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Covers of moduli surfaces

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Abstract

Which algebraic surfaces occur as covers of moduli spaces for curves? We show that surfaces of a special type over number fields do, with possible exception for non-elliptic K3 surfaces.

1. Introduction

Unramified covers of the moduli spaces for smooth curves are defined over number fields by a result of Weil [Wei56]. Conversely, is every quasi-projective variety over $\bar{\mathbb{Q}}$ birational to a cover of a moduli space for curves? In the one-dimensional case this question has been answered affirmatively by Belyi [Bel79]. His algorithm produces a finite morphism from a given smooth projective curve over $\bar{\mathbb{Q}}$ to the moduli space $\bar{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, unramified over the interior $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. The morphism can be arranged to map a given finite subset of the curve to ∞.

Grothendieck [Gro97] observed that results like this can give rise to a nice elementary description of algebraic varieties over number fields: If a cell decomposition of the moduli space is fixed, then covers are given by combinatorial data, represented by *dessins d'enfants* in the case of curves covering $\mathcal{M}_{0,4}$.

2. Two-dimensional moduli spaces

Let $\mathcal{M}_{g,n}$ be the stack of smooth n-pointed genus g projective curves and $\bar{\mathcal{M}}_{g,n}$ its completion as described by Knudsen [Knu83]. Note that the n marked points are ordered. The quotient by the obvious action of the symmetric group S_n is the stack $\mathcal{M}_{g,[n]}$ of curves with a distinguished n-element subset.

The map $\mathcal{M}_{g,n} \to \mathcal{M}_{g,[n]}$ is an unramified cover. Further covers tend to arise as moduli spaces for curves with various extra structures, such as level structures [BP00] and Hurwitz spaces.

We are working in characteristic 0, by default over the algebraic numbers. However, by Weil's result mentioned above, it makes no difference to allow complex coefficients for covers. A cover of $\mathcal{M}_{g,[n]}$ is then the same as a quotient $\mathcal{T}_{g,n}/H$ of Teichmüller space $\mathcal{T}_{g,n}$ by a cofinite subgroup H of its holomorphic automorphism group, at least if $2g + n \geqslant 5$.

The dimension of $\mathcal{M}_{g,n}$ is $3g - 3 + n$. In dimension 2 we have $\mathcal{M}_{0,5}$ and $\mathcal{M}_{1,2}$. In contrast to higher dimensions, the two-dimensional Teichmüller spaces $\mathcal{T}_{0,5}$ and $\mathcal{T}_{1,2}$ are isomorphic. In the tower of covers

$$\mathcal{T}_{1,2} = \mathcal{T}_{0,5} \to \mathcal{M}_{0,5} \to \mathcal{M}_{1,[2]} \to \mathcal{M}_{0,[5]},$$

the map from $\mathcal{M}_{0,5}$ to $\mathcal{M}_{1,[2]}$ associates to a five-pointed line its double cover E branched in the first four marked points and marked with the preimages $p, \bar{p}$ of the fifth marked point in the line; there are ways of setting up a morphism of stacks from this. The map from $\mathcal{M}_{1,[2]}$ to $\mathcal{M}_{0,[5]}$ associates

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to a marked curve $(E, \{p, \bar{p}\})$ its quotient by the involution $t \mapsto \bar{t}$ with $\bar{t} = p + \bar{p} - t$ in $\mathrm{Pic}^1(E)$. This quotient is a line marked by the branch points and the image of p and $\bar{p}$.

Let $P = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let Q be the complement of the diagonal in $P \times P$. To $(p, p') \in Q$ we associate the five-pointed curve $(\mathbb{P}^1; 0, 1, \infty, p, p')$. This gives $\mathcal{M}_{0,5} \cong Q$ endowed with a universal family of five-pointed lines. We will identify $\mathcal{M}_{0,5}$ with its fine moduli space Q. The corresponding model of $\bar{\mathcal{M}}_{0,5}$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in the three points $(0, 0)$, $(1, 1)$ and (∞, ∞). The complement of $\mathcal{M}_{0,5} = Q$ in this surface is the union of ten smooth rational curves of self-intersection -1.

3. Bundles and abelian surfaces

We are now going to realize some classes of surfaces over $\bar{\mathbb{Q}}$ as covers of $Q = \mathcal{M}_{0,5}$ up to birational equivalence.

Product surfaces $C \times C'$ cover $\mathbb{P}^1 \times \mathbb{P}^1$ without ramification over Q: Just put together Belyi morphisms (cf. § 1) for the two factors. Because we consider surfaces up to birational equivalence, this technique applies to all ruled surfaces.

Proposition 1. *Any fibre bundle (for the étale topology) of fibre genus $g \geqslant 2$ is birational to a cover of $\mathcal{M}_{0,5}$.*

Proof. Let $X \to C$ be a bundle with fibre F. Let $\bar{F} := F/\mathrm{Aut}(F)$ and let $V \subset \bar{F}$ be the branch locus of $F \to \bar{F}$. The natural morphism $X \to \bar{F} \times C$ is branched precisely over $V \times C$. The desired morphism $X \to \mathbb{P}^1 \times \mathbb{P}^1$ arises from Belyi morphisms for the curves C and $\bar{F}$, the latter chosen such that V be mapped to $\{0, 1, \infty\}$. □

Proposition 2. *Any abelian surface is birational to a cover of $\mathcal{M}_{0,[5]}$.*

Proof. If the abelian surface A is isogenous to the product of two elliptic curves then we are done. Otherwise A is the Jacobian of a curve C of genus two. Then A is birationally equivalent to the symmetric square $\mathrm{Sym}^2 C = (C \times C)/S_2$ of the curve. From a Belyi morphism β: $C \to \mathbb{P}^1$ we get a map $\mathrm{Sym}^2 C \to \mathrm{Sym}^2 \mathbb{P}^1$ unramified over the image of Q.

Consider the subgroup $S_2 \subset S_5$ acting on $\mathcal{M}_{0,5}$ by interchanging the last two marked points. This corresponds to the involution $(p, p') \mapsto (p', p)$ on Q. Since it acts freely, the quotient scheme Q/S_2, which is the image of Q in $\mathrm{Sym}^2 \mathbb{P}^1$, represents the quotient stack Q/S_2, which covers $\mathcal{M}_{0,[5]}$. □

4. A criterion for fibred surfaces

It would be nice to have variants of Belyi's algorithm [Bel79, Bel02] which work over function fields, because then we could apply the following criterion.

Theorem 1. *Let X be a smooth irreducible surface over $\bar{\mathbb{Q}}$ and $X \to C$ a morphism with connected fibres onto a curve C. Let $K = K(C)$ be the function field of the curve. If the generic fibre X_K admits a morphism $X_K \to \mathbb{P}^1_K$ branched at most over four K-rational points, then there is a finite cover of $\mathcal{M}_{0,5}$ birationally equivalent to X.*

Proof. The generic fibre X_K is smooth and geometrically integral. By assumption we have g_K: $X_K \to \mathbb{P}^1_K$ branched over four K-rational points. Call them 0, 1, ∞ and t.

We may consider t as a morphism $C \to \mathbb{P}^1$ because a rational map on a smooth curve is defined everywhere. If the map is constant, i.e. $t \in \bar{\mathbb{Q}}$, then X is birationally a cover of $\mathbb{P}^1 \times C$ branched over horizontal and vertical fibres, and we may proceed as in Proposition 1. Assume now that t is transcendental over $\bar{\mathbb{Q}}$.

Over some open part $C_0 \subset C$ the fibration $X_0 := X \times_C C_0$ is smooth, g_K extends to a morphism $g_0\colon X_0 \to \mathbb{P}^1$ and the restriction of g_0 to any fibre $X(c)$, $c \in C_0$, is finite and ramifies at most over 0, 1, ∞ and $t(c)$. Blow up X until g_0 extends to a morphism $g\colon X \to \mathbb{P}^1$. The exceptional locus of the blow-up will lie over $C \smallsetminus C_0$.

Belyi provides a morphism $b\colon \mathbb{P}^1 \to \mathbb{P}^1$ which maps not only its own ramification points but also the points 0, 1 and ∞, the branch points of t and the image $t(C \smallsetminus C_0)$ to $\{0, 1, \infty\}$. Define a map $h = (h_1, h_2)\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ by

$$h_1 = b \circ t \circ (X \to C), \quad h_2 = b \circ g.$$

Denote by $X' \subset X$ the preimage of the open part $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$ from § 2. We claim that the restriction $h'\colon X' \to Q$ of h is finite and étale.

It is proper because h is. Fix a point $x \in X'$. It projects to a point $c \in C_0$ because otherwise $h_1(x) = b(t(c)) \in \{0, 1, \infty\}$. So the restriction of h_2 to the fibre $X'(c)$ is quasi-finite. However, then h' cannot blow down a curve through x because the fibres of h_1 are unions of fibres over C.

If h' is not étale at x then either $b \circ t$ ramifies at c or $h_2|X'(c)$ ramifies at x. In the first case we have $h_1(x) \in \{0, 1, \infty\}$ by the choice of b. In the latter case we distinguish two subcases.

a) b is ramified at $g(x)$. Then $h_2(x) \in \{0, 1, \infty\}$.

b) $g|X'(c)$ ramifies at x. Then $g(x) = 0$, 1, ∞ or $t(c)$; thus $h_2(x) = 0$, 1, ∞ or $b(t(c))$. In the last case $h(x)$ is on the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. □

5. Elliptic fibrations

The surface X in this section fibres over a curve C and the general fibre is a smooth curve of genus 1. Denote by $K := K(C)$ the function field (or generic point) of the base curve. So the generic fibre $X_K = X \times_C K$ is a smooth and geometrically irreducible curve of genus 1 over K.

Theorem 2. *Any elliptic fibration is birational to a cover of $\mathcal{M}_{0,5}$.*

Proof. We want to construct a morphism $X_K \to \mathbb{P}^1_K$ as required by Theorem 1. As X_K is an unramified cover of its Picard curve $\operatorname{Pic}^0 X_K$ by means of any polarization, we are reduced to the case that X_K is an elliptic curve with zero section $o \in X(K)$. Let $q\colon X_K \to \mathbb{P}^1_K$ be the quotient by group inversion.

The ramification locus of q is defined over K as it is the kernel of $x \mapsto x + x$. So the branch locus of q is a K-subvariety of $\mathbb{P}^1$. Over an algebraic closure $\bar{K}$ of K it consists of four points $q(o)$, z_1, z_2, z_3.

The branch points z_1, z_2, z_3 are permuted by a subgroup H of $\operatorname{Aut}(\mathbb{P}^1)$. Over $\bar{K}$ this subgroup is isomorphic to the symmetric group S_3. Its elements might not be K-rational; though, the set of the three involutions, the set of the two 3-cycles and the subgroup scheme H are defined over K.

Let $q_2 : \mathbb{P}^1_K \to \mathbb{P}^1_K$ the quotient by H. Its ramification locus over $\bar{K}$ splits into three H-orbits:

- the fixed points of the 3-cycles are conjugate by any of the involutions;
- the z_i are conjugate by definition of H;
- the other three fixed points of the involutions are then conjugate, too.

Every orbit is defined over K. So their images by q_2 are K-rational. These are the branch points of q_2. Call them 0, ∞ and 1728. The point $q(o)$ is then mapped to the j-invariant of X_K.

The composite map $q_2 \circ q\colon X_K \to \mathbb{P}^1_K$ ramifies precisely over the K-rational points 0, 1728, $j(X_K)$ and ∞. □

Remark 1. At this point we have realized all surfaces of Kodaira dimension 1 as well as all Enriques and hyperelliptic surfaces. Additionally there are K3 surfaces which admit elliptic fibrations. We are left with non-elliptic K3 and surfaces of general type.

Remark 2. We considered $\mathcal{M}_{0,5}$ as a subset of $\mathbb{P}^1 \times \mathbb{P}^1$. Alternatively $\mathcal{M}_{0,5}$ can be embedded into the projective plane. It is isomorphic to the complement of a configuration of lines known as the complete square: the union of the six lines connecting four points in general position. Special covers of this representative of $\mathcal{M}_{0,5}$ have been extensively studied, see e.g. [BHH87]. Thus a wide variety of examples is known.

Remark 3. Most of our constructions realize an open part of a given complete surface X as a cover of $\mathcal{M}_{0,[5]}$. To see that we cannot always realize X itself as a cover of $\bar{\mathcal{M}}_{0,[5]}$, note that if X is simply connected then every morphism $X \to \bar{\mathcal{M}}_{0,[5]}$ lifts to $X \to \bar{\mathcal{M}}_{0,5}$. On the other hand a surjective morphism $X \to \bar{\mathcal{M}}_{0,5}$ can only exist if X contains curves with negative self-intersection. It follows that $\mathbb{P}^2$ is not a cover of $\bar{\mathcal{M}}_{0,[5]}$.

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