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ESTIMATES FOR THE HEAT KERNEL ON $\mathrm{SL}(n,\mathbf{R})/\mathrm{SO}(n)$

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ABSTRACT. In [1], Jean-Philippe Anker conjectures an upper bound for the heat kernel of a symmetric space of noncompact type. We show in this paper that his prediction is verified for the space of positive definite $n \times n$ real matrices.

Introduction. Our notation will reflect that used in [7]. Given a symmetric space G/K of noncompact type, let $P_t(e^H)$ be the fundamental solution of the heat kernel where $H \in \mathbf{a}$, a Cartan subalgebra of the Lie algebra of G.

In [1], Jean-Philippe Anker gives an upper bound for the heat kernel P_t for the symmetric space $\mathbf{U}(p,q)/\mathbf{U}(p)\times\mathbf{U}(q)$. He shows that there exists a constant C such that for $t>0$ and $H\in\mathbf{a}^+$, then

(1)
$$P_t(e^H) \le Ce^{-\gamma^2 t}t^{-q/2}e^{-r^2(H)/(4t)} \prod_{\alpha\in\Sigma_0^+}\left(1+\frac{1+\alpha(H)}{t}\right)^{(m_\alpha+m_{2\alpha})/2-1} \prod_{\alpha\in\Sigma_0^+}\big(1+\alpha(H)\big)e^{-\rho(H)}$$

where $r^2(H)=\langle H,H\rangle$, ρ is the half-sum of the positive roots, $\gamma^2=\langle\rho,\rho\rangle$ and $q=l+2|\Sigma_0^+|$ ("the dimension at infinity"). Here l is the rank of the symmetric space and Σ_0^+ is the set of positive indivisible roots. The inner product $\langle\cdot,\cdot\rangle$ defined on $\mathbf{a}$ is a fixed multiple of the Killing form of the Lie algebra of G. We use the corresponding inner product on the set of linear functionals on $\mathbf{a}$.

Anker then conjectures that this upper bound holds for all symmetric spaces of noncompact type. This is true in the complex case as can be seen from the expression of the heat kernel as given by R. Gangolli in [5]. It also holds for all symmetric spaces of rank 1 as pointed out in [1] by Anker.

At the conference in Nancy in the honour of Pierre Eymard (May 1994), Anker gave an overview of the state of affairs in regard to his conjecture. Some general but weaker results have been obtained in the normal real case using the work of Mogens Flensted-Jensen in [6] (see also the doctoral dissertation of Maurice Chayet in [3]). In [2], Anker gives upper bounds that apply to all symmetric spaces of noncompact type but are weaker than his conjecture. In [8], we prove Anker's conjecture for the spaces $\mathbf{SU}^*(2n)/\mathbf{Sp}(n)$ and $E_{6(-26)}/F_4$. In [7], we show that for the spaces $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2)$ and $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3)$, $P_t(H)$ is bounded above and below by constant multiples of the right

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hand side of (2). The corresponding result for the heat kernel of a real hyperbolic space has been obtained by E. B. Davies and N. Mandouvalos (see [4, Theorem 5.7.2]). This has led Anker to a further prediction that a lower bound of the same type will also hold. He announced that he could verify his initial conjecture and this new prediction for all symmetric spaces of noncompact type in the case when $\|H\| \leq C(1+t)$.

This paper is concerned with proving the upper bound given in (2) for the space $SL(n,\mathbf{R})/SO(n)$. In [7], we give a recursive formula for the inverse of the Abel transform for $SL(n,\mathbf{R})/SO(n)$. Our strategy is inspired from the one Anker uses in [1]. In Theorem 3.1 of this paper, we show that the spherical functions as given by equation (2), can be extended analytically to the complex domain $\Omega_\pi = \{H \in \mathbf{a_C} : |\Im\alpha(H)| < \pi$ for each root $\alpha\}$ ($\Re z$ is the real part of z and $\Im z$ is the imaginary part of z). This result is interesting in itself. The corresponding result is easily seen to be true in the complex case and cannot be improved there. Using the spherical Fourier transform and an appropriate bound for the spherical functions, we show in Proposition 3.2 that the derivatives of the heat kernel by invariant differential operators are also analytic in the same domain.

We then use the expression for the heat kernel given in [7] to give an explicit formula for the heat kernel (and derivatives) on $\Omega_\eta^+ = \{H \in \mathbf{a_C} : |\Im\alpha(H)| < \eta, \Re\alpha(H) > 0$ for each root $\alpha\}$ with $0 < \eta < \pi$. Choosing $\epsilon > 0$ small enough, we prove our estimate on $\Omega^+_{\eta,\epsilon} = \{H \in \mathbf{a_C} : |\Im\alpha| < \eta, \Re\alpha(H) > 0, |\alpha(H)| \geq \epsilon$ for each root $\alpha\}$ (Proposition 3.9). Using symmetry and continuity, these estimates are also valid on $\Omega_{\eta,\epsilon} = \{H \in \mathbf{a_C} : |\Im\alpha| < \eta, |\alpha(H)| \geq \epsilon$ for each root $\alpha\}$. These detours allow us to avoid the difficulty when estimating the heat kernel near the boundary of a Weyl chamber of $\mathbf{a}$. Indeed, by extending the function to an analytic one, we can use the maximum modulus principle for polydisks to prove the estimate on $\Omega_\eta = \{H \in \mathbf{a_C} : |\Im\alpha(H)| < \eta$ for each root $\alpha\}$ and thus circumvent the problem.

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1. **Preliminaries.** The symmetric space $SL(n,\mathbf{R})/SO(n)$ can be realized as the space of positive definite $n \times n$ matrices of determinant 1 over the real numbers. We consider instead $\mathbf{Pos}(n,\mathbf{R})$, the space of positive definite $n\times n$ matrices over the real numbers. The correspondence between the results that interest us about the former space and the later one is straightforward.

In that context, $\mathbf{a}$ denotes the $n\times n$ diagonal matrices with real entries, $\mathbf{a}^+$ the diagonal matrices with strictly decreasing entries and $\mathbf{a_C}$ represents the diagonal matrices with complex entries. If $H \in \mathbf{a}$, then H_i will denote the i-th diagonal entry of H. We will use ξ to denote a $(n-1)\times(n-1)$ diagonal matrix with diagonal entries $\xi_1,\ldots,\xi_{n-1}$. A function f defined on a set of diagonal matrices is said to be *symmetric* if the values f takes do not depend on the order of the diagonal entries.

If λ is a linear functional on $\mathbf{a}$, then the corresponding spherical function is

$$\phi_\lambda(e^H) = \int_K e^{(i\lambda-\rho)\left(H(e^H k)\right)}\,dk. \tag{2}$$

The Abel transform of f is $F_f(e^H) = e^{\rho(H)} \int_N f(e^H n)\, dn$.

In [7], we introduce a new transform.

DEFINITION 1.1. The "False Abel Inverse Transform": if $H \in \mathbf{a}^+$ then

$$G(1, f; H) = f(e^H) \text{ and,} \quad \text{for } n \geq 2,$$

$$G(n, f; H) = \int_{H_{n-1}}^{H_{n-2}} \cdots \int_{H_1}^{\infty} G(n-1, f_{\mathrm{tr}\, H}; \xi) \prod_{i<j} \sinh(H_i - H_j) \cdot \prod_{i=1}^{n-1} \Big[\prod_{j=1}^{i-1} \sinh^{-1/2}(H_j - \xi_i) \prod_{j=i}^{n} \sinh^{-1/2}(\xi_i - H_j) \Big]\, d\xi$$

where $f_{\mathrm{tr}\, H}(e^{\xi}) = f\big(\exp(\mathrm{diag}[\xi, \mathrm{tr}\, H - \mathrm{tr}\, \xi])\big)$.

The interest of this transform comes from the following results ([7, Corollary 6.2 and Corollary 6.3]).

THEOREM 1.2. *If f is a smooth K-invariant function on $\mathbf{Pos}(n, \mathbf{R})$ of compact support then there exists a constant C such that $f = CG\big(n, \partial(\pi) F_f; \cdot\big)$. Here $\partial(\pi) = \prod_{i<j} (\frac{\partial}{\partial H_i} - \frac{\partial}{\partial H_j})$.*

PROOF. Since this is proven in detail in an earlier paper ([7, Corollary 6.2]), we only give here a brief sketch of the argument. The first step is to show that for a smooth function f that decreases sufficiently quickly at infinity, $\Delta G(n, f; \cdot) = G(n, \Gamma(\Delta) f; \cdot)$ where Δ is the radial part of the Laplacian and $\Gamma(\Delta) = \sum_{i=1}^{n} \frac{\partial^2}{\partial^2 H_i^2} - \gamma^2$. The next step is to show that there exists a positive constant independent of the linear functional $i\nu$, ν real, such that $G(n, \sum_{s\in W} (\det s) e^{is\nu}; e^H) = C\big(\pi(\nu)\big)^{-1} |c(\nu)|^{-2} \phi_\nu(e^H)$ where W is the Weyl group and c is Harish-Chandra c-function ($\partial(\pi) e^{i\nu(H)} = \pi(i\nu) e^{i\nu(H)} = K\pi(\nu) e^{i\nu(H)}$). Finally, if f is a smooth Weyl-invariant function then using the Euclidean Plancherel formula $f(H) = C \int_{\mathbf{a}^*} \hat{f}(\nu) e^{i\nu(H)}\, d\nu$ ($\mathbf{a}^*$ is the dual space of $\mathbf{a}$ over $\mathbf{R}$) we observe that

$$\begin{aligned} f(H) &= \frac{1}{|W|} \sum_{s\in W} f(s^{-1} \cdot H) = \frac{C}{|W|} \sum_{s\in W} \int_{\mathbf{a}^*} \hat{f}(\nu) e^{i\nu(s^{-1}\cdot H)}\, d\nu \\ &= \frac{C}{|W|} \sum_{s\in W} \int_{\mathbf{a}^*} \hat{f}(\nu) e^{is\nu(H)}\, d\nu, \end{aligned}$$

$$\big(\partial(\pi) f\big)(H) = C' \int_{\mathbf{a}^*} \pi(\nu) \hat{f}(\nu) \sum_{s\in W} (\det s) e^{is\nu(H)}\, d\nu.$$

Hence, writing $\tilde{f}$ for the spherical transform of K-invariant function f,

$$\begin{aligned} G\big(n, \partial(\pi) F_f; e^H\big) &= C \int_{\mathbf{a}^*} \widehat{F_f}(\nu) \phi_\nu(e^H) |c(\nu)|^{-2}\, d\nu \\ &= C' \int_{\mathbf{a}^*} \tilde{f}(\nu) \phi_\nu(e^H) |c(\nu)|^{-2}\, d\nu = C'' f \end{aligned}$$

(we used the fact that F_f is Weyl-invariant, that $\widehat{F_f} = \tilde{f}$ and the Plancherel formula for the symmetric space).

COROLLARY 1.3. *We write $W_t(H) = e^{-\gamma^2 t} t^{-n/2} \prod_{i<j} \big((H_i - H_j)/t\big) \exp\big(-r^2(H)/(4t)\big)$. Then $G(n, W_t; \cdot)$ is a nonzero multiple of the fundamental solution of the heat equation for $\mathbf{Pos}(n, \mathbf{R})$.*

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PROOF. The Abel transform of the heat kernel is known to be $Ce^{-\gamma^2 t}t^{-n/2}\exp\big(-r^2(H)/(4t)\big)$ (see for instance [1]). The result follows then from the previous theorem.

2. General results and definitions.

In this section, we put together the technical results and definitions we will need in the rest of the paper.

DEFINITION 2.1. For any nonzero complex number z, $z = |z|e^{i\arg z}$ with $-\pi < \arg z \le \pi$. If z is not a negative number, then $\sqrt{z}$ is the principal branch of the square root: $\sqrt{z} = |z|^{1/2}e^{i(\arg z)/2}$.

The next result will be used extensively in our estimates.

LEMMA 2.2. *The function* $(\sinh z)/z$ *is analytic in the domain* $|\Im z| < \pi$ *(more precisely, it has a removable singularity at* 0*). In that domain, it is nonzero and does not take negative values. There exists* $C = C_\eta > 0$ *such that if* $|\Im z| \le \eta < \pi$ *and* $\Re z \ge 0$ *then* $C^{-1}\frac{|z|}{1+|z|}\exp(\Re z) \le |\sinh z| \le C\frac{|z|}{1+|z|}\exp(\Re z)$*. If in addition,* $|z| \ge \epsilon > 0$*, then there exists* $C = C_{\eta,\epsilon}$ *such that* $C^{-1}\exp(\Re z) \le |\sinh z| \le C\exp(\Re z)$*.*

PROOF. The power series for $f(z) = (\sinh z)/z$, $\sum_{k=0}^{\infty} z^{2k}/(2k+1)!$, converges for all z. In particular, $f(0) = 1$. In what follows, $z = x + iy$ with $x, y \in \mathbf{R}$ and $|y| < \pi$. We now show that $f(z)$ is nonzero and does not take negative values. Since f is an even function with $f(x) > 0$ if $x \in \mathbf{R}$, we may assume that $0 < y < \pi$. The equation $(\sinh z)/z = -c^2$, $c \ge 0$, can be written as $\sinh x\cos y + i\cosh x\sin y = -c^2x - ic^2y$ which is not possible since $\cosh x\sin y > 0$. Now assume that $x \ge 0$ and $-\pi < -\eta \le y \le \eta < \pi$. If $x \le 1$, the inequality $C^{-1}\frac{|z|}{1+|z|}\exp(\Re z) \le |\sinh z| \le C\frac{|z|}{1+|z|}\exp(\Re z)$ is implied by the inequality $\tilde{C}^{-1} \le |(\sinh z)/z| < \tilde{C}$ over the compact set $0 \le x \le 1$, $|y| \le \eta$ which is clear since $(\sinh z)/z$ is analytic and nonzero on that set. If $x \ge 1$ then $\sinh z = (1/2)e^z(1 - e^{-2z})$ with $|e^z| = e^{\Re z}$, $|1 - e^{-2z}| \le 1 + |e^{-2z}| = 1 + e^{-2\Re z} \le 2$ and $|1 - e^{-2z}| \ge 1 - |e^{-2z}| = 1 - e^{-2\Re z} \ge 1 - e^{-2} > 0$. Therefore, if $x \ge 1$, $\frac{1-e^{-2}}{2}e^{\Re z} \le |\sinh z| \le e^{\Re z}$. The two cases put together prove the result. The last part of the lemma is straightforward.

DEFINITION 2.3. Let $r \ge 0$ and $z \in \mathbf{C}$. Define $D(z,r) = \{w \in \mathbf{C} : |w - z| < r\}$, $\bar{D}(z,r) = \{w \in \mathbf{C} : |w - z| \le r\}$ and $S(z,r) = \{w \in \mathbf{C} : |w - z| = r\}$.

DEFINITION 2.4. Let $\eta > 0$ and $\epsilon > 0$. We define $\Omega_\eta = \{H \in \mathbf{a}_{\mathbf{C}} : |\Im(H_i - H_j)| < \eta$ for all $i,j\}$, $\Omega_{\eta,\epsilon} = \{H \in \Omega_\eta : |H_i - H_j| \ge \epsilon$ for all $i \ne j\}$, $\Omega_\eta^+ = \{H \in \Omega_\eta : \Re(H_i - H_{i+1}) > 0$ for each $i\}$ and $\Omega_{\eta,\epsilon}^+ = \Omega_\eta^+ \cap \Omega_{\eta,\epsilon}$.

PROPOSITION 2.5. *Suppose* $n \ge 2$, $\eta > 0$ *and* $0 < \epsilon < \eta/(2^{n+1}n!)$*. Write* $r_n = 2^n n!\,\epsilon$ *for simplicity. For* $H \in \Omega_{\eta-2r_n}$*, there exist* $s_1, \ldots, s_n$ *such that*

1. $0 < s_i < r_n$ *for each i.*
2. $\prod_{i=1}^{n} S(H_i, s_i) \subset \Omega_{\eta,\epsilon}$.

Note that $\eta - 2r_n > 0$ and that (2) implies that $\prod_{i=1}^{n} \bar{D}(H_i, s_i) \subset \Omega_\eta$.

PROOF. We use induction on n. Let $n = 2$ and $H \in \Omega_{\eta-2r_2}$. If $|H_1 - H_2| \geq 2\epsilon$, we take $s_1 = s_2 = \epsilon/2$. If $|H_1 - H_2| < 2\epsilon$, we take $s_1 = \epsilon/2$ and $s_2 = 7\epsilon/2 < r_2$. If $F_i \in S(H_i, s_i)$ then $|F_1 - F_2| \geq \epsilon$ and $|\Im(F_1 - F_2)| \leq |\Im(F_1 - H_1)| + |\Im(H_1 - H_2)| + |\Im(F_2 - H_2)| < \epsilon/2 + (\eta - 2r_2) + 7\epsilon/2 < \eta$.

Suppose the result is true for $n-1$, $n \geq 3$. We will use the symbol $\hat{}$ to distinguish the sets that for $n-1$ correspond to Ω_η and $\Omega_{\eta,\epsilon}$. Let $H \in \Omega_{\eta-2r_n}$; since $r_{n-1} < r_n$, $\mathrm{diag}[H_2, \ldots, H_n] \in \hat{\Omega}_{\eta-2r_{n-1}}$. Therefore there exist $s_2, \ldots, s_n$ such that

1. $0 < s_i < r_{n-1}$ for each $i > 1$.
2. $\prod_{i=2}^n S(H_i, s_i) \subset \hat{\Omega}_{\eta,\epsilon}$.

Let $I_1 = \{i > 1 : D(H_i, s_i) \cap D(H_1, 3\epsilon/2) \neq \emptyset\}$. If $I_1 = \emptyset$, put $s_1 = \epsilon/2$ and stop; otherwise, let $I_2 = \{i > 1 : D(H_i, s_i) \cap D(H_1, 2r_{n-1} + 7\epsilon/2) \neq \emptyset\}$. If $I_2 = I_1$, then put $s_1 = 2r_{n-1} + 5\epsilon/2$ and stop. In general, if $I_k \neq I_{k-1}$, then set $I_{k+1} = \{i > 1 : D(H_i, s_i) \cap D(H_1, 2kr_{n-1} + (4k+3)\epsilon/2) \neq \emptyset\}$. If $I_{k+1} = I_k$, then put $s_1 = 2kr_{n-1} + (4k+1)\epsilon/2$ and stop. Otherwise continue. Clearly the process must stop with k at most $n-1$. In any case, $s_1 \leq 2(n-1)r_{n-1} + (4(n-1)+1)\epsilon/2 < r_n$. Let $F_i \in S(H_i, s_i)$ for $i = 1, \ldots, n$. The condition $|F_i - F_j| \geq \epsilon$ for $i \neq j$ is satisfied by the induction hypothesis if $i > 1$ and $j > 1$ and by construction if $i = 1$, $j > 1$. If $i > 1$, $j > 1$ then by the induction hypothesis, $|\Im(F_i - F_j)| < \eta$. Finally, if $j > 1$, then $|\Im(F_1 - F_j)| \leq |\Im(F_1 - H_1)| + |\Im(H_1 - H_j)| + |\Im(H_j - F_j)| < s_1 + (\eta - 2r_n) + s_j < \eta$.

PROPOSITION 2.6. *Let $s_1 > 0, \ldots, s_n > 0$. Suppose that f is analytic on $\prod_{i=1}^n D(x_i, s_i)$ and continuous on $A = \prod_{i=1}^n \bar{D}(x_i, s_i)$. Let $\partial A = \prod_{i=1}^n S(x_i, s_i)$. Then* $\max_{z\in A} |f(z)| = \max_{z\in\partial(A)} |f(z)|$.

PROOF. We need only to use Cauchy's integral formula for the polydisk.

The following lemma and its proof are inspired from the result stating that the roots of the derivative P' of a complex polynomial P are in the convex hull of the roots of P.

LEMMA 2.7. *Suppose $z_1, \ldots, z_n \in \mathbf{C}$. Let $P(z) = \sum_{p=1}^n \beta_p \prod_{j\neq p}(z - z_j)$ where $\beta_1 \geq 0, \ldots, \beta_n \geq 0$ are not all zero. Then the zeros of P belong to the convex hull of $z_1, \ldots, z_n$.*

PROOF. Let $Q(z) = \prod_{i=1}^n (z - z_i)$. We have $\frac{P(z)}{Q(z)} = \sum_{p=1}^n \frac{\beta_p}{z - z_p} = \sum_{p=1}^n \frac{\beta_p}{|z-z_p|^2}(\bar{z} - \bar{z}_p)$. If z is a zero of P but not a zero of Q then $0 = \sum_{p=1}^n \frac{\beta_p}{|z-z_p|^2}(\bar{z} - \bar{z}_p)$ and, taking conjugates, $\sum_{p=1}^n \frac{\beta_p}{|z-z_p|^2} z = \sum_{p=1}^n \frac{\beta_p}{|z-z_p|^2} z_p$ which implies the result.

DEFINITION 2.8. The elementary symmetric polynomials in the $n-1$ variables $y_1, \ldots, y_{n-1} \in \mathbf{C}$ are $e_q(y) = \sum_{i_1<\cdots<i_q} y_{i_1} \cdots y_{i_q}$ where $q = 1, \ldots, n-1$. Given $z_1, \ldots, z_n \in \mathbf{C}$, we will denote $e_q^{(p)}(z) = e_q(z_1, \ldots, z_{p-1}, z_{p+1}, \ldots, z_n)$.

COROLLARY 2.9. *Suppose $z_1, \ldots, z_n \in \mathbf{C}$. Fix $\beta_1 \geq 0, \ldots, \beta_n \geq 0$ such that $\sum_{p=1}^n \beta_p = 1$. If we define, modulo the order, $y_1, \ldots, y_{n-1}$ by $e_q(y) = \sum_{p=1}^n \beta_p e_q^{(p)}(z)$ then each y_i belong to the convex hull of $z_1, \ldots, z_n$.*

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PROOF. It suffices to expand both sides of the equality $\sum_{p=1}^{n}\beta_p\prod_{j\neq p}(z-z_j)=\prod_{i=1}^{n-1}(z-y_i)$ in powers of z and use Lemma 2.7.

The proof of the following result was communicated to us by Professor A. M. Davie from the University of Edinburgh.

LEMMA 2.10. *Suppose D is a symmetric domain in $\mathbf{C}^n$ (i.e. invariant under any permutation of the coordinates) and suppose $f: D\longrightarrow \mathbf{C}^n$ is analytic and symmetric (i.e. invariant under any permutation of its variables). Then there exists a domain E in $\mathbf{C}^n$ and an analytic function $F: E\longrightarrow \mathbf{C}^n$ such that $f(z_1,\ldots,z_n)=F(e_1,\ldots,e_n)$ where $e_1,\ldots,e_n$ are the elementary symmetric polynomials in $z_1,\ldots,z_n$.*

PROOF. Let E be the image of D under the mapping $(z_1,\ldots,z_n)\longrightarrow(e_1,\ldots,e_n)$ and define $h(e_1,\ldots,e_n)=f(z_1,\ldots,z_n)$ where $z_1,\ldots,z_n$ are the roots of the equation $z^n-e_1z^{n-1}+e_2z^{n-2}-\cdots+(-1)^ne_n=0$. Since the roots depend continuously on the coefficients, h is continuous. It is also analytic when the roots are distinct. Thus h is continuous on E and analytic outside a proper subvariety and is therefore analytic on E.

3. Estimates.

THEOREM 3.1. *For every linear functional λ on $\mathbf{a}$, the spherical function ϕ_λ can be extended analytically to Ω_π. It is then symmetric in that domain. If $0<\eta<\pi$ and $\lambda(H)=\sum_{i=1}^{n}a_iH_i$ with $a_i\in\mathbf{R}$ for each i, then there exists a positive continuous function C_η independent of λ such that $|\phi_\lambda(e^H)|\leq C_\eta(H)\exp(-\sum_{i=1}^{n}a_i\Im H_i)\exp(\sum_{i<j}\eta|a_i-a_j|)$ on Ω_η.*

PROOF. We prove the result using induction. The result is clearly true for $n=1$ since then $\phi_\lambda(e^H)=e^{i\lambda(H)}$. Suppose $\lambda(H)=\sum_{i=1}^{n}a_iH_i$ with $a_i\in\mathbf{C}$ for all i. Let $\lambda_0(\xi)=\sum_{i=1}^{n-1}(a_i-a_n)\xi_i$ and let

$$f(\xi,H)=e^{ia_n(H_1+\cdots+H_n)}\phi_{\lambda_0}(e^\xi)\prod_{i<j}\frac{\sinh(\xi_i-\xi_j)}{\xi_i-\xi_j}\prod_{i=1}^{n-1}\left[\prod_{j=1}^{i}\sqrt{\frac{H_j-\xi_i}{\sinh(H_j-\xi_i)}}\prod_{j=i+1}^{n}\sqrt{\frac{\xi_i-H_j}{\sinh(\xi_i-H_j)}}\right].$$

If we adapt [9, Theorem 5.3] to our situation ($m=1$, $r=n$ and $\phi_\lambda(e^H)=\Phi_s(e^{2H})$ with $s_k=\frac{ia_k-(n+1-2k)}{2}$), and let $H_1>H_2>\cdots>H_n$, we then have

$$\begin{aligned}\phi_\lambda(e^{H_1},\ldots,e^{H_r})&=\frac{\Gamma(n/2)}{\big(\Gamma(1/2)\big)^n}e^{ia_n(H_1+\cdots+H_n)}\int_{H_n}^{H_{n-1}}\cdots\int_{H_2}^{H_1}\phi_{\lambda_0}(e^\xi)\prod_{i<j}\sinh(\xi_i-\xi_j)\\&\quad\cdot\prod_{i=1}^{n-1}\left[\prod_{j=1}^{i}\sinh^{-1/2}(H_j-\xi_i)\prod_{j=i+1}^{n}\sinh^{-1/2}(\xi_i-H_j)\right]d\xi_1\cdots d\xi_{n-1}\\&=\frac{\Gamma(n/2)}{\big(\Gamma(1/2)\big)^n}\int_{H_n}^{H_{n-1}}\cdots\int_{H_2}^{H_1}f(\xi,H)\frac{\prod_{i<j}(\xi_i-\xi_j)\,d\xi_1\cdots d\xi_{n-1}}{\prod_{j=1}^{i}\sqrt{H_j-\xi_i}\prod_{j=i+1}^{n}\sqrt{\xi_i-H_j}}.\end{aligned}$$

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Now $f(\xi, H)$ is a symmetric function of ξ and of H which is analytic in ξ and H as long as

(3) $$|\Im(\xi_i - \xi_j)| < \pi \text{ and } |\Im(\xi_i - H_j)| < \pi \quad \text{for all } i, j.$$

Let $\beta_p = \prod_{i=1}^{n-1}(\xi_i - H_p) / \prod_{i \neq p}(H_i - H_p)$ for $p = 1, \ldots, n$. Then $\sum_{p=1}^n \beta_p = 1$ and

$$\phi_\lambda(e^{H_1}, \ldots, e^{H_r}) = \frac{\Gamma(n/2)}{\big(\Gamma(1/2)\big)^n} \int_\sigma F(\sum_{p=1}^n \beta_p e_1^{(p)}(H), \ldots, \sum_{p=1}^n \beta_p e_{n-1}^{(p)}(H), H) \frac{d\beta_1 \cdots d\beta_{n-1}}{\sqrt{\beta_1 \cdots \beta_n}}$$

where $\sigma = \{(\beta_1, \ldots, \beta_n) : \beta_1 \geq 0, \ldots, \beta_n \geq 0 \text{ and } \sum_{p=1}^n \beta_p = 1\}$ and F corresponds to f by virtue of Lemma 2.10. This formula allows us to give an analytic continuation of ϕ_λ to Ω_π. Indeed, for any $H \in \Omega_\pi$, $\big(\sum_{p=1}^n \beta_p e_1^{(p)}(H), \ldots, \sum_{p=1}^n \beta_p e_{n-1}^{(p)}(H)\big)$ corresponds to ξ with each ξ_i in the convex hull of $H_1, \ldots, H_n$ by the corollary to Lemma 2.7. This means that the conditions in equation (3) are satisfied. It is also clear that ϕ_λ is a symmetric function. Suppose now that $a_k \in \mathbf{R}$ for each k and let $0 < \eta < \pi$. By the induction hypothesis, there exists a positive continuous function $C_\eta(\xi)$ such that whenever $H \in \Omega_\eta$,

$$\begin{aligned}
&|e^{ia_n(H_1+\cdots+H_n)}\phi_{\lambda_0}(e^\xi)| \\
&\quad \leq C_\eta(\xi)|e^{ia_n(H_1+\cdots+H_n)}| \exp\big(-\sum_{i=1}^{n-1}(a_i - a_n)\Im\xi_i\big) \exp(\sum_{i<j<n} \eta|a_i - a_j|) \\
&\quad = C_\eta(\xi) \exp(-\sum_{i=1}^n a_i \Im H_i) \exp(\sum_{i<j<n} \eta|a_i - a_j|) \exp\big(-\sum_{i=1}^{n-1}(a_i - a_n)\Im(\xi_i - H_i)\big) \\
&\quad \leq \tilde{C}_\eta(\xi) \exp(-\sum_{i=1}^n a_i \Im H_i) \exp(\sum_{i<j} \eta|a_i - a_j|).
\end{aligned}$$

If we use this with Lemma 2.2, we find that

$$|f(\xi, H)| \leq \hat{C}_\eta(\xi, H) \exp(-\sum_{i=1}^n a_i \Im H_i) \exp(\sum_{i<j} \eta|a_i - a_j|)$$

where $\hat{C}_\eta(\xi, H)$ is continuous. This is all we need.

PROPOSITION 3.2. *Let p be a skew-symmetric polynomial in n variables. Then $G(n, pe^{-r^2/(4t)}; \cdot)$ can be extended analytically on Ω_π and is a symmetric function there.*

PROOF. For $\lambda(H) = \sum_{k=1}^n a_k H_k$, let $\pi(\lambda) = \prod_{i<j}(a_i - a_j)$. Let $q(\lambda, t)e^{-\lambda^2 t}$ be the Euclidean Fourier transform of $pe^{-r^2/(4t)}$. It is clear that q is a polynomial which is skew-symmetric in the variables $a_1, \ldots, a_n$. We have, for $H \in \mathbf{a}^+$ (see [7, (19)]),

$$G(n, pe^{-r^2/(4t)}; H) = C \int_{\mathbf{a}^*} \frac{q(\lambda, t)}{\pi(\lambda)} e^{-\lambda^2 t} \phi_\lambda(e^H) |c(\lambda)|^{-2} \, d\lambda$$

where C is a constant and $\mathbf{a}^*$ is the set of real-valued linear functionals on $\mathbf{a}$. Theorem 3.1 allows us the analytic continuation of the right hand side of this equation.

LEMMA 3.3. *If p is a skew-symmetric polynomial in n variables then $(pe^{-r^2/(4t)})_{\operatorname{tr} H} = \exp\big(-(\operatorname{tr} H - \operatorname{tr} \xi)^2/(4t)\big) p_{\operatorname{tr} H}(\xi) e^{-r^2(\xi)/(4t)}$ and $p_{\operatorname{tr} H}$ is a skew-symmetric polynomial in the variables $\xi_1, \ldots, \xi_{n-1}$.*

PROOF. Recall that $f_{\operatorname{tr} H}(e^\xi) = f\big(\exp(\operatorname{diag}[\xi, \operatorname{tr} H - \operatorname{tr}\xi])\big)$. Therefore

$$\begin{aligned}(pe^{-r^2/(4t)})_{\operatorname{tr} H}(e^\xi) &= p(\xi_1,\ldots,\xi_{n-1},\operatorname{tr} H - \operatorname{tr}\xi)\\ &\qquad \cdot \exp\big(-(\xi_1^2+\cdots+\xi_{n-1}^2 + (\operatorname{tr} H - \operatorname{tr}\xi)^2)/(4t)\big)\\ &= \exp\big(-(\operatorname{tr} H - \operatorname{tr}\xi)^2/(4t)\big) p(\xi_1,\ldots,\xi_{n-1},\operatorname{tr} H - \operatorname{tr}\xi)\\ &\qquad \cdot \exp\big(-(\xi_1^2+\cdots+\xi_{n-1}^2)/(4t)\big).\end{aligned}$$

That $p_{\operatorname{tr} H}(\xi) = p(\xi_1,\ldots,\xi_{n-1},\operatorname{tr} H - \operatorname{tr}\xi)$ is a skew-symmetrix polynomial in the variables $\xi_1,\ldots,\xi_{n-1}$ is clear.

Now, we need an explicit formula for $G(n, pe^{-r^2/(4t)}; H)$ on a suitable portion of Ω_π in order to compute our estimates. We will proceed in several steps. Lemma 3.4 will give us a formula which is a natural extension of Definition 1.1. This formula involves integrating over the path from H_1 to ∞ and, for $2 \le i \le n-1$, over the path from H_i to H_{i-1}. It is however rather difficult to estimate. Lemma 3.5 gives a modified version of the formula which will be easier to handle. Finally, after giving the construction of the appropriate paths in Lemma 3.6, we will be ready to compute our estimates in Proposition 3.9.

LEMMA 3.4. *Suppose $n \ge 2$ and $0 < \eta < \pi$. For $H \in \Omega_\eta^+$, choose continuous simple paths $\Xi_1,\ldots,\Xi_{n-1}$ satisfying the following conditions (we denote a point of the path Ξ_k by ξ_k).*

1. *The path Ξ_1 starts at H_1 and its real part increases to ∞.*
2. *For $2 \le k \le n-1$, Ξ_k is a path from H_k to H_{k-1}.*
3. *For $i \le j$, $\xi_i - H_j \in \{z \in \mathbf{C} : |\Im z| < \eta\} - [0,\infty)$ (unless $i = j$ and $\xi_i = H_i$).*
4. *For $j < i$, $H_j - \xi_i \in \{z \in \mathbf{C} : |\Im z| < \eta\} - [0,\infty)$ (unless $j = i-1$ and $\xi_i = H_{i-1}$).*
5. *$|\Im(\xi_i - \xi_j)| < \eta$ for each i, j.*

Let $f = pe^{-r^2/(4t)}$ where p is a skew-symmetric polynomial in n variables and suppose that on these paths, we have

$$|g(\xi,H)| \le C(H)P(\xi_1,\ldots,\xi_{n-1})e^{-\sum_{i=1}^{n-1}\Re(\xi_i - H_i)} \tag{4}$$

where C does not depend on ξ, P is bounded by a polynomial and

$$g(\xi,H) = G(n-1, f_{\operatorname{tr} H};\xi)\prod_{i<j}\frac{\sinh(\xi_i-\xi_j)}{\xi_i-\xi_j}\prod_{i=1}^{n-1}\left[\prod_{j=1}^{i-1}\sqrt{\frac{H_j-\xi_i}{\sinh H_j - \xi_i}}\prod_{j=i}^{n}\sqrt{\frac{\xi_i - H_j}{\sinh(\xi_i - H_j)}}\right].$$

Then for $H \in \Omega_\eta^+$, we have

$$G(n,f;H) = \int_{\Xi_{n-1}}\cdots\int_{\Xi_1} g(\xi,H)\frac{\prod_{i<j}(\xi_i-\xi_j)}{\prod_{i=1}^{n-1}\big[\prod_{j=1}^{i-1}\sqrt{H_j-\xi_i}\prod_{j=i}^{n}\sqrt{\xi_i - H_j}\big]}\,d\xi.$$

We will see in Proposition 3.9 that the condition in equation (4) is necessarily satisfied (it is a consequence of the induction step in the proof of that result).

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PROOF. If we refer to Definition 1.1, we only have to show that the above expression is analytic in Ω_η^+ for any choice of paths satisfying the above conditions. Note that Lemma 2.2, Proposition 3.2 and Lemma 3.3 ensures that $g(\xi, H)$ is analytic along these paths.

For each $k \geq 2$, the integration over the path Ξ_k can be written as

$$\int_{\Xi_k} h(H, \xi) \frac{d\xi_k}{\sqrt{H_{k-1} - \xi_k}\sqrt{\xi_k - H_k}}$$

where $h(\xi, H)$ is analytic in H and ξ. This is clearly differentiable and, by Cauchy's theorem, does not depend on the particular choice of path. A similar argument can be given for $k = 1$ (the factor $e^{-\Re(\xi_1 - H_1)}$ plays an important role there).

LEMMA 3.5. *Suppose $n \geq 2$ and $0 < \eta < \pi$. For $H \in \Omega_\eta^+$, choose continuous simple paths $\Xi_1, \ldots, \Xi_{n-1}$ satisfying the following conditions (we denote a point of the path Ξ_k by ξ_k).*

1. *For $1 \leq k \leq n-1$, the path Ξ_k starts at H_k and its real part tends to ∞.*
2. *For $i < j$, $\xi_i - H_j \in \{z \in \mathbf{C} : |\Im z| < \eta\} - [0, \infty)$ (unless $i = j$ and $\xi_i = H_i$).*
3. *For $j < i$, $H_j - \xi_i \in \{z \in \mathbf{C} : |\Im z| < \eta\} - [0, \infty)$.*
4. *$|\Im(\xi_i - \xi_j)| < \eta$ for each i, j.*

Let $f = pe^{-r^2/(4t)}$ where p is a skew-symmetric polynomial in n variables and suppose $g(\xi, H)$ is as in Lemma 3.4 and satisfies condition (4). Then

$$G(n, f; H) = \int_{\Xi_{n-1}} \cdots \int_{\Xi_1} g(\xi, H) \frac{\prod_{i<j}(\xi_i - \xi_j)}{\prod_{i=1}^{n-1}\left[\prod_{j=1}^{i-1}\sqrt{H_j - \xi_i}\prod_{j=i}^{n}\sqrt{\xi_i - H_j}\right]}\, d\xi. \tag{5}$$

PROOF. We start by the formula obtained in Lemma 3.4 with the paths described there.

If we assume for a moment that $\xi_1 - H_1$ is never a positive real number and replace the path Ξ_2 by the path $\Xi_2 \cup \Xi_1$ then the result remains the same. Indeed, the integration on the additional portion is zero since the integrand is skew-symmetric in ξ_1 and ξ_2. Using Cauchy's theorem, the path $\Xi_2 \cup \Xi_1$ can be replaced by any path satisfying the conditions of the lemma. For the same reason, the condition that $\xi_1 - H_1$ is never a positive real number can be relaxed (it was put there so that $H_1 - \xi_2$ would not be a negative number).

We repeat the process for $i = 3, \ldots, n-1$ using the fact that Ξ_{i-1} is a path from H_{i-1} to ∞ and that the integrand is skew-symmetric in ξ_i and ξ_{i-1}.

Finally, we make a specific choice of paths. In addition to take care of the conditions set in Lemma 3.5, the shape of the path will address the condition raised in Lemma 3.7.

LEMMA 3.6. *Let $n \geq 2$. Suppose $0 < \eta < \pi$, $0 < \epsilon < \eta$ and $H \in \Omega_{\eta,\epsilon}^+$. For a given k, $1 \leq k \leq n-1$, let $\theta_k = -\pi/4$ if $\Im(H_k - H_n) > 0$ and $\theta_k = \pi/4$ otherwise. It is possible to choose γ_k such that $\epsilon/(10n) \leq \gamma_k \leq \epsilon/2$ and such that the path Ξ_k composed of the segment from H_k to $H_k + \gamma_k e^{i\theta_k}$, followed by the horizontal line $H_k + \gamma_k e^{i\theta_k} + t$, $t \geq 0$ does not intersect any balls $D\big(H_i, \epsilon/(10n)\big)$ for $i \neq k$ (see Figure 1 below).*

PROOF. Consider the ball $D(H_k, \epsilon/2)$ with $k < n$: this ball does not intersect any balls $D\big(H_i, \epsilon/(10n)\big)$ with $i \neq k$ since $H \in \Omega^+_{\eta,\epsilon}$.

In Figure 1, the corridor between the two horizontal lines $H_k + \epsilon e^{i\theta_k}/(10n) + t$, $t \geq 0$ and $H_k + \epsilon e^{i\theta_k}/2 + t$, $t \geq 0$ has a width of $\epsilon\big(1/2 - 1/(10n)\big)\sqrt{2}/2$. There are at most $n-2$ balls $D\big(H_i, \epsilon/(10n)\big)$ that could possibly intersect with this corridor since we must have $i < k < n$. The total width of $n-2$ balls of radius $\epsilon/(10n)$ is $2(n-2)\epsilon/(10n) < \epsilon\big(1/2 - 1/(10n)\big)\sqrt{2}/2$. It is then clearly possible to draw a line $H_k + \gamma_k e^{i\theta_k} + t$, $t \geq 0$ that will not intersect any such balls.

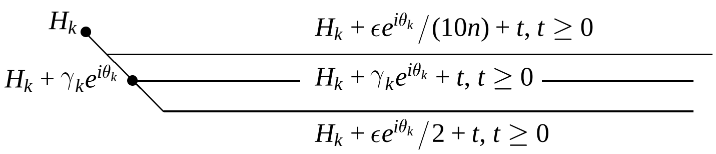

FIGURE 1: From H_k to ∞

LEMMA 3.7. *Suppose that, for each i, $\xi_i = H_i$ or $|\arg(\xi_i - H_i)| \leq \pi/4$. Then* $\Re\big(r^2(H) - r^2(\xi) - (\operatorname{tr} H - \operatorname{tr} \xi)^2\big) \leq -2\Re \sum_{j=1}^{n-1}(\xi_j - H_j)(H_j - H_n)$.

PROOF. Observe that

$$\begin{aligned}
r^2(H) \;-\; & r^2(\xi) - (\operatorname{tr} H - \operatorname{tr} \xi)^2 \\
&= -2\sum_{k=1}^{n-1}(\xi_k - H_{k+1})\Big(\sum_{j=1}^{k}(\xi_j - H_j)\Big) \\
&= -2\sum_{k=1}^{n-1}(H_k - H_{k+1})\Big(\sum_{j=1}^{k}(\xi_j - H_j)\Big) - 2\sum_{k=1}^{n-1}(\xi_k - H_k)\Big(\sum_{j=1}^{k}(\xi_j - H_j)\Big) \\
&= -2\sum_{j=1}^{n-1}(\xi_j - H_j)\Big(\sum_{k=j}^{n-1}(H_k - H_{k+1})\Big) - 2\sum_{k=1}^{n-1}(\xi_k - H_k)\Big(\sum_{j=1}^{k}(\xi_j - H_j)\Big) \\
&= -2\sum_{j=1}^{n-1}(\xi_j - H_j)(H_j - H_n) - 2\sum_{k=1}^{n-1}\sum_{j=1}^{k}(\xi_k - H_k)(\xi_j - H_j).
\end{aligned}$$

Note that if $z \neq 0$, then $\Re z \geq 0$ if and only if $|\arg z| \leq \pi/2$. The rest follows easily.

LEMMA 3.8. *Let $p(H) = \sum_{i=1}^{n} a_j H_j + b$. Then there exists a constant C independent of H and ξ (but which depends on p) such that* $|p_{\operatorname{tr} H}(\xi)| \leq C(1 + |p(H)|)\prod_{i=1}^{n-1}(1 + |\xi_i - H_i|)$.

PROOF.

$$\begin{aligned}
|p_{\operatorname{tr} H}(\xi)| &= \Big|\sum_{j=1}^{n-1} a_j \xi_j + a_n(\operatorname{tr} H - \operatorname{tr} \xi) + b\Big| \\
&= \Big|\sum_{j=1}^{n} a_j H_j + b + \sum_{j=1}^{n-1}(a_j - a_n)(\xi_j - H_j)\Big|
\end{aligned}$$

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$$
\begin{aligned}
&\le |\sum_{j=1}^{n} a_j H_j + b| + |\sum_{j=1}^{n-1} (a_j - a_n)(\xi_j - H_j)| \\
&\le (1 + |\sum_{j=1}^{n} a_j H_j + b|)\big(1 + |\sum_{j=1}^{n-1} (a_j - a_n)(\xi_j - H_j)|\big) \\
&\le C(1 + |\sum_{j=1}^{n} a_j H_j + b|) \prod_{j=1}^{n-1} (1 + |\xi_j - H_j|).
\end{aligned}
$$

We are now at the heart of the matter.

PROPOSITION 3.9. *Suppose* $0 < \eta < \pi$. *Let* p *be a skew-symmetric polynomial in* n *variables. Write* $p = p_1 \cdots p_N$ *where* $p_i(H) = \sum_{j=1}^{n} a_{ij} H_j + b_i$. *Then there exists a constant* $C_\eta > 0$ *such that for* $H \in \Omega_\eta$,

$$
|e^{r^2(H)/(4t)} G(n, pe^{-r^2/(4t)}; H)| \le C_\eta \prod_{i=1}^{N} \big(1 + |p_i(H)|\big) \prod_{i<j} \left(1 + \frac{1 + |H_i - H_j|}{t}\right)^{-1/2} \cdot e^{-\Re \sum_{i<j} (H_i - H_j)/2}.
$$

PROOF. We prove the result by induction on n. The result is clearly true for $n = 1$. Suppose $n \ge 2$ and let η be such that $0 < \eta < \pi$. Choose η' such that $\eta < \eta' < \pi$. We prove first our inequality on $\Omega^+_{\eta',\epsilon}$ where $\epsilon > 0$ is chosen so that $\eta' - 2^{n+1} n!\, \epsilon \ge \eta$ and $\eta' + \epsilon < \pi$.

We construct our paths according to Lemma 3.6. As usual ξ_i denotes a point on the path Ξ_i. The choice of θ_k will ensure that for some $K > 0$, $\Re(\xi_k - H_k)(H_k - H_n) \ge K\Re(\xi_k - H_k)|H_k - H_n|$ on the first part of the path and that $\Re(\xi_k - H_k)(H_k - H_n) \ge K|H_k - H_n|$ on the second part of the path. Also, we have $|\xi_k - H_j| \ge \epsilon/(10n)$ for $k \ne j$. Moreover, either $\xi_k = H_k$ or $|\arg(\xi_k - H_k)| \le \pi/4$. This is a condition we need to apply Lemma 3.7: $\Re\big(r^2(H) - r^2(\xi) - (\operatorname{tr} H - \operatorname{tr} \xi)^2\big)/(4t) \le -2\Re \sum_{j=1}^{n-1} (\xi_j - H_j)(H_j - H_n) \le 0$.

By the induction hypothesis, Lemma 3.3 and Lemma 3.8, there exists a constant C such that whenever $|\Im(\xi_i - \xi_j)| < \eta' + \epsilon$, a condition which is satisfied on our paths, we have

$$
\begin{aligned}
&\left|e^{r^2(H)/(4t)} G\big(n-1, (pe^{-r2/(4t)})_{\operatorname{tr} H}; \xi\big)\right| \\
&= \exp\Big(\Re\big(r^2(H) - r^2(\xi) - (\operatorname{tr} H - \operatorname{tr} \xi)^2\big)/(4t)\Big) |e^{r^2(\xi)/(4t)} G(n-1, p_{\operatorname{tr} H} e^{-r^2/(4t)}; \xi)| \\
&\le C \exp\Big(\Re\big(r^2(H) - r^2(\xi) - (\operatorname{tr} H - \operatorname{tr} \xi)^2\big)/(4t)\Big) \prod_{i=1}^{N} (1 + |(p_i)_{\operatorname{tr} H}(\xi)|) \\
&\qquad \cdot \prod_{i<j} \left(1 + \frac{1 + |\xi_i - \xi_j|}{t}\right)^{-1/2} e^{-\Re \sum_{i<j} (\xi_i - \xi_j)/2}
\end{aligned}
$$

$$
\le C \exp\Big(-\Re \sum_{j=1}^{n-1} (\xi_j - H_j)(H_j - H_n)/(2t)\Big) \prod_{i=1}^{N} \big(1 + |p_i(H)|\big) \prod_{i=1}^{n-1} (1 + |\xi_i - H_i|)^N \cdot \prod_{i<j} \left(1 + \frac{1 + |\xi_i - \xi_j|}{t}\right)^{-1/2} e^{-\Re \sum_{i<j} (\xi_i - \xi_j)/2}. \tag{6}
$$

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Repeated applications of Lemma 2.2 show that

$$(7)\qquad e^{-\Re\sum_{i<j}(\xi_i-\xi_j)/2}\Big|\prod_{i<j}\frac{\sinh(\xi_i-\xi_j)}{\xi_i-\xi_j}\prod_{i=1}^{n-1}\Big[\prod_{j=1}^{i-1}\sqrt{\frac{H_j-\xi_i}{\sinh H_j-\xi_i}}\prod_{j=i}^{n}\sqrt{\frac{\xi_i-H_j}{\sinh(\xi_i-H_j)}}\Big]\Big|$$
$$\le C\prod_{i<j}\frac{1}{1+|\xi_i-\xi_j|}\prod_{i=1}^{n-1}\Big[e^{-\Re(\xi_i-H_i)}\prod_{j=1}^{i-1}\sqrt{1+|H_j-\xi_i|}\prod_{j=i}^{n}\sqrt{1+|\xi_i-H_j|}\Big].$$

If we now use equations (5), (6), (7) and the fact that $|\xi_k-H_j|\ge\epsilon/(10n)$ whenever $j\ne k$, we see that we only need to show that

$$\int_{\Xi_{n-1}}\cdots\int_{\Xi_1}\prod_{i<j}\Big(1+\frac{1+|\xi_i-\xi_j|}{t}\Big)^{-1/2}$$
$$\cdot\prod_{i=1}^{n-1}\Big[\frac{(1+|\xi_i-H_i|)^{N+1/2}}{\sqrt{|\xi_i-H_i|}}e^{-\Re(\xi_i-H_i)}\exp\big(-\Re(\xi_i-H_i)(H_i-H_n)/(2t)\big)\Big]|d\xi|$$

is bounded by a constant multiple of $\prod_{i<j}(1+\frac{1+|H_i-H_j|}{t})^{-1/2}$ on $\Omega_{\eta',\epsilon}$ (of course, this constant will depend on ϵ and η').

Fix $k<n$. Let $a_i=H_i$ if $i<k$ and $a_i=\xi_i$ otherwise ($i\le n-1$). Let $b_i=H_i$ if $i\le k$ and $b_i=\xi_i$ otherwise ($i\le n-1$). If we suppose that the corresponding inequality has been proven for $\tilde k<k$, it suffices to show that for some constant C,

$$\int_{\Xi_k}\prod_{i<j}\Big(1+\frac{1+|a_i-a_j|}{t}\Big)^{-1/2}\frac{(1+|\xi_k-H_k|)^{N+1/2}}{\sqrt{|\xi_k-H_k|}}$$
$$\cdot e^{-\Re(\xi_k-H_k)}\exp\big(-\Re(\xi_k-H_k)(H_k-H_n)/(2t)\big)|d\xi_k|$$
$$\le C\max\Big\{\prod_{i<j}\Big(1+\frac{1+|H_i-H_j|}{t}\Big)^{-1/2},$$
$$\prod_{i<j}\Big(1+\frac{1+|b_i-b_j|}{t}\Big)^{-1/2}\cdot\Big(1+\frac{1+|H_k-H_n|}{t}\Big)^{-1/2}\Big\}$$

and to notice that for some constant C,

$$\int_{\Xi_k}\frac{(1+|\xi_k-H_k|)^{N+1/2}}{\sqrt{|\xi_k-H_k|}}e^{-\Re(\xi_k-H_k)}\exp\big(-\Re(\xi_k-H_k)(H_k-H_n)/(2t)\big)|d\xi_k|\le C.$$

Observe that for $H\in\Omega^+_{\eta',\epsilon}$ and $k\le i<j<j\le n$, $|H_k-H_n|\ge C|H_i-H_j|$ for some constant $C>0$. Therefore, given $K>0$, $e^{-K|H_k-H_n|/(2t)}\le e^{-K'\sum_{k\le i<j}|H_i-H_j|/(2t)}=\prod_{k\le i<j}e^{-K'|H_i-H_j|/(2t)}\le A\prod_{k\le i<j}(1+\frac{1+|H_i-H_j|}{t})^{-1/2}$ for some constant A.

We now divide the path Ξ_k in two parts (see Figure 1): $\gamma_{k,1}$ for the oblique segment and $\gamma_{k,2}$ for the horizontal line. Let $s_k=\Re(\xi_k-H_k)$. On the path $\gamma_{k,1}$, we use the fact that ξ_k is close to H_k and also that $\Re(\xi_k-H_k)(H_k-H_n)\ge Ks_k|H_k-H_n|$ for some $K>0$. On the path $\gamma_{k,2}$, we use the fact that s_k is not small and that $\Re(\xi_k - H_k)(H_k - H_n) \geq K|H_k - H_n|$ for some $K > 0$.

$$\int_{\gamma_{k,1}} \prod_{i<j}\left(1 + \frac{1 + |a_i - a_j|}{t}\right)^{-1/2} \frac{(1 + |\xi_k - H_k|)^{N+1/2}}{\sqrt{|\xi_k - H_k|}} \cdot e^{-\Re(\xi_k - H_k)} \exp\bigl(-\Re(\xi_k - H_k)(H_k - H_n)/(2t)\bigr)|d\xi_k|$$
$$\leq C\prod_{i<j}\left(1 + \frac{1 + |b_i - b_j|}{t}\right)^{-1/2} \int_0^\infty \frac{(1 + s_k)^{N+1/2}}{\sqrt{s_k}} e^{-s_k} e^{-Ks_k|H_k - H_n|/(2t)}\, ds_k$$
$$\leq C'\prod_{i<j}\left(1 + \frac{1 + |b_i - b_j|}{t}\right)^{-1/2}\left(1 + \frac{1 + |H_k - H_n|}{t}\right)^{-1/2}.$$

Finally,

$$\int_{\gamma_{k,2}} \prod_{i<j}\left(1 + \frac{1 + |a_i - a_j|}{t}\right)^{-1/2} \frac{(1 + |\xi_k - H_k|)^{N+1/2}}{\sqrt{|\xi_k - H_k|}} e^{-\Re(\xi_k - H_k)} \cdot \exp\bigl(-\Re(\xi_k - H_k)(H_k - H_n)/(2t)\bigr)|d\xi_k|$$
$$\leq C\prod_{i<j<k}\left(1 + \frac{1 + |H_i - H_j|}{t}\right)^{-1/2} \int_0^\infty \frac{(1 + s_k)^{N+1/2}}{\sqrt{s_k}} e^{-s_k} e^{-K|H_k - H_n|/(2t)}\, ds_k$$
$$\leq C'\prod_{i<j}\left(1 + \frac{1 + |H_i - H_j|}{t}\right)^{-1/2}.$$

This concludes the proof for $H \in \Omega^+_{\eta',\epsilon}$.

We can extend the bound to $\Omega_{\eta',\epsilon}$ using the symmetry and continuity of $e^{r^2(H)/(4t)}G(n, pe^{-r^2/(4t)}; H)$.

Since, $\eta \leq \eta' - 2^{n+1}n!\,\epsilon$, if $H \in \Omega_\eta$ then Proposition 2.5 allows us to find s_i, $1 \leq i \leq n$, such that $0 < s_i \leq 2^n n!\,\epsilon$ and $\prod_{i=1}^n S(H_i, s_i) \subset \Omega_{\eta',\epsilon}$. By the maximum modulus principle (Proposition 2.6), this means that the maximum of $|e^{r^2(H)/(4t)}G(n, pe^{-r^2/(4t)}; H)|$ on $\prod_{i=1}^n \overline{D}(H_i, s_i)$ is attained on $\prod_{i=1}^n S(H_i, s_i)$. The desired inequality is then valid for every element of Ω_η with possibly a larger constant (the size of that constant is limited by the fact that $s_i \leq 2^n n!\,\epsilon$ for each i).

THEOREM 3.10. *Anker's conjecture, the upper bound given in (2), is valid for the space* $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$.

PROOF. According to the corollary to Theorem 1.2 we have $P_t(e^H) = Ce^{-\gamma^2 t}t^{-n/2-n(n-1)/2}G(n, pe^{-r^2/(4t)}; H)$ with $p(H) = \prod_{i<j}(H_i - H_j)$ for some constant C. Moreover, the positive roots are $H_i - H_j$ with $i < j$ and their multiplicity is 1. Since $\mathbf{a}^+ \subset \Omega_\eta$ whenever $0 < \eta < \pi$, it suffices to apply Proposition 3.9 with any choice of η with $0 < \eta < \pi$.

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4. **Conclusion.** In addition to confirming the validity of Anker's conjecture for the space $\mathrm{SL}(n, \mathbf{R}) / \mathrm{SO}(n)$, Lemma 3.9 can be used to provide upper bounds for the derivatives of the heat kernel by invariant differential operators. With the paper [8], we can now say that Anker's conjecture on the upper bound has been verified for all the symmetric spaces of noncompact type corresponding to the root system A_{n-1}.

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