J. Austral. Math. Soc. **71** (2001), 159–168

A NOTE ON MINIMAL COVERINGS OF GROUPS BY SUBGROUPS

R. A. BRYCE and L. SERENA

To Laci Kovács on his 65th birthday

(Received 30 March 2001)

Communicated by J. Cossey

Abstract

A *cover* for a group is a finite set of subgroups whose union is the whole group. A cover is *minimal* if its cardinality is minimal. Minimal covers of finite soluble groups are categorised; in particular all but at most one of their members are maximal subgroups. A characterisation is given of groups with minimal covers consisting of abelian subgroups.

2000 *Mathematics subject classification*: primary 20E99, 20F99.

1. Introduction

A *cover* for a group G is a finite collection of proper subgroups whose set-theoretic union is all of G. A cover $\mathscr{A}$ is *irredundant* if no proper subcollection of $\mathscr{A}$ is a cover for G. A *minimal* cover is one of least cardinality among covers for the group; it is necessarily irredundant. The size of a minimal cover of a group G we denote $\sigma(G)$. The idea of a minimal cover is due to Cohen [3] and the terminology to Tomkinson [12] who showed that, in a finite soluble group G, $\sigma(G) = |V| + 1$, where V is a chief factor of G with least order among chief factors of G with multiple complements.

In this note we first add some detail to Tomkinson's result by proving the following theorem.

THEOREM 1. *Every minimal cover of a non-cyclic, finite, soluble group contains at most one non-maximal subgroup.*

Rather more precise information is given below in Theorem 7 and its corollaries.

It is natural to ask questions about groups covered by subgroups with restricted properties. Thus, for example, it is known that a group is centre-by-finite if and only if it is coverable by abelian subgroups: see [8]; and hypercentral-by-finite if and only if coverable by nilpotent subgroups: see [11]. These results are trivial in the case of finite groups. However if we ask for *minimal* covers whose members have restricted properties the question is no longer so easy. By way of example we prove the following result.

THEOREM 2. *A non-abelian group G has a minimal cover consisting of abelian subgroups if and only if its central factor group G/Z is either*

(1) *monolithic, with non-central, elementary abelian monolith K/Z of prime-power order p^a having cyclic complements, and with K abelian; or*

(2) *elementary abelian of order p^2 for some prime number p;*

and, for each prime number $q < p^a$ or $q < p$, as the case may be, every finite factor group of G has cyclic Sylow q-subgroups.

2. Notations and quotations

If $\mathscr{A} = \{A_i : 1 \leq i \leq n\}$ is a cover for a group G and if D is the intersection of all the members of $\mathscr{A}$ then we write $D_G := \text{core}_G(D)$ and call it the *core* of $\mathscr{A}$, denoted core $\mathscr{A}$. For $N \trianglelefteq G$ and $N \subseteq D$ we write $\mathscr{A}/N := \{A_i/N : 1 \leq i \leq n\}$; it is a cover of G/N, irredundant if and only if $\mathscr{A}$ is irredundant, and minimal if, but not necessarily only if, $\mathscr{A}$ is minimal. If $g \in G$ then we write $\mathscr{A}^g := \{A_i^g : 1 \leq i \leq n\}$, the *conjugate* of $\mathscr{A}$ via g. Plainly $\mathscr{A}$ is irredundant (respectively minimal) if, and only if, every conjugate of $\mathscr{A}$ is irredundant (respectively minimal).

A natural partial order on covers of G is defined as follows. If $\mathscr{A} = \{A_i : 1 \leq i \leq n\}$ and $\mathscr{B} = \{B_j : 1 \leq j \leq r\}$ are covers of G and if there is a one-to-one function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, r\}$ such that $A_i \subseteq B_{f(i)}$ $(1 \leq i \leq n)$ we write $\mathscr{A} \preceq \mathscr{B}$; it is easily checked that $\preceq$ is a partial order. Notice that every minimal cover of a group is dominated in this partial order by a minimal cover consisting of maximal subgroups. We shall term *minimising* a minimal cover that dominates no other cover. Of course every minimal cover dominates a minimising cover.

We state here, for ease of reference, the result of Tomkinson [12] referred to earlier.

PROPOSITION 3. *If G is a finite soluble group then $\sigma(G) = |V| + 1$, where V is of least order among chief factors of G with multiple complements.*

We examine in more detail just how minimal covers arise in soluble groups G. Let p be a prime number and $r > 1$ an integer for which the order of p modulo r is a. Let

https://doi.org/10.1017/S1446788700002809 Published online by Cambridge University Press

C be a cyclic group of order r and let K be a faithful $\mathbb{Z}_p C$-module of dimension a, so that K is simple. Then construct $H = CK$ as the semi-direct product of C and K, the latter thought of as an elementary abelian group of order p^a. It is easy to see that every minimal cover of H consists of the p^a conjugates of C together with a subgroup containing K, the only member of the cover that need not be maximal and the only one that is not core-free.

Continuing the same notation, but allowing $r = 1$, we next consider the $\mathbb{Z}_p C$-module $V := K \oplus K$. It has $p^a + 1$ non-zero simple submodules. Consequently the natural semidirect product $L = CV$ has $p^a + 1$ subgroups, all containing C and all isomorphic to H, and these form a cover of L. This cover is minimal by Proposition 3. None of its members is core-free. This is our second example.

We shall say that a minimal cover $\mathscr{A}$ of a finite group G is of *type* 1 if $G/\operatorname{core}\mathscr{A}$ is isomorphic to a group of the type described in the penultimate paragraph; and of *type* 2 if $G/\operatorname{core}\mathscr{A}$ is of the type described in the last paragraph.

3. Proofs of Theorem 1 and extensions

It will be convenient to introduce the following notation. If $\mathscr{A}$ is a cover of the finite soluble group G then we write $\mathscr{A} = \mathscr{A}_0 \cup \mathscr{A}_1$ where $\mathscr{A}_0$ consists of the members of $\mathscr{A}$ that are not maximal in G and $\mathscr{A}_1$ consists of the members of $\mathscr{A}$ that are maximal in G. Moreover, if $\mathscr{D}$ is a subset of $\mathscr{A}_0$, and if N is minimal normal in G, we write $\mathscr{D}^*(N) := \{AN : A \in \mathscr{D}\}$ and $\mathscr{D}(N) := \mathscr{D}^*(N) \cup (\mathscr{A}_0 \backslash \mathscr{D}) \cup \mathscr{A}_1$. Notice that, for every $A \in \mathscr{A}_0$, $AN \neq G$ and therefore $\mathscr{D}(N)$ is a minimal cover of G whenever $\mathscr{A}$ is minimal.

LEMMA 4. *Let G be a finite, soluble but non-cyclic group, and $\mathscr{A}$ a minimal cover for it. Then $\mathscr{A}_1 \neq \emptyset$.*

PROOF. We suppose, in order to obtain a contradiction, that G is a non-cyclic finite soluble group of smallest order with respect to having a minimal cover $\mathscr{A}$ in which $\mathscr{A}_1$ is empty. Let N be minimal normal in G. Now $\mathscr{A}(N)/N$ is a minimal cover for G/N so it follows that, for some $A_0 \in \mathscr{A}$, $N \not\subseteq A_0$. Also, for some $a \in A_0$,

$$aN \notin \bigcup_{B \in \mathscr{A} \backslash \{A_0\}} BN/N.$$

Choose $x \in N \backslash A_0$. Then $ax \notin A_0$ so $ax \in B$ for some $B \in \mathscr{A}$ $(B \neq A_0)$. But then $aN \in BN/N \neq A_0N/N$, a contradiction. □

LEMMA 5. *Suppose G is a finite, non-cyclic, soluble group with $\sigma(G) = n$. If $\mathscr{A}$ is a minimal cover of G, and if $\mathscr{A}_1$ contains a core-free maximal subgroup of G, then $|\mathscr{A}_0| \leq 1$.*

https://doi.org/10.1017/S1446788700002809 Published online by Cambridge University Press

PROOF. Let $A_1 \in \mathscr{A}_1$ be core-free. Since G is finite and soluble there is a minimal normal subgroup N_1 of G for which $G = A_1N_1$. Indeed N_1 is the unique minimal normal subgroup of G, and in particular $|N_1| > 2$, and the complements of N_1 in G form a single conjugacy class of subgroups of G. Moreover a maximal subgroup of G is core-free if and only if it is a complement of N_1 in G. Note that N_1 has more than one complement or else A_1 is not core-free.

It follows from Proposition 3 that

$$n \leq |N_1| + 1.$$

Suppose that $|\mathscr{A}_0| > 1$ and let $\mathscr{A}' := \mathscr{A}_0(N_1)$, a minimal cover of G. We can find

$$h \in A_1 \setminus \bigcup_{A \in \mathscr{A}' \setminus \{A_1\}} A.$$

Then, for all $x \in N_1 \setminus \{1\}$,

$$hx \in \bigcup \mathscr{A}_1$$

or else h is in $\mathscr{A}_0^*(N_1)$, a contradiction. Now $|\mathscr{A}_1 \setminus \{A_1\}| < |N_1| - 1$. Hence, for some distinct $x, x' \in N_1 \setminus \{1\}$, and some $A \in \mathscr{A}_1 \setminus \{A_1\}$, $hx, hx' \in A$ whence $1 \neq x^{-1}x' \in A \cap N_1$ and therefore $N_1 \subseteq A$. However this leads to $h \in A$, a contradiction. Therefore $|\mathscr{A}_0| \leq 1$, as required. □

LEMMA 6. *If G is a finite, non-cyclic, soluble group with a minimal cover containing no core-free maximal subgroup of G, then $|\mathscr{A}_0| \leq 1$.*

PROOF. We suppose that $|\mathscr{A}_0| \geq 2$ and obtain a contradiction. Let G be a group of least order with this property satisfying the hypotheses.

Let N be a minimal normal subgroup of G. Firstly note that N is not contained in every member of $\mathscr{A}$. This is because the cover $\mathscr{A}/N$ of G/N either has a core-free maximal subgroup, in which case, by Lemma 5, it has at most one non-maximal subgroup; or $\mathscr{A}/N$ has no core-free maximal subgroups of G/N and so, by the minimality of G, has at most one non-maximal member, whence the minimality of G gives a contradiction. We deduce from this that there is no minimal normal subgroup contained in every member of $\mathscr{A}_1$. For, if N is minimal normal, and if $N \subseteq A$ $(A \in \mathscr{A}_1)$ let $A_0 \in \mathscr{A}_0$ with $N \not\subseteq A_0$ and

$$\mathscr{A}^{(1)} := (\mathscr{A}_0 \setminus \{A_0\})(N),$$

a minimal cover of G. Choose

$$h \in A_0 \setminus \cup (\mathscr{A}^{(1)} \setminus \{A_0\}),$$

and $x \in N \setminus A_0$. Then hx belongs to no member of $\mathscr{A}$, a contradiction. In fact there are at least two members of $\mathscr{A}_1$ not containing N. If N were in all but one member

https://doi.org/10.1017/S1446788700002809 Published online by Cambridge University Press

of $\mathscr{A}_1$ then N would be in all but one member of the irredundant cover $\mathscr{A}_0(N)$ and therefore in all of them, by [1, Lemma 2.2 (b)], also a contradiction.

It follows that N has more than one complement in G and hence, from Proposition 3,

$$\sigma(G) \leq |N| + 1.$$

If $A \in \mathscr{A}_1$ does not contain N then let $h \in A \setminus \cup(\mathscr{A}_0(N) \setminus \{A\})$ and $x \in N \setminus \{1\}$. Then $hx \notin A \cup \bigcup \mathscr{A}_0$. But

$$\left|\{A\} \cup \bigcup \mathscr{A}_0\right| \geq 3$$

and so hx is in one of the at most $|N| - 2$ subgroups in $\mathscr{A}_1 \setminus \{A\}$. But there are $|N| - 1$ possible choices for x so, for some different $x, x' \in N \setminus \{1\}$, hx, hx' belong to the same member, B say, of $\mathscr{A}_1$ with $B \neq A$, whence

$$1 \neq x^{-1}x' \in B \cap N,$$

so $N \subseteq B$ and therefore $h \in B$, a contradiction.

The assumption that $|\mathscr{A}_0| \geq 2$ is thus proved false. □

The proof of Theorem 1 is now complete because every minimal cover contains a maximal subgroup by Lemma 4, and either some maximal subgroup in the cover is core-free, the case covered by Lemma 5, or none are, the case covered by Lemma 6.

The next theorem gives more detail concerning minimal covers in finite soluble groups.

THEOREM 7. *A minimal cover for a finite soluble group is either of type* 1 *or of type* 2.

First we prove a useful lemma.

LEMMA 8. *Let $\mathscr{A}$ be a core-free minimal cover of a finite soluble group G and N be a minimal normal subgroup of G. Then N is contained in a unique member of $\mathscr{A}$ and intersects trivially all the other members of $\mathscr{A}$. Moreover, $|N| = |\mathscr{A}| - 1$.*

PROOF. Let $\mathscr{A} = \{A_i : 1 \leq i \leq n\}$. First of all we suppose that $\mathscr{A}$ consists of maximal subgroups, that is $\mathscr{A} = \mathscr{A}_1$. Notice that N intersects non-trivially, and therefore is contained in, at least one member of $\mathscr{A}$. Moreover, by [1, Lemma 2.2 (b)], there are at least two members of $\mathscr{A}$ not containing N. Hence, for some t satisfying $1 \leq t \leq n - 2$, and re-numbering if necessary, we may suppose that $N \subseteq A_i$ $(1 \leq i \leq t)$ but, since A_j is maximal whenever $j > t$, $N \cap A_j = 1$ $(t + 1 \leq j \leq n)$. Now since $\mathscr{A}$ is irredundant as a cover for G, there exists $a \in A_{t+1}$ not belonging to A_k for $k \neq t+1$. Then, for all $n \in N \setminus \{1\}$, $an \in \bigcup_{i=t+2}^{n} A_i$. Consequently, if $|N|-1 > n-(t+1)$ we would have, by the pigeon-hole principle, distinct $x, x' \in N \setminus \{1\}$ and some $j \in \{t+2, \dots, n\}$, for which $ax, ax' \in A_j$. Then $1 \neq x^{-1}x' \in A_j \cap N = 1$, a contradiction. It follows that $|N| \leq n-t$. Moreover, N has multiple complements in G. Therefore, by Proposition 3, $|N| \geq n-1$ and so $t \leq 1$. Of course N is in at least one member of $\mathscr{A}$ so $t = 1$. This completes the proof when $\mathscr{A}$ has no non-maximal subgroup.

In the case that $A_1 \in \mathscr{A}$, say, is not maximal in G, $A_1N \neq G$, so there is a maximal subgroup A_1^* containing A_1N. Also $\mathscr{A}^* := (\mathscr{A} \setminus \{A_1\}) \cup \{A_1^*\}$ is a minimal cover of G and, by Theorem 1, it consists of maximal subgroups. Therefore, by what we have already proved, N is in a unique member of $\mathscr{A}^*$. Hence N is not contained in, nor does it intersect non-trivially, members of $\mathscr{A}$ other than A_1, so $N \subseteq A_1$. This completes the proof of Lemma 8. □

COROLLARY 9. *A non-maximal subgroup in a core-free minimal cover for a finite soluble group G contains the Fitting subgroup $F(G)$.*

PROOF. We use the notation of the lemma. First of all note that the Frattini subgroup $\Phi(G) \subseteq D := \cap\mathscr{A}$ because, by [1, Lemma 2.2 (b)] and Theorem 1, D is the intersection of the maximal members of $\mathscr{A}$. Since $\Phi(G) \trianglelefteq G$, $\Phi(G) = 1$. Then $F(G)$ is the socle of G. If N is minimal normal in G, and if A_1 is the non-maximal member of $\mathscr{A}$ then, considering $\mathscr{A}^*$ as in the proof of the lemma, we see that $N \subseteq A_1^*$ so N is in no other member of $\mathscr{A}$ other than A_1. It follows that $N \subseteq A_1$ for all minimal normal N. That is $F(G) \subseteq A_1$, as required. □

PROOF OF THEOREM 7. Let G be a finite soluble group with $\sigma(G) = n$, and let $\mathscr{A}$ be a minimal cover for G. Assume, as we may do, that $\mathscr{A}$ is core-free. Either $\mathscr{A}$ contains a core-free maximal subgroup, or it does not. In the first case G is monolithic, say N is the monolith. Hence, by Lemma 8, there are $n-1 = |N|$ members of $\mathscr{A}$ all complementing N. Since N has a unique conjugacy class of $|N| = n-1$ complements therefore, $\mathscr{A}$ consists of the members of this class and one other subgroup of G containing N. Now we show that C is cyclic. By [5], chief factors of G above N have order less than $|N|$; and since $\mathscr{A}$ is a minimal cover with $|N|+1$ members it must be that such chief factors are either central or Frattini. This shows that C is nilpotent. However, if not cyclic C, and therefore G, has a factor group isomorphic to $C_q \times C_q$ for some prime q, and therefore a cover of $q+1$ subgroups. But $q < |N|$, a contradiction. (This argument is used by Tomkinson [12].) Thus $\mathscr{A}$ is of type 1.

In the other case, that is where $\mathscr{A}$ contains no core-free maximal subgroup, all members of $\mathscr{A}$ are maximal: for, by Corollary 9, a minimal cover with a non-maximal member necessarily contains core-free maximal subgroups. Choose $A \in \mathscr{A}$

https://doi.org/10.1017/S1446788700002809 Published online by Cambridge University Press

and suppose that $N \subseteq A$ is minimal normal. By Lemma 8 $|N| = n - 1$. Let $S := F(G)$. Then $S \cap A = N$. For, if $A \neq B \in \mathscr{A}$ then $BN = G$. Hence $S = N \times T$ where $T := S \cap B \trianglelefteq G$. Then $1 \neq T \nsubseteq A$ so $AT = G$ and $A \cap T \trianglelefteq G$. However A, B contain no common minimal normal subgroup, so $A \cap T \subseteq \mathrm{core}_G(A \cap B) = 1$. That is $A \cap S = N(A \cap T) = N$. In like manner $T = B \cap S$ is a minimal normal subgroup of G, so S is a direct product of two minimal normal subgroups of G.

We have proved that every member of $\mathscr{A}$ intersects S in a minimal normal subgroup of G contained in no other member of $\mathscr{A}$. Hence S contains a minimal normal subgroup U of G other than N, T. From this we see that N, T are G-isomorphic.

Since A, B are inconjugate maximal subgroups of G, $C := A \cap B$ is maximal in at least one of them, say in A, by [4, Corollary 16.7]. Now $CN = A$ so $CS = (CN)T = AT = G$ and so $C \cap S \trianglelefteq G$ whence $C \cap S = 1$. Then, since N, T are G-isomorphic,

$$C_C(N) = C_C(T) = C_C(S) \leq S \cap C = 1.$$

By Gaschütz [5] a chief factor of G above S has order less than $|N|$. By an argument used earlier in this proof we see that $C \cong G/S$ is cyclic. It follows that G has a type 2 minimal cover, and it remains to show that $\mathscr{A}$ is such a cover.

To this end we let $C = \langle c \rangle$, $\mathscr{A} = \{A_1, A_2, \dots, A_n\}$. Observe that G is a Frobenius group with kernel S and complement C: every element of G is contained either in S or in a conjugate of C. Now no $A_i \in \mathscr{A}$ is core-free so, by Lemma 8, the subgroups $V_i := S \cap A_i$ $(1 \leq i \leq n)$ are precisely the minimal normal subgroups of G. Moreover each A_i contains a conjugate of C: for, A_i is maximal and does not contain S so $G = A_iS = CS$ whence $A_i/S \cap A_i \cong G/S \cong C$. Therefore A_i contains a conjugate of C, using the Schur-Zassenhaus Theorem. It follows that the subsets S_i of S defined by

$$S_i := \{v \in S : c^v \in A_i\} \quad (1 \leq i \leq n)$$

are not empty. What is more, S is the union of the S_is.

We show that S_i is a coset of V_i in S. First, if $v_1, v_2 \in S_i$ then

$$[c, v_1v_2^{-1}] = [c, v_1v_2^{-1}]^{v_2} = (c^{-1}c^{v_1v_2^{-1}})^{v_2} = c^{-v_2}c^{v_1} \in A_i \cap V = V_i.$$

However the function $v \mapsto [c, v]$ is a one-to-one C-homomorphism under which $V_i \to V_i$. Hence $v_1v_2^{-1} \in V_i$ so, for some $w_i \in V$, $S_i \subseteq V_iw_i$. The converse inclusion is easily proved, so we have that $S_i = V_iw_i$ $(1 \leq i \leq n)$.

Finally, by [1, Lemma 2.6(b)], the S_is form an irredundant cover of S and they meet in a (unique) common point. That is, for some $r \in V$, $S_i = V_ir$ $(1 \leq i \leq n)$, so $c^r \in A_i$ $(1 \leq i \leq n)$. We have proved that $\mathscr{A}$ is, in this case, of type 2.

This completes the proof of Theorem 7. □

The following corollary follows easily.

COROLLARY 10. *Let the finite soluble group G have a minimal cover $\mathscr{A}$ with intersection D.*

(a) *If $\mathscr{A}$ is of type 1 then $D = D_G$ and, if $\mathscr{A}$ is minimising, then it is the unique minimising cover of G with core D_G.*

(b) *If $\mathscr{A}$ is of type 2 it is minimising and there is a unique conjugacy class of minimal covers with core D_G.*

4. Minimal covering with nilpotent subgroups

Here we consider a characterisation of groups minimally covered with nilpotent or abelian subgroups. A point of terminology: we shall call a (minimal) cover consisting of abelian (nilpotent) groups an abelian (nilpotent, minimal) cover. The first result applies to soluble, but not necessarily finite, groups.

THEOREM 11. *Let G be a non-nilpotent soluble group.*

(a) *A nilpotent minimal cover $\mathscr{A}$ for G is minimising of type 1 and, for some positive k, its intersection is $\zeta_k(G)$. Moreover, if $K/\zeta_k(G)$ is the socle of $G/\zeta_k(G)$ then, for every prime $q < |K/\zeta_k(G)|$, the Sylow q-subgroups of every finite factor group of G are cyclic.*

(b) *Conversely, if for some positive k, $G/\zeta_k(G)$ is monolithic with elementary abelian monolith $K/\zeta_k(G)$ with cyclic complements, and if for every prime $q < |K/\zeta_k(G)|$ the Sylow q-subgroups of every finite factor group of G are cyclic, then G has a unique nilpotent minimal cover.*

PROOF. (a) Let the intersection of $\mathscr{A}$ be D. If $\mathscr{A}$ is of type 1 then, by Corollary 10, $D_G = D$ and $\mathscr{A}$ is minimising since it is nilpotent. On the other hand, if $\mathscr{A}$ is of type 2 then the nilpotence of the members of $\mathscr{A}/D_G$ requires that $D = D_G$ and that G/D_G be elementary of order p^2.

Now, by Tomkinson [11, Theorem A], for some positive k, $D = D_G \subseteq \zeta_k(G)$. Since G is not nilpotent therefore, $\mathscr{A}$ is of type 1 and then, since $\zeta_1(G/D) = 1$, $D = \zeta_k(G)$.

Next let N be normal and of finite index in G. Then, by [7], $N \cap D$ is also of finite index. Let p be the prime dividing K/D. Since $K/D \cap N$ is nilpotent it has a unique Sylow p-subgroup $P/D \cap N$. Also G/P is a nilpotent p'-group because each of its chief factors is central. If for some prime $q \mid |G/P|$, the Sylow q-subgroup of G/P were not cyclic then G would have a factor group isomorphic to $C_q \times C_q$ and so $|K/D| + 1 = \sigma(G) \leq q + 1$. This completes the proof of (a).

https://doi.org/10.1017/S1446788700002809 Published online by Cambridge University Press

(b) First we prove the existence statement noting first of all that G has a cover $\mathscr{B}$ with intersection $\zeta_k(G)$: its members are K and the $|K/\zeta_k(G)|$ subgroups C containing $\zeta_k(G)$ for which $C/\zeta_k(G)$ are complements in $G/\zeta_k(G)$. All the members of $\mathscr{B}$ are nilpotent since each is an abelian extension of $\zeta_k(G)$. It remains to prove that $\mathscr{B}$ is minimal. To this end let $\mathscr{A}$ be a minimal cover of G and D its intersection. Note that $G/D_G \cap \zeta_k(G)$ is finite, using [7], and has exactly one non-central factor in a chief series. If $\mathscr{A}$ were of type 2 with G/D_G not nilpotent, it would have at least two non-central chief factors in every chief series. Hence either $\mathscr{A}$ is of type 1 or, if of type 2, G/D_G is elementary abelian of order q^2 for some prime q. In the second case $q \geq |K/\zeta_k(G)|$ since, by hypothesis, for primes q less than $|K/\zeta_k(G)|$, $G/D_G \cap \zeta_k(G)$ has cyclic Sylow q-subgroups. Of course then $q = |K/\zeta_k(G)|$ so $\mathscr{B}$ is minimal.

If $\mathscr{A}$ is of type 1 a chief series of $G/\zeta_k(G) \cap D_G$ through $\zeta_k(G)/\zeta_k(G) \cap D_G$ has exactly one non-central factor so the same is true of a chief series of $G/\zeta_k(G) \cap D_G$ through $D_G/\zeta_k(G) \cap D_G$. Consequently, writing L/D_G for the socle of G/D_G, we see that $|\mathscr{B}| = |K/\zeta_k(G)| + 1 = |L/D_G| + 1 = |\mathscr{A}| = \sigma(G)$, so $\mathscr{B}$ is minimal.

We now prove that $\mathscr{B}$ is the unique nilpotent minimal cover of G. To this end we suppose that $\mathscr{A}$ is a nilpotent minimal cover of G. By (a) its core is $\zeta_l(G)$ for some positive l and $G/\zeta_l(G)$ has non-central monolith with cyclic complements. Since both $G/\zeta_k(G)$ and $G/\zeta_l(G)$ have trivial centre it follows that $l = k$ and then that $\mathscr{A} = \mathscr{B}$. □

Proof of Theorem 2. Let $\mathscr{A} = \{A_1, \ldots, A_n\}$ be an abelian minimal cover of G. Since G is not abelian we may, and we temporarily do, suppose that $Z := \zeta_1(G) \subseteq A_i$ $(1 \leq i \leq n)$. Also, since $\mathscr{A}$ is a minimal covering, $\langle A_i, A_j \rangle = G$ $(1 \leq i < j \leq n)$. Then $A_i \cap A_j \subseteq Z$ and so $A_i \cap A_j = Z$ $(1 \leq i < j \leq n)$. Let $\bar{G} := G/Z$, $\bar{A}_i := A_i/Z$ $(1 \leq i \leq n)$, and $\bar{\mathscr{A}} := \mathscr{A}/Z$. Then $\bar{\mathscr{A}}$ is a minimal cover of $\bar{G}$ and $\bar{A}_i \cap \bar{A}_j = 1$ $(1 \leq i < j \leq n)$. That is $\bar{\mathscr{A}}$ is a partition of $\bar{G}$. $\bar{G}$ is finite by [7]. □

By a result of Suzuki [10] a finite insoluble group with a partition is isomorphic to one of the following:

(1) $PSL_2(q)$.
(2) $PGL_2(q)$.
(3) $Sz(2^n)$.

We use this to prove

Lemma 12. *A group with an abelian minimal cover is soluble.*

Proof. It is sufficient to show that, if G is a group with an abelian minimal cover then, in the notation introduced above, $\bar{G}$ is soluble. By [2] minimal coverings of groups of types (1) and (2) all contain non-abelian subgroups so $\bar{G}$ is neither of these.

In the case (3), that is $\bar{G} \cong Sz(2^n)$, a Sylow 2-subgroup S would be non-abelian, so S would not be not in any $\bar{A}_i$ and the set $\{S \cap \bar{A}_i : 1 \leq i \leq n\}$ would be a partition of S. A 2-group with a partition has proper Hughes subgroup. However $H_2(S) = S$, its involutions being all in the derived group of S. Thus case (3) is also dismissed. It follows that G is soluble. □

Now from Theorem 1 and [1, Lemma 2.2 (b)], Z is actually contained in the given A_i, and in fact $D = Z$. Either G is nilpotent, or it is not nilpotent. In the second case therefore, by Theorem 11, for some k, $Z = D = \zeta_k(G)$ so $k = 1$, and G/D is monolithic with elementary abelian monolith K/D with cyclic complements. Note that K is a member of $\mathscr{A}$, so abelian. Also all Sylow q-subgroups of finite factor groups of G are cyclic for $q < |K/D|$. In the case that G is nilpotent $\mathscr{A}$ is of type 2 and $G/D \cong C_p \times C_p$. Hence the conditions of Theorem 2 are necessary.

When G is not nilpotent the sufficiency of the stated conditions follows at once from Theorem 11. When G is nilpotent the result is easy and the details are left to the reader.

References

[1] R. A. Bryce, V. Fedri and L. Serena, 'Covering groups with subgroups', *Bull. Austral. Math. Soc.* **55** (1997), 469–476.

[2] ———, 'Subgroup coverings of some linear groups', *Bull. Austral. Math. Soc.* **60** (1999), 227–238.

[3] J. H. E. Cohen, 'On n-sum groups', *Math. Scand.* **75** (1994), 44–58.

[4] K. Doerk and T. Hawkes, *Finite soluble groups* (Walter de Gruyter, Berlin, 1992).

[5] W. Gaschütz, 'Existenz und Konjugiertsein von Untergruppen, die in endlichen auflösbaren Gruppen durch gewisse Indexschranken definiert sind', *J. Algebra* **53** (1978), 1–20.

[6] O. Kegel, 'Die Nilpotenz der H_p-Gruppen', *Math. Z.* **75** (1960/61), 373–376.

[7] B. H. Neumann, 'Groups covered by finitely many cosets', *Publ. Math. Debrecen* **3** (1954), 227–242.

[8] ———, 'A problem of Paul Erdös on groups', *J. Austral. Math. Soc. Ser. A* **21** (1976), 467–472.

[9] D. Passman, *Permutation groups* (W. A. Benjamin, Inc., New York, 1968).

[10] M. Suzuki, 'On a finite group with a partition', *Arch. Math.* **12** (1961), 241–254.

[11] M. J. Tomkinson, 'Hypercentre-by-finite groups', *Publ. Math. Debrecen* **40** (1992), 313–321.

[12] ———, 'Groups as the union of proper subgroups', *Math. Scand.* **81** (1997), 189–198.

School of Mathematical Sciences
Australian National University
Canberra ACT 0200
Australia
e-mail: bob.bryce@maths.anu.edu.au

Dipartimento di Matematica
e Applicazioni per l'Architettura
Università di Firenze
Piazza Ghiberti 27
50122 Firenze
Italia
e-mail: serena@math.unifi.it