathfrak{x}_{\beta,\lambda},$$

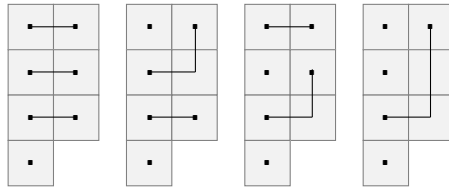
where $\mathfrak{x}_{\beta,\lambda}$ is $(-1)^\sigma$ or 0 according to the conditions in Proposition 2.2. ■

Example 3.39 Let $\lambda = [2, 2, 2, 1]$. Then $s_\lambda \in \text{Sym} \subseteq \text{QSym}$ can be expanded in the basis $\{\mathfrak{S}_\alpha^*\}_\alpha$:

$$s_{2221} = \mathfrak{S}_{2221}^* - \mathfrak{S}_{1321}^* - \mathfrak{S}_{2131}^* + \mathfrak{S}_{1141}^*,$$

since only the permutations $\sigma \in \{1234, 2134, 1324, 2314\}$ contribute to the sum in the expansion of s_{2221} . There are potentially 24 terms in this sum, but for the partition $[2, 2, 2, 1]$ it is easy to reason that $\sigma_4 = 4$ and $\sigma_1 < 3$.

These combinatorics arise in the paper of Egge, Loehr, and Warrington [8] when they describe how to obtain a Schur expansion given a quasi-symmetric fundamental expansion. In their language, these are called “special rim hook tableau”.



Recall that by Proposition 2.2 we know that precisely one term in the right-hand side of the expansion of Theorem 3.38 is equal to a partition, and we have the following procedure for going from the expansion of a symmetric function F in the \mathfrak{S}_α^* basis to the Schur expansion.

Corollary 3.40 *If F is symmetric, and the \mathfrak{S}^* -expansion of F is $\sum c_\alpha \mathfrak{S}_\alpha^*$, then the Schur expansion of F is $F = \sum_{\lambda \vdash n} c_\lambda s_\lambda$, where the second sum is taken over all partitions λ .*

Example 3.41 By Proposition 3.16, we have that the expansion of

$$h_{22} = \mathfrak{S}_{22}^* - \mathfrak{S}_{13}^* + \mathfrak{S}_{31}^* + \mathfrak{S}_4^*.$$

To recover the Schur expansion of h_{22} in terms of Schur functions we can throw away all terms not indexed by partitions, and then $h_{22} = s_{22} + s_{31} + s_4$.

3.7.1 Dual Immaculate Pieri Conjecture

Similarly to the left Pieri rule of the immaculate basis, we conjecture that the dual immaculate basis has a multiplicity-free signed Pieri rule. Explicitly

$$F_i \mathfrak{S}_\alpha^* = \sum_{\beta} (-1)^{\text{sign}(\alpha, \beta)} \mathfrak{S}_\beta^*,$$

for some collection of β and some statistic sign.

Example 3.42 We let $\alpha = [2, 1, 2]$ and $i = 2$. Then

$$F_2 \mathfrak{S}_{212}^* = -\mathfrak{S}_{1312}^* - \mathfrak{S}_{142}^* + \mathfrak{S}_{2212}^* + \mathfrak{S}_{3112}^* + \mathfrak{S}_{322}^* + \mathfrak{S}_{412}^*.$$

3.8 The Product of Immaculate Functions

In general, the product of two immaculate functions does not expand positively in the immaculate basis. However, for certain products we have a positive expansion.

Conjecture 3.43 ([3]) If λ is a partition, then the coefficients $c_{\alpha, \lambda}^\beta$ appearing in

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_{\beta} c_{\alpha, \lambda}^\beta \mathfrak{S}_\beta,$$

are non-negative integers.

Example 3.44 We give an example of Theorem 3.43 with $\alpha = [1, 2]$ and $\lambda = [3, 1]$:

$$\begin{aligned} \mathfrak{S}_{12} \mathfrak{S}_{31} &= \mathfrak{S}_{1231} + \mathfrak{S}_{1321} + \mathfrak{S}_{133} + \mathfrak{S}_{1411} + \mathfrak{S}_{142} + \mathfrak{S}_{151} + \mathfrak{S}_{2221} + \mathfrak{S}_{223} \\ &+ \mathfrak{S}_{2311} + 2\mathfrak{S}_{232} + 2\mathfrak{S}_{241} + \mathfrak{S}_{25} + \mathfrak{S}_{3211} + \mathfrak{S}_{322} + 2\mathfrak{S}_{331} + \mathfrak{S}_{34} + \mathfrak{S}_{421} + \mathfrak{S}_{43}. \end{aligned}$$

Example 3.45 We need not look very far to find mixed negative signs in a product of two immaculate functions when the right one is not indexed by a partition. For instance,

$$\mathfrak{S}_1 \mathfrak{S}_{13} = \mathfrak{S}_{113} - \mathfrak{S}_{221} - \mathfrak{S}_{32}.$$

But products of two immaculate functions indexed by two compositions can potentially be much more complicated. For instance,

$$\begin{aligned} \mathfrak{S}_{11} \mathfrak{S}_{1313} &= \mathfrak{S}_{111313} - \mathfrak{S}_{122113} + \mathfrak{S}_{122221} + \mathfrak{S}_{122232} - \mathfrak{S}_{13213} - \mathfrak{S}_{212113} + \mathfrak{S}_{212221} \\ &+ \mathfrak{S}_{21232} - \mathfrak{S}_{221113} + \mathfrak{S}_{221221} + \mathfrak{S}_{22132} + \mathfrak{S}_{222121} - \mathfrak{S}_{22213} + \mathfrak{S}_{222211} \\ &+ \mathfrak{S}_{22222} + \mathfrak{S}_{22231} - \mathfrak{S}_{23113} + \mathfrak{S}_{23221} - \mathfrak{S}_{31213} - \mathfrak{S}_{32113} + \mathfrak{S}_{32221} - \mathfrak{S}_{3313}. \end{aligned}$$

The image of the immaculate function indexed by $[1, 3, 2, 1, 3]$ under the forgetful map is $\chi(\mathfrak{S}_{13213}) = s_{22222}$. We do not have a rule to predict its coefficient of -1 in the above expansion. This example should indicate how surprising Theorem 3.43 is, given that there are 11 positive terms and 9 negative terms in an example of a product $\mathfrak{S}_\alpha \mathfrak{S}_\beta$ when the immaculate function \mathfrak{S}_β is not indexed by a partition.

3.9 The Immaculate Poset, Paths and Skew Immaculate Tableaux

We create a labelled poset on the set of all compositions, which we call the immaculate poset \mathfrak{P} . We place an arrow from α to β if $\beta \subset_1 \alpha$ (equivalently, $\mathfrak{S}_\beta H_1$ expands in the immaculate basis as $\mathfrak{S}_\alpha + \dots$). Such a cover implies that α and β differ by a single box. We give a label m to this cover, where m is the row containing said box, and denote this by $\alpha \xrightarrow{m} \beta$.

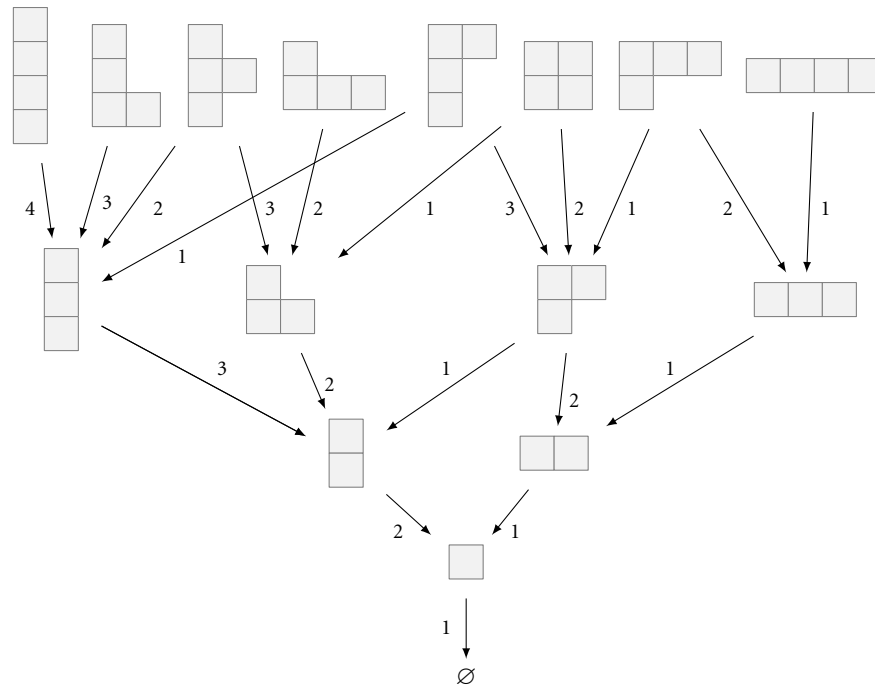


Figure 1: The first few levels of \mathfrak{P}

Maximal chains on this poset from α to \emptyset are equivalent to immaculate standard tableaux and maximal chains on an interval from α to β are what we will call skew immaculate tableaux of shape α/β . We can visualize a path

$$\{\alpha = \beta^{(0)} \xrightarrow{m_1} \beta^{(1)} \xrightarrow{m_2} \dots \xrightarrow{m_k} \beta^{(k)} = \beta\}$$

as a labelled composition diagram of outer shape α by labeling a cell in row m_i with a $k - i + 1$. The cells representing β will not have a label and the labels must increase from left to right in a row.

Example 3.46 Consider the example of a skew immaculate tableau of shape $[2, 2, 2]/[1, 2]$ given by the diagram below. We show the labelled composition on the left and the corresponding representation as a path to the right.

$$\begin{array}{|c|c|} \hline & 2 \\ \hline & \\ \hline 1 & 3 \\ \hline \end{array} \longleftrightarrow \{[2, 2, 2] \xrightarrow{3} [2, 2, 1] \xrightarrow{1} [1, 2, 1] \xrightarrow{3} [1, 2]\}.$$

A path $P = \{\alpha = \beta^{(0)} \xrightarrow{m_1} \beta^{(1)} \xrightarrow{m_2} \dots \xrightarrow{m_k} \beta^{(k)} = \beta\}$ in \mathfrak{P} will be called a *horizontal k -strip* if $m_1 \leq m_2 \leq \dots \leq m_k$. The Pieri rule for the immaculate functions from Theorem 3.5 says that there is a horizontal k -strip from α to β if and only if \mathfrak{S}_α appears in the expansion of $\mathfrak{S}_\beta H_k$.

The *descent composition* of a word $\ell_1, \ell_2, \dots, \ell_n$ is the composition

$$[i_1, i_2 - i_1, \dots, i_j - i_{j-1}, n - i_j],$$

where $i_1 < i_2 < \dots < i_j$ are the descents of the word; that is, the elements in $\{1, \dots, n - 1\}$ such that $\ell_{i_a} > \ell_{i_a+1}$. For example, the descent composition of the word $3, 2, 1, 1, 4, 1$ is $[1, 1, 3, 1]$. For a path

$$P = \{\beta_0 \xrightarrow{m_1} \beta_1 \xrightarrow{m_2} \dots \xrightarrow{m_n} \beta_n\},$$

the descent composition $\alpha(D(P))$ associated with P is the reverse of the descent composition of the word m_1, m_2, \dots, m_n . When P is a standard tableau of shape $[\emptyset, \alpha]$, this notion of descent set is equivalent to the definition of descent from Definition 3.20. Equivalently, the descent composition describes how P can be decomposed into horizontal k -strips of maximal lengths.

3.10 Skew Dual Immaculate Quasi-symmetric Functions

Following notions in [6], since there is a positive right Pieri rule on the immaculate basis of NSym , there is a natural way of constructing skew dual immaculate elements of QSym . For an interval $\{\gamma : \beta \subseteq \gamma \subseteq \alpha\} \subseteq \mathfrak{P}$, we then define the skew dual immaculate function as

$$\mathfrak{S}_{\alpha/\beta}^* = \sum_{\gamma} \langle \mathfrak{S}_\beta H_\gamma, \mathfrak{S}_\alpha^* \rangle M_\gamma.$$

By Theorem 3.5 and the notion of standardization from Definition 3.11, the coefficient $\langle \mathfrak{S}_\alpha H_\gamma, \mathfrak{S}_\alpha^* \rangle$ is equal to the number of skew standard immaculate tableaux of shape α/β with descent composition coarser than γ .

Right multiplication by H_r on an element \mathfrak{S}_β is a *Pieri operator* [6] on the poset of compositions defined in the previous section. By [6, Theorem 2.3] we have an expansion of these elements into other bases by the use of the dual pairing between NSym and QSym.

Proposition 3.47 For $\{\gamma : \beta \subseteq \gamma \subseteq \alpha\}$ an interval of \mathfrak{P} ,

$$\mathfrak{S}_{\alpha/\beta}^* = \sum_{\gamma} \langle \mathfrak{S}_\beta R_\gamma, \mathfrak{S}_\alpha^* \rangle F_\gamma = \sum_{\gamma} \langle \mathfrak{S}_\beta \mathfrak{S}_\gamma, \mathfrak{S}_\alpha^* \rangle \mathfrak{S}_\gamma^*.$$

The coefficients $\mathfrak{c}_{\beta,\gamma}^\alpha = \langle \mathfrak{S}_\beta \mathfrak{S}_\gamma, \mathfrak{S}_\alpha^* \rangle$ are those that appear in the expansion

$$\mathfrak{S}_\beta \mathfrak{S}_\gamma = \sum_{\alpha} \mathfrak{c}_{\beta,\gamma}^\alpha \mathfrak{S}_\alpha.$$

Theorem 3.43 states that certain of these coefficients will be positive, but they are not positive in general.

The expansion of these elements in the fundamental and monomial bases are positive. An argument similar to that given for Theorem 3.25 and Proposition 3.37 shows that the skew immaculate tableaux can be used to give a combinatorial expansion of these elements in the fundamental basis.

Proposition 3.48 Let $\{\gamma : \beta \subseteq \gamma \subseteq \alpha\}$ be an interval of \mathfrak{P} , then

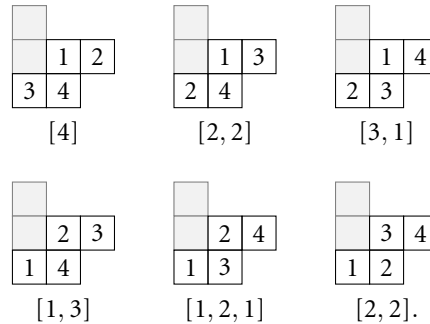
$$\mathfrak{S}_{\alpha/\beta}^* = \sum_P F_{\alpha(D(P))}$$

where the sum is over all paths P in \mathfrak{P} from α to β (alternatively, skew standard immaculate tableaux of shape α/β).

Proof Let $K_{\alpha/\beta,\gamma} = \langle \mathfrak{S}_\beta H_\gamma, \mathfrak{S}_\alpha^* \rangle$, which we have already noted is equal to the number of skew immaculate tableaux of shape α/β whose descent composition is coarser than γ . Let $L_{\alpha/\beta,\tau}$ be the number of skew immaculate tableaux of shape α/β whose descent composition is equal to τ . Clearly we have $K_{\alpha/\beta,\gamma} = \sum_{\tau \geq \gamma} L_{\alpha/\beta,\tau}$ and by Möbius inversion this is equivalent to $\sum_{\gamma \geq \tau} (-1)^{\ell(\tau) - \ell(\gamma)} K_{\alpha/\beta,\gamma} = L_{\alpha/\beta,\tau}$.

$$\begin{aligned} \langle R_\tau, \mathfrak{S}_{\alpha/\beta}^* \rangle &= \langle \mathfrak{S}_\beta R_\tau, \mathfrak{S}_\alpha^* \rangle = \sum_{\gamma \geq \tau} (-1)^{\ell(\tau) - \ell(\gamma)} \langle \mathfrak{S}_\beta H_\gamma, \mathfrak{S}_\alpha^* \rangle \\ &= \sum_{\gamma \geq \tau} (-1)^{\ell(\tau) - \ell(\gamma)} K_{\alpha/\beta,\gamma} = L_{\alpha/\beta,\tau}. \quad \blacksquare \end{aligned}$$

Example 3.49 There are 6 paths from $[1, 3, 2]$ to $[1, 1]$. We give their representation as a tableau diagram in the table below. Below each of the diagrams we also give the descent composition of the path:



The fundamental expansion of this element is positive and is the sum of 6 terms corresponding to these tableaux, but the expansion in the dual immaculate basis is not positive in this case:

$$\mathfrak{S}_{132/11}^* = F_{121} + F_{13} + 2F_{22} + F_{31} + F_4 = -\mathfrak{S}_{13}^* + \mathfrak{S}_{22}^* + \mathfrak{S}_{31}^* + \mathfrak{S}_4^*.$$

4 Hall–Littlewood Basis for NSym

As an application and demonstration of the power of the immaculate basis, we use this section to build lifts of Hall–Littlewood functions inside NSym. We begin by reminding the reader of a definition of Hall–Littlewood symmetric functions inside Sym defined by creation (or vertex) operators.

4.1 Hall–Littlewood Symmetric Functions

The Hall–Littlewood symmetric functions P_λ , first studied by Hall, are symmetric functions with a parameter q (i.e., elements of $\text{Sym}[q] := \mathbb{Q}(q)[h_1, h_2, \dots]$). They generically (q not specialized) form a basis for $\text{Sym}[q]$. When $q = 0$, they specialize to monomial symmetric functions, and at $q = 1$ they specialize to Schur functions. We are interested in the corresponding dual basis elements Q'_μ , which have the property that $\langle P_\lambda, Q'_\mu \rangle = \delta_{\lambda, \mu}$. The elements Q'_μ have the expansion in the Schur basis

$$Q'_\mu = \sum_{\lambda} K_{\lambda\mu}(q) s_\lambda$$

where $K_{\lambda\mu}(q)$ are the q -Kostka polynomials. At $q = 0$ the Q'_λ specialize to Schur functions s_λ , and at $q = 1$ they specialize to homogeneous complete functions h_λ .

We define an operator $\tilde{\mathbf{B}}_m: \text{Sym}[q]_n \rightarrow \text{Sym}[q]_{n+m}$ by

$$\tilde{\mathbf{B}}_m := \sum_{i \geq 0} q^i \mathbf{B}_{m+i} h_i^\perp.$$

Theorem 4.1 (Jing [18], cf. also [25, pp. 237–238]) *If $m \geq \lambda_1$, then $\tilde{\mathbf{B}}_m Q'_\lambda = Q'_{(m, \lambda)}$.*

Definition 4.2 One may define, for any integer tuple $\alpha \in \mathbb{Z}^m$, a symmetric function

$$Q'_\alpha := \tilde{\mathbf{B}}_{\alpha_1} \cdots \tilde{\mathbf{B}}_{\alpha_m}(1).$$

Remark 4.3 The $\tilde{\mathbf{B}}_m$ operators generalize the \mathbf{B}_m operators of Bernstein [37] and are due to Jing [18] who studied them as a “vertex operator” definition of the algebra of Hall–Littlewood polynomials Q_λ . Garsia [9] used a modified version as creation operators for the symmetric functions Q'_λ . Specializations of these operators can be used to create P and Q -Schur functions (see for instance [17, 19, 25]). The commutation relations of these operators make it natural to consider the symmetric functions for all compositions or integer tuples, but only the Q'_λ for λ a partition are known to be positive when expanded in the Schur basis. Generalizations were considered for Macdonald symmetric functions [20, 22] as well as a technique for proving the polynomiality of the Macdonald–Kostka coefficients. Shimozono and Zabrocki [32] considered compositions of these operators that were indexed by tuples of integers. These functions were recently studied further in [15] where it was shown that

$$(4.1) \quad \sum_{\alpha \models n} (-q)^{n-\ell(\alpha)} Q'_\alpha = e_n[X].$$

In that case, Hall–Littlewood functions indexed by compositions were used to understand the action of the operator ∇ introduced in [4] on a spanning set of the symmetric functions.

4.2 A New Hall–Littlewood Basis for NSym

We start by building operators $\tilde{\mathbb{B}}_m : \text{NSym}[q] \rightarrow \text{NSym}[q]$, defined for $m \in \mathbb{Z}$ by

$$(4.2) \quad \tilde{\mathbb{B}}_m(f) = \sum_{i \geq 0} q^i \mathbb{B}_{m+i} F_i^\perp(f).$$

We may now define our new basis of NSym; they are the result of applying successive $\tilde{\mathbb{B}}_m$ operators.

Definition 4.4 If $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_m] \in \mathbb{Z}^m$, then we define

$$Q'_\alpha = \tilde{\mathbb{B}}_{\alpha_1} \tilde{\mathbb{B}}_{\alpha_2} \cdots \tilde{\mathbb{B}}_{\alpha_m}(1).$$

Example 4.5 We will calculate $Q'_{4,2}$ and expand this in the \mathfrak{S} -basis. First note that $Q'_2 = \tilde{\mathbb{B}}_2(1) = \mathbb{B}_2(1) = \mathfrak{S}_2$. Next we apply $\tilde{\mathbb{B}}_4$:

$$Q'_{4,2} = \tilde{\mathbb{B}}_4 \mathfrak{S}_2 = \mathbb{B}_4 \mathfrak{S}_2 + q \mathbb{B}_5 F_1^\perp(\mathfrak{S}_2) + q^2 \mathbb{B}_6 F_2^\perp(\mathfrak{S}_2) = \mathfrak{S}_{4,2} + q \mathfrak{S}_{5,1} + q^2 \mathfrak{S}_6.$$

It is worth presenting a further example that seems to give a convincing reason why the immaculate basis and the basis Q'_α merit further study. By definition, we have the property that $\chi(Q'_\alpha) = Q'_\alpha$. We know that for a partition λ , the Q'_λ form a basis of Sym and are Schur positive. It seems (see Conjecture 4.16) that the elements Q'_λ are immaculate positive. What is surprising is that there are more terms in the expansion of $Q'_\lambda \in \text{NSym}$ than in the expansion of $Q'_\lambda \in \text{Sym}$, and so through surjection under χ , many of the positive terms of Q'_λ cancel.

Example 4.6 The basis element $Q'_{1111} \in \text{NSym}$ has the following expansion in the immaculate basis:

$$Q'_{1111} = \mathfrak{S}_{1111} + q\mathfrak{S}_{112} + (q + q^2)\mathfrak{S}_{121} + q^3\mathfrak{S}_{13} + (q + q^2 + q^3)\mathfrak{S}_{211} \\ + (q^2 + q^3 + q^4)\mathfrak{S}_{22} + (q^3 + q^4 + q^5)\mathfrak{S}_{31} + q^6\mathfrak{S}_4.$$

The image of χ on this element is the Hall–Littlewood symmetric function:

$$Q'_{1111} = s_{1111} + (q + q^2 + q^3)s_{211} + (q^2 + q^4)s_{22} + (q^3 + q^4 + q^5)s_{31} + q^6s_4.$$

For any partition λ , $\chi(\mathfrak{S}_\lambda) = s_\lambda$. For the other immaculate basis elements of NSym , $\chi(\mathfrak{S}_{112}) = \chi(\mathfrak{S}_{121}) = 0$ and $\chi(\mathfrak{S}_{13}) = -s_{22}$.

4.3 A Right-Pieri Type Rule on the Non-commutative Hall–Littlewood Basis

We start this section by developing a Pieri rule for the Q' -basis of $\text{NSym}[q]$.

In the same way that we derived Lemma 3.4 from Lemma 2.6, we can also derive the following result.

Lemma 4.7 For $f \in \text{NSym}$ and $s, m \geq 0$,

$$\tilde{\mathbb{B}}_m(fH_s) = \sum_{k=0}^s q^k \tilde{\mathbb{B}}_{m+k}(f)H_{s-k} - \sum_{k=0}^{s-1} q^k \tilde{\mathbb{B}}_{m+k+1}(f)H_{s-k-1}.$$

Proof We explicitly calculate the left-hand side and apply Lemma 2.6 (where $F_j = 0$ for $j < 0$)

$$\tilde{\mathbb{B}}_m(fH_s) = \sum_{i \geq 0} q^i \mathbb{B}_{m+i} F_i^\perp(fH_s) = \sum_{i \geq 0} q^i \mathbb{B}_{m+i} \left(\sum_{k=0}^s F_{i-k}^\perp(f)H_{s-k} \right) \\ = \sum_{k=0}^s \sum_{i \geq 0} q^i \mathbb{B}_{m+i} (F_{i-k}^\perp(f)H_{s-k}).$$

Now by Lemma 3.4 this last expression is equal to

$$= \sum_{k=0}^s \sum_{i \geq 0} q^i (\mathbb{B}_{m+i} F_{i-k}^\perp(f)H_{s-k} - \mathbb{B}_{m+i+1} F_{i-k}^\perp(f)H_{s-k-1}).$$

If we let $j = i - k$ and shift the indices of the sum, we have

$$= \sum_{k=0}^s q^k \sum_{j \geq 0} q^j (\mathbb{B}_{m+k+j} F_j^\perp(f)H_{s-k} - \mathbb{B}_{m+k+1+j} F_j^\perp(f)H_{s-k-1}) \\ = \sum_{k=0}^s q^k \tilde{\mathbb{B}}_{m+k}(f)H_{s-k} - \sum_{k=0}^{s-1} q^k \tilde{\mathbb{B}}_{m+k+1}(f)H_{s-k-1}.$$

The second sum in the last equation terminates at $k = s - 1$, since for $k = s$, $H_{s-k-1} = H_{-1} = 0$. ■

For two compositions α, β , with $\alpha_i \leq \beta_i$ for all i , we let $n(\alpha, \beta)$ denote the number of rows of α that are strictly shorter than the same row of β (i.e., $n(\alpha, \beta) = |\{i : \alpha_i < \beta_i\}|$).

Theorem 4.8 *The Q'_α have a right-Pieri type rule given by*

$$Q'_\alpha H_s = \sum_{\alpha \subset_s \beta} (1 - q)^{n(\alpha, \beta)} Q'_\beta.$$

Proof We will prove this using a double induction. First we use induction on s . The case $s = 0$ is trivially true as $H_0 = 1$. We now assume it is true for all $j < s$ and prove it for s . For a given s we make a second inductive hypothesis on the length of the composition α . Again, for compositions of length 0 it is trivially true, so we assume it to hold for length $\ell \leq \ell(\alpha)$ and prove it for $\ell(\alpha) + 1$. We apply Lemma 4.7 to yield

$$\begin{aligned} (4.3) \quad Q'_{[m, \alpha]} H_s &= [\tilde{\mathbb{B}}_m(Q'_\alpha)] H_s \\ &= \tilde{\mathbb{B}}_m(Q'_\alpha H_s) - \sum_{k=1}^s q^k \tilde{\mathbb{B}}_{m+k}(Q'_\alpha) H_{s-k} + \sum_{k=0}^{s-1} q^k \tilde{\mathbb{B}}_{m+k+1}(Q'_\alpha) H_{s-k-1}. \end{aligned}$$

When we apply the inductive hypotheses, (4.3) becomes:

$$\begin{aligned} (4.4) \quad &= \sum_{\alpha \subset_s \beta} (1 - q)^{n(\alpha, \beta)} Q'_{[m, \beta]} - \sum_{k=1}^s \sum_{[m+k, \alpha] \subset_{s-k} \gamma} q^k (1 - q)^{n([m+k, \alpha], \gamma)} Q'_\gamma \\ &\quad + \sum_{k=0}^{s-1} \sum_{[m+k+1, \alpha] \subset_{s-k-1} \delta} q^k (1 - q)^{n([m+k+1, \alpha], \delta)} Q'_\delta. \end{aligned}$$

We need to show that the coefficient of a term Q'_η is $(1 - q)^{n([m, \alpha], \eta)}$. If $\eta = [m, \zeta]$, then the only coefficient in (4.4) will come from the first sum; forcing $\beta = \zeta$ yields $(1 - q)^{n(\alpha, \zeta)} = (1 - q)^{n([m, \alpha], \eta)}$.

If $\eta = [m + a, \zeta]$ with $s \geq a \geq 1$, then the contribution to Q'_η will come from the second and third summands in (4.4). The coefficient coming from the second summand is

$$\begin{aligned} & - \sum_{k=1}^a q^k (1 - q)^{n([m+k, \alpha], \eta)} \\ &= - \sum_{k=1}^{a-1} q^k (1 - q)^{1+n(\alpha, \zeta)} - q^a (1 - q)^{n(\alpha, \zeta)} \\ &= -(1 - q)^{n(\alpha, \zeta)} \left(\left(\sum_{k=1}^{a-1} q^k (1 - q) \right) + q^a \right) = -q(1 - q)^{n(\alpha, \zeta)}. \end{aligned}$$

The coefficient coming from the third summand is

$$\begin{aligned} & \sum_{k=0}^{a-1} q^k (1-q)^{n([m+k+1, \alpha], \eta)} \\ &= \sum_{k=0}^{a-2} q^k (1-q)^{1+n(\alpha, \zeta)} - q^{a-1} (1-q)^{n(\alpha, \zeta)} \\ &= (1-q)^{n(\alpha, \zeta)} \left(\left(\sum_{k=0}^{a-2} q^k (1-q) \right) + q^{a-1} \right) = (1-q)^{n(\alpha, \zeta)}. \end{aligned}$$

Combining these coefficients gives $(1-q)^{n(\alpha, \zeta)+1} = (1-q)^{n([m, \alpha], \eta)}$. ■

Example 4.9 Similar to Example 3.8, we compute

$$\begin{aligned} Q'_{23} H_3 &= Q'_{233} + (1-q)Q'_{242} + (1-q)Q'_{251} + (1-q)Q'_{26} + (1-q)Q'_{332} \\ &+ (1-q)^2 Q'_{341} + (1-q)^2 Q'_{35} + (1-q)Q'_{431} + (1-q)^2 Q'_{44} + (1-q)Q'_{53}. \end{aligned}$$

As an immediate corollary, repeated application of our Pieri rule gives an explicit formula for the expansion of the H basis. For an immaculate tableau T , we let $n(T) = \sum_{j=0}^{\ell(\text{sh}(T))} d_j(T) - 1$, where $d_j(T)$ is the number of distinct entries in row j . Equivalently, $n(T)$ counts the number of times that a distinct letter l exists in a row that does not start with l .

Theorem 4.10 For a composition β ,

$$H_\beta = \sum_T (1-q)^{n(T)} Q'_{\text{shape}(T)},$$

the sum over all immaculate tableaux of content β .

Proof This follows from repeated application of Theorem 4.8. ■

Example 4.11 The five immaculate tableaux with shape $[4, 2, 3]$ and content $[3, 1, 2, 3]$ from Example 3.10 have $n(T) = 2, 3, 3, 3, 1$ respectively:

1	1	1	3	1	1	1	3	1	1	1	4	1	1	1	4	1	1	1	2
2	3			2	4			2	3			2	4			3	3		
4	4	4		3	4	4		3	4	4		3	3	4		4	4	4	

Therefore,

$$H_{3123} = \dots + ((1-q) + (1-q)^2 + 3(1-q)^3) Q'_{423} + \dots$$

Theorem 4.12 For generic q , the Q'_α form a basis of $\text{NSym}[q]$. When α is a partition, $\chi(Q'_\alpha) = Q'_\alpha$. Moreover, the Q'_α interpolate between the complete homogeneous basis (at specialization $q = 1$) and the immaculate basis (at specialization $q = 0$).

Proof It is easy to see that the Q'_α expand into the immaculate basis upper unitriangularly with respect to lexicographic ordering. The second statement follows from Theorem 4.1 and the fact that $\chi \circ \mathbb{B}_m = \tilde{\mathbb{B}}_m \circ \chi$, which follows from the fact that $\chi \circ \mathbb{B}_m = \mathbb{B}_m \circ \chi$, which is easy to check on the immaculate basis. The only immaculate tableau T that has $n(T) = 0$ and content β is the tableau with β_i many i 's in row i . Substituting $q = 1$ in Theorem 4.10 then proves that Q'_β is H_β . Finally, substituting $q = 0$ into the definition of $\tilde{\mathbb{B}}_m$ shows that $\tilde{\mathbb{B}}_m = \mathbb{B}_m$ in this case. ■

Equation (4.2) defines $\tilde{\mathbb{B}}_m$ in terms of the \mathbb{B}_n operators. We can also give a similar expression for the \mathbb{B}_n operators in terms of $\tilde{\mathbb{B}}_m$.

Proposition 4.13 For $m \in \mathbb{Z}$,

$$\mathbb{B}_m = \sum_{i \geq 0} (-q)^i \tilde{\mathbb{B}}_{m+i} F_{1^i}^\perp.$$

Proof Our calculations below will use the identity that $\sum_{i=0}^n (-1)^i F_{n-i} F_{1^i} = 0$ for $n > 0$ (see for instance Stanley [33], where $F_{n-i} = h_{n-i}$ and $F_{1^i} = e_i$). We begin with the left-hand side of the expression and then group terms together with $i + j = n$

$$\begin{aligned} \sum_{i \geq 0} (-q)^i \tilde{\mathbb{B}}_{m+i} F_{1^i}^\perp &= \sum_{i, j \geq 0} (-q)^i q^j \tilde{\mathbb{B}}_{m+i+j} F_j^\perp F_{1^i}^\perp \\ &= \sum_{n \geq 0} \sum_{i=0}^n (-1)^i q^n \tilde{\mathbb{B}}_{m+n} F_{n-i}^\perp F_{1^i}^\perp = \mathbb{B}_m. \end{aligned} \quad \blacksquare$$

This formula potentially gives us a recursive means of computing an element of the immaculate basis in terms of the Q' -basis. Although it is not obvious how this should be done for all compositions, for certain compositions there are remarkably simple expressions. In particular, the analogue of equation (4.1) also holds for our lifted Hall–Littlewood functions, and they are proved here using roughly the analogous proof to that provided in [15].

Proposition 4.14 For $n \geq 1$, $\mathfrak{S}_{1^n} = \sum_{\alpha \vdash n} (-q)^{n-\ell(\alpha)} Q'_\alpha$. More generally, for $1 \leq k \leq n$,

$$\mathfrak{S}_{k, 1^{n-k}} = \sum_{\substack{\alpha \vdash n \\ \alpha_1 \geq k}} (-q)^{n-k+1-\ell(\alpha)} Q'_\alpha.$$

Proof For the first statement we proceed by induction on n . The base case is that $\mathfrak{S}_n = H_n = Q'_n$, which has already been established. Then

$$\mathfrak{S}_{1^n} = \mathbb{B}_1 \mathfrak{S}_{1^{n-1}} = \sum_{i \geq 0} (-q)^i \tilde{\mathbb{B}}_{1+i} F_{1^i}^\perp \mathfrak{S}_{1^{n-1}} = \sum_{i \geq 0} (-q)^i \tilde{\mathbb{B}}_{1+i} \mathfrak{S}_{1^{n-i-1}},$$

since $F_{1^i}^\perp \mathfrak{S}_{1^{n-1}} = \mathfrak{S}_{1^{n-1-i}}$ by Corollary 3.31. Next we apply the inductive hypothesis:

$$\begin{aligned} \sum_{i \geq 0} (-q)^i \widetilde{\mathbb{B}}_{1+i} \mathfrak{S}_{1^{n-1-i}} &= \sum_{i \geq 0} \sum_{\beta \vdash n-i-1} (-q)^i (-q)^{n-i-1-\ell(\beta)} \widetilde{\mathbb{B}}_{1+i} \mathcal{Q}'_\beta \\ &= \sum_{\alpha \vdash n} (-q)^{n-\ell(\alpha)} \mathcal{Q}'_\alpha. \end{aligned}$$

The second expression follows by applying \mathbb{B}_k to the expression just derived and again applying Corollary 3.31:

$$\begin{aligned} \mathfrak{S}_{k,1^{n-k}} &= \mathbb{B}_k \mathfrak{S}_{1^{n-k}} \\ &= \sum_{i \geq 0} (-q)^i \widetilde{\mathbb{B}}_{k+i} F_{1^i}^\perp \mathfrak{S}_{1^{n-k}} = \sum_{i \geq 0} (-q)^i \widetilde{\mathbb{B}}_{k+i} \mathfrak{S}_{1^{n-k-i}} \\ &= \sum_{i \geq 0} \sum_{\beta \vdash n-k-i} (-q)^i (-q)^{n-k-i-\ell(\beta)} \widetilde{\mathbb{B}}_{k+i} \mathcal{Q}'_\beta = \sum_{\substack{\alpha \vdash n \\ \alpha_1 \geq k}} (-q)^{n-k+1-\ell(\alpha)} \mathcal{Q}'_\alpha. \quad \blacksquare \end{aligned}$$

We let $\{\mathcal{P}_\alpha\}$ denote the basis of QSym that is dual to the $\{\mathcal{Q}'_\alpha\}$ basis of NSym. Then by duality, the \mathcal{P}_α are monomial positive in the variable $(1 - q)$.

Theorem 4.15 For a composition α ,

$$\mathcal{P}_\alpha = \sum_T (1 - q)^{n(T)} M_{\text{content}(T)},$$

the sum being over all immaculate tableaux of shape α . Moreover, \mathcal{P}_α interpolates between the monomial basis of QSym (at $q = 1$) and the dual immaculate basis (at $q = 0$).

It is a fundamental combinatorial result of Lascoux and Schützenberger [24] that the function Q'_λ expands positively in the Schur basis,

$$Q'_\lambda = \sum_T q^{\text{charge}(T)} s_{\text{shape}(T)},$$

the sum being over all standard Young tableaux of content λ . We end our paper with a similar conjectured expansion of our new lifted Hall-Littlewood basis into the immaculate basis.

Conjecture 4.16 If λ is a partition, then Q'_λ expands in the immaculate basis \mathfrak{S}_β with coefficients that are positive polynomials in q . More explicitly,

$$Q'_\lambda = \sum_T q^{\text{st}(T)} \mathfrak{S}_{\text{shape}(T)},$$

for some statistic st , over all immaculate tableaux of content λ .

Example 4.17 For $\lambda = [3, 3, 1]$, the expansion of Q'_λ corresponds to the number of immaculate tableaux of content λ .

$$\begin{aligned}
 Q'_{331} = & \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & & \\ \hline \end{array} \mathfrak{S}_{331} & +q \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & & 3 \\ \hline \end{array} \mathfrak{S}_{34} & +q \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & & \\ \hline 3 & & & \\ \hline \end{array} \mathfrak{S}_{421} & + (q^2 + q) \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \\ \hline & & & \\ \hline \end{array} \mathfrak{S}_{43} & + (q^2 + q) \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & \\ \hline & & & \\ \hline \end{array} \mathfrak{S}_{43} \\
 & +q^2 \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & & & & \\ \hline 3 & & & & \\ \hline \end{array} \mathfrak{S}_{511} & + (q^3 + q^2) \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & & & \\ \hline & & & & \\ \hline \end{array} \mathfrak{S}_{52} & + (q^3 + q^2) \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & & & \\ \hline & & & & \\ \hline \end{array} \mathfrak{S}_{52} \\
 & + (q^4 + q^3) \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 3 & & & & & \\ \hline & & & & & \\ \hline \end{array} \mathfrak{S}_{61} & + (q^4 + q^3) \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & & & & & \\ \hline & & & & & \\ \hline \end{array} \mathfrak{S}_{61} & + q^5 \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline & & & & & 3 \\ \hline & & & & & \\ \hline \end{array} \mathfrak{S}_7
 \end{aligned}$$

Remark 4.18 The conjecture has been checked for partitions of size $n \leq 11$. The statement is not true for compositions. The first such example is $\alpha = [1, 1, 3]$:

$$\begin{aligned}
 Q'_{113} = & \mathfrak{S}_{113} + q \mathfrak{S}_{122} + q^2 \mathfrak{S}_{131} + q^2 \mathfrak{S}_{14} + q^2 \mathfrak{S}_{212} + (q^3 + q^2 - q) \mathfrak{S}_{221} \\
 & + (q^4 + q^3) \mathfrak{S}_{23} + q^2 \mathfrak{S}_{311} + (q^5 + q^4 + q^3 - q^2) \mathfrak{S}_{32} + (q^6 + q^5) \mathfrak{S}_{41} + q^7 \mathfrak{S}_5.
 \end{aligned}$$

This example should be compared to the image $\chi(Q'_{113}) = Q'_{113}$ that may be calculated either by applying the forgetful map to the right-hand side or from Definition 4.2:

$$Q'_{113} = (q^3 - q)s_{221} + q^4s_{311} + (q^5 + q^4 - q^2)s_{32} + (q^6 + q^5)s_{41} + q^7s_5.$$

Remark 4.19 The immaculate and the dual immaculate bases are now available in the latest version of Sage. The first, third, and fifth authors, with the help of Florent Hivert and Nicolas Thiéry, have put the non-commutative and quasi-symmetric functions into Sage.

A Appendix: Matrices for $n = 4$

In this appendix we show some examples of the transition matrices. $M(A, B)$ denotes the transition matrix between bases A and B . The rows and columns are indexed by compositions of 4 in lexicographic order (i.e., $[1, 1, 1, 1]$ is the top row and the leftmost column). In the matrix $M(s, \mathfrak{S}^*)$, the rows are indexed by partitions of 4, again in lexicographic order.

$$M(H, \mathfrak{S}) = \begin{pmatrix} 1 & 1 & 2 & 1 & 3 & 3 & 3 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\mathfrak{S}, H) = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(R, \mathfrak{S}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\mathfrak{S}, R) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(Q', \mathfrak{S}) = \begin{pmatrix} 1 & q & q^2 + q & q^3 & q^3 + q^2 + q & q^4 + q^3 + q^2 & q^5 + q^4 + q^3 & q^6 \\ 0 & 1 & q & q^2 & q^2 & q^3 + q^2 & q^4 + q^3 & q^5 \\ 0 & 0 & 1 & q & q & q^2 + q & q^3 + q^2 & q^4 \\ 0 & 0 & 0 & 1 & 0 & q & q^2 & q^3 \\ 0 & 0 & 0 & 0 & 1 & q & q^2 + q & q^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & q & q^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\mathfrak{S}, \mathcal{Q}') = \begin{pmatrix} 1 & -q & -q & q^2 & -q & q^2 & q^2 & -q^3 \\ 0 & 1 & -q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -q & -q & q^2 - q & q^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -q & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -q & -q & q^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\mathcal{Q}', R) = \begin{pmatrix} 1 & q & q^2 & q^3 & q^3 & q^4 & q^5 & q^6 \\ 0 & 1 & q-1 & q^2 & q^2 - q & q^3 - q + 1 & q^4 - q^2 + q - 1 & q^5 \\ 0 & 0 & 1 & q & q-1 & q^2 - 1 & q^3 - q + 1 & q^4 \\ 0 & 0 & 0 & 1 & 0 & q-1 & q^2 - q & q^3 \\ 0 & 0 & 0 & 0 & 1 & q & q^2 & q^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & q-1 & q^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(R, \mathcal{Q}') = \begin{pmatrix} 1 & -q & -q & q^2 & -q & q^2 & q^2 & -q^3 \\ 0 & 1 & -q+1 & -q & -q+1 & q^2 - 2q & q^2 - q & q^2 \\ 0 & 0 & 1 & -q & -q+1 & q^2 - 2q + 1 & q^2 - 2q & q^2 \\ 0 & 0 & 0 & 1 & 0 & -q+1 & -q+1 & -q \\ 0 & 0 & 0 & 0 & 1 & -q & -q & q^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -q+1 & -q \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(s, \mathfrak{S}^*) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\mathcal{Q}', H) = \begin{pmatrix} 1 & q-1 & q^2-1 & q^3-1 & q^4-1 & q^5-1 & q^6-1 & q^7-1 & q^8-1 & q^9-1 \\ 0 & 1 & q-1 & q^2-1 & q^3-1 & q^4-1 & q^5-1 & q^6-1 & q^7-1 & q^8-1 \\ 0 & 0 & 1 & q-1 & q^2-1 & q^3-1 & q^4-1 & q^5-1 & q^6-1 & q^7-1 \\ 0 & 0 & 0 & 1 & q-1 & q^2-1 & q^3-1 & q^4-1 & q^5-1 & q^6-1 \\ 0 & 0 & 0 & 0 & 1 & q-1 & q^2-1 & q^3-1 & q^4-1 & q^5-1 \\ 0 & 0 & 0 & 0 & 0 & 1 & q-1 & q^2-1 & q^3-1 & q^4-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & q-1 & q^2-1 & q^3-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & q-1 & q^2-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & q-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(H, \mathcal{Q}') = \begin{pmatrix} 1 & -q+1 & -2q+1 & q^2-2q+1 & 3q^2-6q+3 & 3q^2-6q+3 & -q^3+3q^2-3q+1 \\ 0 & 1 & -q+1 & -q+1 & -q+1 & q^2-3q+2 & q^2-2q+1 \\ 0 & 0 & 1 & -q+1 & -q+1 & q^2-3q+2 & q^2-2q+1 \\ 0 & 0 & 0 & 1 & 0 & -q+1 & -q+1 \\ 0 & 0 & 0 & 0 & 1 & -2q+2 & q^2-2q+1 \\ 0 & 0 & 0 & 0 & 0 & -q+1 & -q+1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -q+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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