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QUADRATIC OPERATORS DO NOT GENERATE MAXIMAL LEFT IDEALS OF THE WEYL ALGEBRA

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Abstract. We prove that if $n \geq 2$ then $A_n d$ cannot be a maximal left ideal of the n-th Weyl algebra A_n over $\mathbb{C}$ for almost every operator d of degree 2 of A_n.

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In Section 5.2 of [**1**], Bernstein and Lunts claim that an operator of A_n of degree 2 (with respect to the Bernstein filtration) cannot generate a maximal left ideal of A_n. For the sake of simplicity they give a proof under the assumption that the operator of degree 2 has a generic principal symbol. Moreover, their proof is more of a sketch than a complete proof, and it is not really elementary. In this note we give a completely elementary proof of a result that includes the generic result proved by Bernstein and Lunts. For unexplained terminology and for more information on the Weyl algebra see [**2**].

Throughout this note we will denote by B_k the k-th component of the Bernstein filtration of the complex Weyl algebra A_n. Recall that A_n is generated over $\mathbb{C}$ by n commuting variables $x_1, \ldots, x_n$ and their partial differential operators, denoted respectively by $\partial_1, \ldots, \partial_n$. We will say that an element of A_n is a *homogeneous operator of degree* $r \geq 1$ if it can be written as a linear combination of standard monomials $x^\alpha \partial^\beta$ for which $|\alpha| + |\beta| = r$. The principal symbol of $d \in A_n$ with respect to the Bernstein filtration will be denoted by $\sigma(d)$. Thus, setting $\sigma(x_i) = y_i$ and $\sigma(\partial_i) = \xi_i$ we have $\sigma(x^\alpha \partial^\beta) = y^\alpha \xi^\beta$. Therefore, by assuming that the monomials of a symbol are always to be written in the form $y^\alpha \xi^\beta$, we conclude that a homogeneous operator is uniquely determined by its symbol.

Let V_1 be the complex vector space with basis $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_n$. One easily checks that

$$[x^\alpha \partial^\beta, x_i] = \beta_i x^\alpha \partial^{\beta - e_i} \quad \text{and} \quad [x^\alpha \partial^\beta, \partial_i] = -\alpha_i x^{\alpha - e_i} \partial^\beta,$$

where e_i is the n-vector with 1 in the i-th entry and zeros elsewhere. Thus a homogeneous operator P of degree 2 induces a vector space endomorphism ad_d of V_1 by mapping $u \in V_1$ to $[P, u]$. Moreover if P has degree 1, then the same rule determines an element of the dual $V_1^\star$.

The symbols of operators of degree 2 in n variables are quadratic polynomials in $2n$ variables and are thus parametrized by $\mathbb{C}^N$, where $N = (2n+1)(n+1)$. Let $\mathcal{O}_n$ be

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the set of points in $\mathbb{C}^N$ that correspond to polynomials p that satisfy both of the following conditions:

(1) p is irreducible;
(2) if P is the homogeneous operator of degree 2 whose symbol is p then $\mathrm{ad}_P^n \neq 0$.

One immediately checks that $\mathcal{O}_n$ is a Zariski open set of $\mathbb{C}^N$. Moreover if $n \geq 2$, then $\mathcal{O}_n$ is not empty because the symbol of $P = x_1\partial_1 + x_2\partial_2$ is irreducible, and $[P, x_1] = x_1$. In particular, conditions (1) and (2) hold 'generically', in the sense of [**1**], when $n \geq 2$.

Theorem. *If d is an operator of degree 2 of A_n whose symbol corresponds to a point in $\mathcal{O}_n$, then $A_n d$ is not a maximal left ideal of A_n.*

Proof. We may write d in the form $d = d_2 + d_1 + d_0$, where d_i is a homogeneous operator of degree i. Since the vector space endomorphism ad_{d_2} is not nilpotent by condition (2), it follows that it must have a non-zero eigenvalue $\lambda \in \mathbb{C}$. Thus there must exist $0 \neq v \in V_1$, an operator of degree 1, such that $[d_2, v] = \lambda v$. Choose $c \in \mathbb{C}$ such that $[d_1, v] = c\lambda$, and write $u = v + c$. A straightforward computation shows that $[d, u] = \lambda u$. We claim that $A_n d + A_n u$ is a proper left ideal of A_n. Since this ideal contains $A_n d$ properly, it follows that $A_n d$ is not maximal.

Suppose, by contradiction, that $A_n d + A_n u$ is not a proper ideal of A_n; in other words $A_n d + A_n u = A_n$. Choose $a, b \in A_n$ such that $ad + bu = 1$, with a of the smallest possible degree. Note that neither a nor b can be zero. Taking principal symbols we have that
$$\sigma(a)\sigma(d) + \sigma(b)\sigma(u) = 0.$$

Since $\sigma(u)$ has degree 1 and $\sigma(d)$ is irreducible by condition (1), we conclude that there exist $\alpha, a', b' \in A_n$, $\alpha \neq 0$, such that $a = -\alpha u + a'$ and $b = \alpha d + b'$, where a' has smaller degree than a, and b' has smaller degree than b. Hence

$$1 = (-\alpha u + a')d + (\alpha d + b')u,$$

so that

$$1 = \alpha[d, u] + (a'd + b'u).$$

Since $[d, u] = \lambda u$ we end up with

$$1 = a'd + (b' + \lambda\alpha)u,$$

which contradicts the minimality of the degree of a. Therefore $A_n d + A_n u \neq A_n$, and the proof is complete.

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