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Almost sure convergence of the L^4 norm of Littlewood polynomials

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Abstract. This paper concerns the L^4 norm of Littlewood polynomials on the unit circle which are given by

$$q_n(z) = \sum_{k=0}^{n-1} \pm z^k;$$

i.e., they have random coefficients in $\{-1, 1\}$. Let

$$||q_n||_4^4 = \frac{1}{2\pi} \int_0^{2\pi} |q_n(e^{i\theta})|^4 d\theta.$$

We show that $||q_n||_4/\sqrt{n} \to \sqrt[4]{2}$ almost surely as $n \to \infty$. This improves a result of Borwein and Lockhart (2001, *Proceedings of the American Mathematical Society* 129, 1463–1472), who proved the corresponding convergence in probability. Computer-generated numerical evidence for the a.s. convergence has been provided by Robinson (1997, *Polynomials with plus or minus one coefficients: growth properties on the unit circle*, M.Sc. thesis, Simon Fraser University). We indeed present two proofs of the main result. The second proof extends to cases where we only need to assume a fourth moment condition.

1 Introduction

The study of Littlewood polynomials enjoys a long and outstanding history [22]. A polynomial with all coefficients in $\{-1, 1\}$ is called a Littlewood polynomial, and we denote by $\mathcal{L}_n$ the family of Littlewood polynomials of degree $n-1$; that is,

$$\mathcal{L}_n := \left\{ q_n : q_n(z) = \sum_{k=0}^{n-1} \varepsilon_k z^k, \quad \varepsilon_k \in \{-1, 1\} \right\}.$$

The L^p norm of Littlewood polynomials on the unit circle has been a fascinating and classical subject of many studies over the last 60 years [7, 8, 10, 11, 20, 21, 23, 24]. Here, the L^p norm $(1 \le p < \infty)$ of $q_n \in \mathcal{L}_n$ on the unit circle is given by

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$$\|q_n\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} |q_n(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}},$$

while $\|q_n\|_\infty$ is the supremum of $|q_n(z)|$ on the unit circle. The L^1, L^4, and L^∞ norms hold special significance in the field of analysis.

Littlewood [20, Section 6] asked how slowly the L^4 norm of $q_n \in \mathcal{L}_n$ can grow with n. Subsequently, the L^4 norm has been studied extensively, which is of interests in communication theory [5, 6, 12, 14, 15, 17]. There are two natural measures which are often used in the investigation of how small the L^4 norm can be. The first one is the ratio $\|q_n\|_4/\|q_n\|_2$; that is, $\|q_n\|_4/\sqrt{n}$. The other is the merit factor

$$\text{MF}(q_n) := \frac{\|q_n\|_2^4}{\|q_n\|_4^4 - \|q_n\|_2^4} = \frac{n^2}{\|q_n\|_4^4 - n^2}, \tag{1}$$

which is considered in the context of the theory of communications [3], where Littlewood polynomials with large merit factor correspond to signals with uniformly distributed energy over frequency. Obviously, a small L^4 norm corresponds to a large merit factor. Moreover, establishing the minimum achievable L^4 norm for Littlewood polynomials has demonstrated substantial importance in theoretical physics [4], where Littlewood polynomials with the largest merit factor correspond to the ground states of Bernasconi's Ising spin model.

Littlewood's question has sparked much interest among mathematicians and physicists (see [15] for a survey of relevant results and historical developments). In 1968, Littlewood [21] constructed a sequence of polynomials, of Rudin–Shapiro type, with asymptotic merit factor 3; that is, the ratio $\|q_n\|_4/\sqrt{n}$ is asymptotically $\sqrt[4]{4/3}$. In 1988, Hødoldt and Jensen [14], working in information theory, showed that this ratio for a sequence of Littlewood polynomials derived from Fekete polynomials is asymptotically $\sqrt[4]{7/6}$, correspondingly, with the asymptotic merit factor 6. They conjectured that the asymptotic value of this ratio cannot be further reduced, and in fact, $\sqrt[4]{7/6}$ has remained the smallest published asymptotic value for more than two decades. In 2013, Jedwab, Katz, and Schmidt [16] made a major discovery indicating that this is not the minimum asymptotic value, thus disproving the abovementioned conjecture. They proved that there exists a sequence of Littlewood polynomials, derived from Fekete polynomials, with the limit of the ratio less than $\sqrt[4]{22/19}$.

The roots of all these explorations actually go back to the original Littlewood's conjecture [20] from 1966 which states that, for all $n \geq 1$, there exists a polynomial $q_n \in \mathcal{L}_n$ such that

$$C_1\sqrt{n} \leq |q_n(z)| \leq C_2\sqrt{n} \tag{2}$$

for all complex z of modulus 1, where C_1 and C_2 are positive absolute constants. Polynomials satisfying (2) are known as flat polynomials. This conjecture was discussed in detail in his well-known 1968 monograph [21, Problem 19], in which he laid out his 30 favorite problems. A significant result by Kahane [18, 19] establishes that there exist ultra-flat polynomials with coefficients of modulus 1, namely, polynomials that satisfy

$$|q_n(z)| = (1 + o(1))\sqrt{n}$$

for all z with $|z| = 1$. Subsequently, an interesting result, due to Beck [2], states that flat polynomials exist with coefficients being 400th roots of unity. However, for the more restrictive class of Littlewood polynomials, much less progress has been made over the next few decades until 2020. Then Littlewood's conjecture was famously confirmed by Balister, Bollobás, Morris, Sahasrabudhe, and Tiba [1], who showed that flat Littlewood polynomials exist and answered the question of Erdős [10, Problem 26].

Newman and Byrnes [24] calculated the expected L^4 norm of $q_n \in \mathcal{L}_n$, specifically,

$$\mathbb{E}(||q_n||_4^4) = 2n^2 - n,$$

and therefore the expected merit factor is 1. Borwein and Lockhart [7] showed that the ratio $||q_n||_4/\sqrt{n}$ converges to $\sqrt[4]{2}$ in probability for $q_n \in \mathcal{L}_n$, equivalently, $\mathrm{MF}(q_n) \to 1$ in probability. In [8], Borwein and Mossinghoff explicitly calculated the L^4 norm of Rudin–Shapiro-like polynomials. In [26], Salem and Zygmund showed that the supremum of Littlewood polynomials on the unit disk lies between $c_1\sqrt{n\log n}$ and $c_2\sqrt{n\log n}$. Indeed, Halász [13] proved that $\lim ||q_n/\sqrt{n\log n}||_\infty = 1$ almost surely.

The aim of this paper is to prove the almost sure convergence of L^4 norm of Littlewood polynomials. Our main result is the following.

Theorem 1 *Let $q_n \in \mathcal{L}_n$. Then*

$$\frac{||q_n||_4}{\sqrt{n}} \to \sqrt[4]{2}$$

almost surely.

This result has been conjectured by Borwein and Lockhart and numerical simulations were performed by Robinson in her master's thesis [25], confirming the rationality of this conjecture and consistent with our theoretical result. An extension of Theorem 1 to a more general case is included in Section 4, where only a fourth moment condition is assumed.

2 Preliminary results

In this section, we present several lemmas before we prove Theorem 1. We let $A_n^m = n!/(n-m)!$ and the combinatorial number of selecting m elements from n distinct elements is denoted by $\binom{n}{m} = n!/(n-m)!m!$. To ensure convenience in calculations, we will adjust the indices from "0 to $n-1$" to "1 to n" at a few places, given that the L^4 norms of $\sum_{k=0}^{n-1} \varepsilon_k z^k$ and $\sum_{k=1}^{n} \varepsilon_k z^k$ on the unit circle are the same.

Lemma 2 *Let $q_n \in \mathcal{L}_n$. The following assertions hold:*

(i) *If n is even, then*

$$\mathbb{E}(||q_n||_4^8) = 4n^4 + \frac{4}{3}n^3 - 19n^2 + \frac{56}{3}n.$$

(ii) *If n is odd, then*

$$\mathbb{E}(||q_n||_4^8) = 4n^4 + \frac{4}{3}n^3 - 19n^2 + \frac{56}{3}n - 4.$$

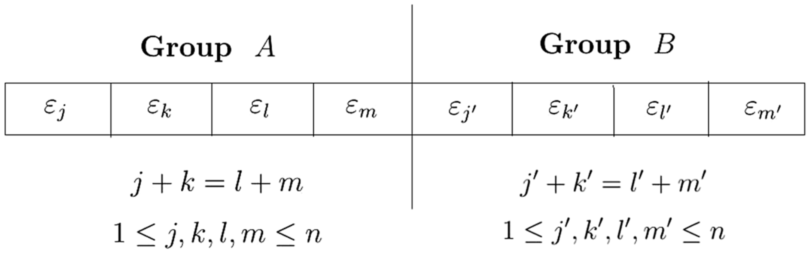

Figure 1: Grouping of parameters for (i) and (ii).

(iii) *If n is even, then*

$$\mathbb{E}(\|q_{n+1}\|_4^4 \cdot \|q_n\|_4^4) = 4n^4 + \frac{28}{3}n^3 - 21n^2 + \frac{53}{3}n.$$

(iv) *If n is odd, then*

$$\mathbb{E}(\|q_{n+1}\|_4^4 \cdot \|q_n\|_4^4) = 4n^4 + \frac{28}{3}n^3 - 21n^2 + \frac{53}{3}n - 4.$$

In particular, the values in (i)–(iv) are positive integers.

Proof By observation,

$$\|q_n\|_4^4 = \sum_{\substack{j+k=l+m \\ 1\le j,k,l,m\le n}} \varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m. \tag{3}$$

Therefore, we obtain

$$\mathbb{E}\big(\|q_n\|_4^8\big) = \mathbb{E}\left(\sum_{\substack{j+k=l+m \\ j'+k'=l'+m' \\ 1\le j,k,l,m,j',k',l',m'\le n}} \varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m\varepsilon_{j'}\varepsilon_{k'}\varepsilon_{l'}\varepsilon_{m'} \right), \tag{4}$$

and

$$\mathbb{E}(\|q_{n+1}\|_4^4 \cdot \|q_n\|_4^4) = \mathbb{E}\left(\sum_{\substack{j+k=l+m \\ j'+k'=l'+m' \\ 1\le j,k,l,m\le n+1 \\ 1\le j',k',l',m'\le n}} \varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m\varepsilon_{j'}\varepsilon_{k'}\varepsilon_{l'}\varepsilon_{m'} \right). \tag{5}$$

Our goal is to count the number of terms with nonvanishing expectation, namely, we concern the terms such that $\mathbb{E}\big(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m\varepsilon_{j'}\varepsilon_{k'}\varepsilon_{l'}\varepsilon_{m'}\big) = 1$.

For (i), as shown in Figure 1, we divide the eight parameters into two groups. The proof proceeds by considering three cases according to the choice of the subscripts j,k,l,m,j',k',l',m'.

Case 1 Each group has two equal pairs of subscripts.

Table 1: Subcases of Case 1 for (i) and (ii).

Subcase	Quantity
$(j=l)\neq(k=m)$ and $(j'=l')\neq(k'=m')$	$A_n^2\cdot A_n^2$
$(j=l)\neq(k=m)$ and $(j'=m')\neq(k'=l')$	$A_n^2\cdot A_n^2$
$(j=l)\neq(k=m)$ and $j'=k'=l'=m'$	$A_n^2\cdot n$
$(j=m)\neq(k=l)$ and $(j'=l')\neq(k'=m')$	$A_n^2\cdot A_n^2$
$(j=m)\neq(k=l)$ and $(j'=m')\neq(k'=l')$	$A_n^2\cdot A_n^2$
$(j=m)\neq(k=l)$ and $j'=k'=l'=m'$	$A_n^2\cdot n$
$j=k=l=m$ and $(j'=l')\neq(k'=m')$	$n\cdot A_n^2$
$j=k=l=m$ and $(j'=m')\neq(k'=l')$	$n\cdot A_n^2$
$j=k=l=m$ and $j'=k'=l'=m'$	$n\cdot n$

We take the Group A as an example. There are three options for subscripts j,k,l,m:

$$(j=l)\neq(k=m), \tag{6}$$

$$(j=m)\neq(k=l), \tag{7}$$

$$j=k=l=m. \tag{8}$$

The three types above correspond to

$$\varepsilon_j\varepsilon_k\varepsilon_j\varepsilon_k,\quad \varepsilon_j\varepsilon_k\varepsilon_k\varepsilon_j,\quad \varepsilon_j\varepsilon_j\varepsilon_j\varepsilon_j,$$

respectively. A similar approach works for Group B. Thus, there are nine different subcases in Case 1 (see Table 1 for the quantity corresponding to each subcase).

Consequently, the sum of these quantities is $4n^4-4n^3+n^2$. Incidentally, Case 1 is the most numerous.

Case 2 Each group has only one pair of equal subscripts.

Since we are concerned with the terms such that $\mathbb{E}\big(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m\varepsilon_{j'}\varepsilon_{k'}\varepsilon_{l'}\varepsilon_{m'}\big)=1$, if one of the Groups A and B has only one pair of equal subscripts, then so does the other group.

Let us take the subcase $j\neq k,\ l=m$ in Group A as an example. Under this assumption, it is worth noting that both j and k are either odd or even. Hence, if j and k have been determined, then there must be $l=m=\frac{j+k}{2}$. Now, correspondingly, Group B has the following options:

$$\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_l,\quad \varepsilon_k\varepsilon_j\varepsilon_l\varepsilon_l,\quad \varepsilon_l\varepsilon_l\varepsilon_j\varepsilon_k,\quad \varepsilon_l\varepsilon_l\varepsilon_k\varepsilon_j.$$

Bearing in mind that n is even, whether j and k are both odd or even, the corresponding quantity is $A_{\frac{n}{2}}^2\cdot 4=n^2-2n$. Hence, for such a subcase, the quantity of options is $2n^2-4n$. Similarly, in the subcase $j=k,\ \ l\neq m$, the quantity is also $2n^2-4n$. Consequently, the quantity of options in Case 2 is $4n^2-8n$.

Case 3 Each group has no pair of equal subscripts.

In this case, we suppose that $j \neq k \neq l \neq m$ for Group A. Recall that $j + k = l + m$, $1 \leq j, k, l, m \leq n$. Let us take $\varepsilon_1\varepsilon_5\varepsilon_2\varepsilon_4$ as a simple example. Since we want $\mathbb{E}\left(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m\varepsilon_{j'}\varepsilon_{k'}\varepsilon_{l'}\varepsilon_{m'}\right) = 1$, if $\varepsilon_1\varepsilon_5\varepsilon_2\varepsilon_4$ appears in Group A, then, correspondingly, one of the following must appear in Group B:

$$\begin{aligned} &\varepsilon_1\varepsilon_5\varepsilon_2\varepsilon_4, \ \ \varepsilon_1\varepsilon_5\varepsilon_4\varepsilon_2, \ \ \varepsilon_5\varepsilon_1\varepsilon_2\varepsilon_4, \ \ \varepsilon_5\varepsilon_1\varepsilon_4\varepsilon_2, \\ &\varepsilon_2\varepsilon_4\varepsilon_1\varepsilon_5, \ \ \varepsilon_2\varepsilon_4\varepsilon_5\varepsilon_1, \ \ \varepsilon_4\varepsilon_2\varepsilon_1\varepsilon_5, \ \ \varepsilon_4\varepsilon_2\varepsilon_5\varepsilon_1. \end{aligned} \tag{9}$$

Therefore, it suffices to consider Group A, and for every choice in Group A, there are eight choices in Group B correspondingly.

Now the problem naturally transforms into considering how many options there are for j, k, l, m to satisfy the following conditions:

$$j + k = l + m, \quad j \neq k \neq l \neq m, \quad 1 \leq j, k, l, m \leq n.$$

For example, we observe that this set of subscripts $\{1, 2, 4, 5\}$ will appear eight times in Group A by adjusting the order reasonably, as shown in (9). For brevity, we first consider the non-repeating case, regardless of the order. For any even number n, the quantity of options in non-repeating case is

$$4\binom{2}{2} + 4\binom{3}{2} + 4\binom{4}{2} + \cdots + 4\binom{\frac{n}{2}-1}{2} + \binom{\frac{n}{2}}{2} = \frac{1}{12}n^3 - \frac{3}{8}n^2 + \frac{5}{12}n.$$

As was mentioned earlier, each set of subscripts has eight different kinds of ordering. Therefore, the quantity of options for Group A is $\frac{2}{3}n^3 - 3n^2 + \frac{10}{3}n$. Combined with the previous analysis for Group B, we obtain that the quantity of options in Case 3 is $\frac{16}{3}n^3 - 24n^2 + \frac{80}{3}n$. Summing the quantities in all cases, we obtain (i), as desired.

A reasoning similar to the proof of (i) leads to (ii). For reader's convenience, we outline the proof. For Case 1, it is exactly the same as before, since it makes no difference whether n is odd or even. For Case 2, there are two subcases for Group A, $j \neq k$, $l = m$ and $j = k$, $l \neq m$. Take the subcase $j \neq k$, $l = m$ as an example. Bearing in mind that n is odd,

- if j and k are both odd, then the corresponding quantity is $A^2_{\frac{n+1}{2}} \cdot 4$,
- if j and k are both even, then the corresponding quantity is $A^2_{\frac{n-1}{2}} \cdot 4$.

Hence, the quantity of options for such a subcase is $2n^2 - 4n + 2$. For the other subcase, the argument is the same as above. Consequently, the quantity of options in Case 2 is $4n^2 - 8n + 4$. For Case 3, as discussed in (i), it suffices to solve the following problem: how many options there are for j, k, l, m to satisfy the following conditions:

$$j + k = l + m, \quad j \neq k \neq l \neq m, \quad 1 \leq j, k, l, m \leq n,$$

where n is odd. Similarly, we still consider the non-repeating case first. For any odd number n, the quantity of options in non-repeating case is

$$4\binom{2}{2} + 4\binom{3}{2} + 4\binom{4}{2} + \cdots + 4\binom{\frac{n-1}{2}-1}{2} + 3\binom{\frac{n-1}{2}}{2} = \frac{1}{12}n^3 - \frac{3}{8}n^2 + \frac{5}{12}n - \frac{1}{8}.$$

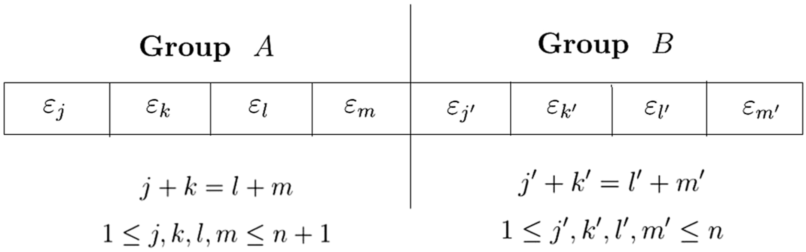

Figure 2: Grouping of parameters for (iii) and (iv).

Table 2: Subcases of Case 1 for (iii) and (iv).

Subcase	Quantity
$(j=l) \neq (k=m)$ and $(j'=l') \neq (k'=m')$	$A_{n+1}^2 \cdot A_n^2$
$(j=l) \neq (k=m)$ and $(j'=m') \neq (k'=l')$	$A_{n+1}^2 \cdot A_n^2$
$(j=l) \neq (k=m)$ and $j'=k'=l'=m'$	$A_{n+1}^2 \cdot n$
$(j=m) \neq (k=l)$ and $(j'=l') \neq (k'=m')$	$A_{n+1}^2 \cdot A_n^2$
$(j=m) \neq (k=l)$ and $(j'=m') \neq (k'=l')$	$A_{n+1}^2 \cdot A_n^2$
$(j=m) \neq (k=l)$ and $j'=k'=l'=m'$	$A_{n+1}^2 \cdot n$
$j=k=l=m$ and $(j'=l') \neq (k'=m')$	$(n+1) \cdot A_n^2$
$j=k=l=m$ and $(j'=m') \neq (k'=l')$	$(n+1) \cdot A_n^2$
$j=k=l=m$ and $j'=k'=l'=m'$	$(n+1) \cdot n$

Therefore, the quantity of options for Group A is $\frac{2}{3}n^3 - 3n^2 + \frac{10}{3}n - 1$. Further, with consideration of Group B, there are $\frac{16}{3}n^3 - 24n^2 + \frac{80}{3}n - 8$ options in Case 3. Consequently, combing three cases, we get the assertion in (ii).

For (iii) and (iv), we divide the eight parameters into two groups as shown in Figure 2 and the proof proceeds by considering three cases as in (i).

We prove (iii) first. Case 1 has the following nine subcases and the corresponding quantity of options for each subcase is shown in Table 2. Therefore, the sum of these quantities in Case 1 is $4n^4 + 4n^3 - n^2 - n$. For Cases 2 and 3, if we want $\mathbb{E}\big(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m\varepsilon_{j'}\varepsilon_{k'}\varepsilon_{l'}\varepsilon_{m'}\big) = 1$, then it suffices to consider $1 \le j,k,l,m \le n$. Therefore, the problems transform to the same one as in (i). Consequently, the quantity of options in Cases 2 and 3 is $4n^2 - 8n$ and $\frac{16}{3}n^3 - 24n^2 + \frac{80}{3}n$, respectively. Hence, we complete the proof of (iii).

Finally, we prove (iv). The proof is divided into three cases as before. Case 1 is the same as in (iii) since it makes no difference whether n is odd or even. For Cases 2 and 3, in fact, it suffices to consider $1 \le j,k,l,m \le n$ and thus follow the same arguments in (ii). Hence, we obtain (iv) and complete the proof of Lemma 2. ∎

Notations Let $T_0 = 0$. For any positive integer $n \ge 1$, define

$$T_n := \frac{\|q_n\|_4^4}{n^2}$$

and

$$X_n := T_n - T_{n-1}.$$

It is worth noting that

$$\sum_{i=1}^{n} X_i = T_n. \tag{10}$$

Lemma 3 *One has*

$$\mathbb{E}(X_n^2) = \frac{16}{n^2} + o\left(\frac{1}{n^2}\right).$$

Proof Since

$$\mathbb{E}(X_n^2) = \frac{\mathbb{E}\left(\|q_n\|_4^8\right)}{n^4} + \frac{\mathbb{E}\left(\|q_{n-1}\|_4^8\right)}{(n-1)^4} - \frac{2\mathbb{E}\left(\|q_n\|_4^4 \cdot \|q_{n-1}\|_4^4\right)}{n^2\,(n-1)^2},$$

by Lemma 2, a direct calculation yields the assertion, as desired. ∎

Lemma 4 *Let i, j be positive integers such that $1 < i < j < \infty$. Then*

$$\left|\mathbb{E}(X_i X_j)\right| \le C\frac{j^5}{(i-1)^8}, \tag{11}$$

where C is a constant independent of i, j.

Proof Observe that

$$\mathbb{E}(X_i X_j) = \frac{\mathbb{E}\left(\|q_i\|_4^4 \cdot \|q_j\|_4^4\right)}{i^2 j^2} - \frac{\mathbb{E}\left(\|q_i\|_4^4 \cdot \|q_{j-1}\|_4^4\right)}{i^2 (j-1)^2} - \frac{\mathbb{E}\left(\|q_{i-1}\|_4^4 \cdot \|q_j\|_4^4\right)}{(i-1)^2 j^2} + \frac{\mathbb{E}\left(\|q_{i-1}\|_4^4 \cdot \|q_{j-1}\|_4^4\right)}{(i-1)^2 (j-1)^2}.$$

Without loss of generality, we assume that i is even. Following similar arguments in the proof of Lemma 2, we obtain the following:

- $\mathbb{E}(\|q_i\|_4^4 \cdot \|q_j\|_4^4) = 4A_i^2 A_j^2 + 2jA_i^2 + 2iA_j^2 + ij + 4i^2 - 8i + \frac{16}{3}i^3 - 24i^2 + \frac{80}{3}i,$
- $\mathbb{E}(\|q_i\|_4^4 \cdot \|q_{j-1}\|_4^4) = 4A_i^2 A_{j-1}^2 + 2(j-1)A_i^2 + 2iA_{j-1}^2 + i(j-1) + 4i^2 - 8i + \frac{16}{3}i^3 - 24i^2 + \frac{80}{3}i,$
- $\mathbb{E}(\|q_{i-1}\|_4^4 \cdot \|q_j\|_4^4) = 4A_{i-1}^2 A_j^2 + 2jA_{i-1}^2 + 2(i-1)A_j^2 + (i-1)j + 4(i-1)^2 - 8(i-1) + 4 + \frac{16}{3}(i-1)^3 - 24(i-1)^2 + \frac{80}{3}(i-1) - 8,$
- $\mathbb{E}(\|q_{i-1}\|_4^4 \cdot \|q_{j-1}\|_4^4) = 4A_{i-1}^2 A_{j-1}^2 + 2(j-1)A_{i-1}^2 + 2(i-1)A_{j-1}^2 + (i-1)(j-1) + 4(i-1)^2 - 8(i-1) + 4 + \frac{16}{3}(i-1)^3 - 24(i-1)^2 + \frac{80}{3}(i-1) - 8.$

Therefore, we have

$$\mathbb{E}(X_i X_j) = \frac{-\frac{32}{3}i^4 j + \frac{64}{3}i^3 j + i^2 j^2 + \frac{16}{3}i^4 - \frac{32}{3}i^3 - ij^2 + \frac{53}{3}i^2 j - \frac{28}{3}i^2 - \frac{109}{3}ij + \frac{56}{3}i}{i^2 j^2 (i-1)^2 (j-1)^2}.$$

Bearing in mind $i < j$, we deduce (11). A similar reasoning leads to the proof when i is odd. The proof is complete now. ∎

3 Proof of Theorem 1

In this section we prove the main theorem. We first introduce some probabilistic tools involved in the proof of Theorem 1. The first lemma is a general method for establishing almost sure convergence, known as the method of subsequences.

Lemma 5 [27] *Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of random variables. Suppose that there exist a random variable Y and an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that*

$$Y_{n_k} \to Y \quad \text{a.s.}$$

and

$$\max_{n_{k-1}<n\leq n_k} |Y_n - Y_{n_{k-1}}| \to 0 \quad \text{a.s.} \quad \text{as} \quad k \to \infty.$$

Then

$$Y_n \to Y \quad \text{a.s.}$$

The next lemma is the so-called Serfling's maximal inequality. Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of random variables. For each $a \geq 0$ and $n \geq 1$, let $F_{a,n}$ be the joint distribution function of $\xi_{a+1}, \dots, \xi_{a+n}$, that is,

$$F_{a,n}(x_1, x_2, \dots, x_n) = \mathbb{P}(\xi_{a+1} \leq x_1, \xi_{a+2} \leq x_2, \dots, \xi_{a+n} \leq x_n),$$

for each $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Let

$$M_{a,n} = \max_{a<k\leq n} \left| \sum_{i=a+1}^{a+k} \xi_i \right|.$$

Then, Serfling's maximal inequality provides a good upper bound for $\mathbb{E}M_{a,n}^2$.

Lemma 6 [27] *Suppose that g is a nonnegative functional defined on the collection of joint distribution functions such that*

$$g(F_{a,k}) + g(F_{a+k,m}) \leq g(F_{a,k+m})$$

for all $1 \leq k < k+m$ and $a \geq 0$,

$$\mathbb{E}\left(\sum_{i=a+1}^{a+n} \xi_i \right)^2 \leq g(F_{a,n})$$

for all $n \geq 1$ and $a \geq 0$. Then

$$\mathbb{E}M_{a,n}^2 \leq \left(\frac{\log(2n)}{\log 2} \right)^2 g(F_{a,n})$$

for all $n \geq 1$ and $a \geq 0$.

Remark It is perhaps clear that the choice of the nonnegative functional g is the crucial part when applying Serfling's maximal inequality.

The following lemma follows from the Chebyshev inequality and the Borel–Cantelli lemma.

Lemma 7 [27] *Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of random variables. Suppose that*

$$\sum_{n=1}^{\infty} \mathbb{E}|\xi_n|^p < \infty$$

for some $p > 0$. Then

$$\xi_n \to 0 \quad \text{a.s.}$$

The following lemma provides a sufficient condition for almost sure convergence, with the main component of its proof being the Borel–Cantelli lemma.

Lemma 8 [9, Proposition 2.6, p. 98] *Suppose that*

$$\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n - \xi| > \varepsilon) < \infty$$

for any $\varepsilon > 0$. Then $\xi_n \to \xi$ almost surely.

Now we are ready to prove Theorem 1.

Proof of Theorem 1 Set $T_n = \|q_n\|_4^4/n^2$ for any positive integer n. The key to the proof is the following:

$$\sum_{k=1}^{\infty} \mathbb{E}\left(\max_{2^{k-1} < n \le 2^k} \left|T_n - T_{2^{k-1}}\right|^2 \right) < \infty. \tag{12}$$

Given the above estimate, the proof can be completed by the method of subsequences as follows. By Markov's inequality and Lemma 2, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{P}\left\{ \left|T_{2^k} - 2\right| > \varepsilon \right\} &\le \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \mathbb{E}\left(T_{2^k} - 2\right)^2 \\ &= \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \left(\frac{16}{3 \cdot 2^k} + O\left(\frac{1}{2^{2k}}\right)\right), \end{aligned}$$

which together with Lemma 8 implies

$$T_{2^k} \to 2 \quad \text{a.s.} \tag{13}$$

In addition, combining (12) and Lemma 7, we get

$$\max_{2^{k-1} < n \le 2^k} \left|T_n - T_{2^{k-1}}\right| \to 0 \quad \text{a.s.} \quad \text{as} \quad k \to \infty. \tag{14}$$

It follows from Lemma 5, (13), and (14) that

$$T_n \to 2 \quad \text{a.s.},$$

as desired.

Now it remains to prove (12). To accomplish this, we use Serfling's maximal inequality. Recall that $X_n := T_n - T_{n-1}$. For each $a \ge 0$ and $n \ge 1$, let $F_{a,n}$ be the joint distribution function of $X_{a+1}, \dots, X_{a+n}$, and $M_{a,n}$ be defined by

$$M_{a,n} = \max_{1\le k\le n}\left|\sum_{i=a+1}^{a+k} X_i\right|. \tag{15}$$

Now we need to construct an appropriate nonnegative functional g, defined on the collection of joint distribution functions, which, after some experimentation, we opt to be

$$g(F_{a,n}) = \sum_{i=a+1}^{a+n} \mathbb{E}(X_i^2) + 2\sum_{i=a+1}^{a+n-1}\sum_{j=i+1}^{a+n} \left|\mathbb{E}(X_iX_j)\right|. \tag{16}$$

Then, a direct calculation shows that the following (17) and (18) hold:

$$g(F_{a,k}) + g(F_{a+k,m}) \le g(F_{a,k+m}) \tag{17}$$

and

$$\mathbb{E}\left(\sum_{i=a+1}^{a+n} X_i\right)^2 \le g(F_{a,n}) \tag{18}$$

for all $a \ge 0, 1 \le k < k+m$ and $n \ge 1$. Thus, by Lemma 6, we obtain

$$\mathbb{E}\left(M_{a,n}^2\right) \le \left(\frac{\log(2n)}{\log 2}\right)^2 g(F_{a,n}). \tag{19}$$

Bearing in mind (10) and combining (15), (16), and (19), we deduce that

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathbb{E}\left(\max_{2^{k-1}<n\le 2^k}\left|T_n - T_{2^{k-1}}\right|^2\right)\\&\quad\le \sum_{k=1}^{\infty} k^2\left(\sum_{i=2^{k-1}+1}^{2^k}\mathbb{E}(X_i^2) + 2\sum_{i=2^{k-1}+1}^{2^k-1}\sum_{j=i+1}^{2^k}\left|\mathbb{E}(X_iX_j)\right|\right).\end{aligned}$$

Set

$$\mathrm{I} := \sum_{k=1}^{\infty} k^2 \sum_{i=2^{k-1}+1}^{2^k}\mathbb{E}(X_i^2)$$

and

$$\mathrm{II} := \sum_{k=1}^{\infty} k^2 \sum_{i=2^{k-1}+1}^{2^k-1}\sum_{j=i+1}^{2^k}\left|\mathbb{E}(X_iX_j)\right|.$$

Next we show that both I and II are finite. Note that

$$\mathrm{I} \asymp \sum_{k=2}^{\infty}(\log k)^2\mathbb{E}(X_k^2), \tag{20}$$

so by Lemma 3, we get I $< \infty$. In addition, Lemma 4 yields

$$\mathrm{II} \le C \sum_{k=1}^{\infty} k^2 \sum_{i=2^{k-1}+1}^{2^k-1} \sum_{j=i+1}^{2^k} \frac{j^5}{(i-1)^8} < \infty,$$

which together with (20) implies (12). This completes the proof. ∎

4 Extension for general random variables

In this section, we extend Theorem 1 to encompass a much more general case, requiring only an assumption on the fourth moment. The main new ingredient is Doob's maximal inequality instead of Serfling's.

Theorem 9 *Let $p_n(z) = \sum_{k=0}^{n-1} \varepsilon_k z^k$, where the random variables $\{\varepsilon_k, k \ge 0\}$ are independent and identically distributed, with mean 0, variance σ^2 and a finite fourth moment $\mathbb{E}(\varepsilon_k^4) < \infty$. Then*

$$\frac{||p_n||_4}{\sqrt{n}} \to \sqrt[4]{2} \cdot \sigma$$

almost surely.

Proof Analogous to (3), we observe that

$$||p_n||_4^4 = \sum_{\substack{j+k=l+m \\ 1 \le j,k,l,m \le n}} \varepsilon_j \varepsilon_k \varepsilon_l \varepsilon_m. \tag{21}$$

Now we decompose $||p_n||_4^4$ into four parts. Let

$$D_n = \{(j,k,l,m) : 1 \le j,k,l,m \le n, j+k = l+m, \mathrm{Card}(\{j,k,l,m\}) = 4\}.$$

That is, the indices j, k, l, and m are mutually different from each other. Let

$$M_n = \sum_{j,k,l,m} \varepsilon_j \varepsilon_k \varepsilon_l \varepsilon_m 1_{\{(j,k,l,m) \in D_n\}}$$

and

$$B_n = \sum_{\substack{l+m=2j \\ 1 \le j,l,m \le n}} \varepsilon_j^2 \varepsilon_l \varepsilon_m 1_{\{l \ne m\}}.$$

Then one has

$$\frac{||p_n||_4^4}{n^2} = \frac{M_n}{n^2} + 2\frac{B_n}{n^2} + 2\frac{\sum_{1 \le j,l \le n} \varepsilon_j^2 \varepsilon_l^2 1_{\{j \ne l\}}}{n^2} + \frac{\sum_{j=1}^n \varepsilon_j^4}{n^2}. \tag{22}$$

By the strong law of large numbers, the last term of (22) converges to 0 almost surely. In addition, the third term of (22) converges to $2\sigma^4$ almost surely by adding $2(\sum_{j=1}^n \varepsilon_j^4)/n^2$ to it and applying SLLN again.

For the first two terms of (22), we observe that $\mathbb{E}B_n = 0$ and a simple estimation yields

$$\mathbb{E}(B_n^2) = 2\mathbb{E}\Bigg(\sum_{1\leq j,l\leq n} \varepsilon_j^4 \varepsilon_l^2 \varepsilon_{2j-l}^2 1_{\{j\neq l, 1\leq 2j-l\leq n\}}\Bigg) = O(n^2).$$

Using Markov's inequality, we obtain

$$\sum_{n=1}^{\infty} \mathbb{P}(B_n^2/n^4 > \delta) \lesssim \frac{1}{\delta}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

for any $\delta > 0$. Then Lemma 8 yields $B_n/n^2 \to 0$ almost surely. Lastly, for the first term of (22), we observe that the sequence $\{M_n\}_{n\geq 1}$ forms a martingale with respect to the standard filtration

$$\{\sigma(\varepsilon_1, \dots, \varepsilon_n); n \geq 1\}.$$

Using $\mathbb{E}M_n = 0$ and the calculations leading to (i) and (ii) of Lemma 2, we have

$$\mathbb{E}(M_n^2) = O(n^3).$$

For any increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$, Doob's maximal inequality implies that for any $\delta > 0$,

$$\mathbb{P}\bigg(\max_{n_k\leq n\leq n_{k+1}} |M_n| > n_k^2\delta\bigg) \leq \frac{\mathbb{E}(M_{n_{k+1}}^2)}{n_k^4\delta^2} = O(n_{k+1}^3/n_k^4).$$

By choosing $n_k = 2^k$, we deduce that

$$\sum_{k=1}^{\infty}\mathbb{P}\bigg(\max_{2^k\leq n\leq 2^{k+1}}\frac{|M_n|}{n^2} > \delta\bigg) \leq \sum_{k=1}^{\infty}\mathbb{P}\bigg(\max_{2^k\leq n\leq 2^{k+1}}\frac{|M_n|}{(2^k)^2} > \delta\bigg) < \infty.$$

By Lemma 8, we get

$$\max_{2^k\leq n\leq 2^{k+1}}\frac{|M_n|}{n^2} \to 0 \quad \text{a.s.} \quad \text{as} \quad k\to\infty,$$

which in turn implies that $M_n/n^2 \to 0$ almost surely. This completes the proof. ∎

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