Revisiting the general cubic: a simplification of Cardano's solution

HSIN-CHIEH LIAO, MARK SAUL and PETER J.-S SHIUE

1. *Introduction*

Given a polynomial equation $x^3 + ax^2 + bx + c = 0$ of degree 3 with real coefficients, we may translate the variable by replacing x with $x - \frac{1}{3}a$ to make the quadratic term vanish. We then obtain a simpler equation $x^3 + px + q = 0$ where $p = -\frac{1}{3}a^2 + b$ and $q = \frac{2}{27}a^3 - \frac{1}{3}ab + c$. Therefore, in order to solve a polynomial equation of degree 3, it is sufficient to solve equations of the form $x^3 + px + q = 0$.

In fact, the three roots of such a polynomial equations are

$$\sqrt[3]{A} + \sqrt[3]{B}, \qquad \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}, \qquad \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B},$$

where $\omega = \frac{1}{2}\left(-1 + \sqrt{3}i\right)$, $A = -\frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}$, $B = -\frac{1}{2}q - \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}$. This is the well-known formula found by the Italian mathematician G. Cardano (1501-1576) [1]. The formula appears in his book *Ars Magna*. For the related historical details, we refer readers to [2, 3].

Inspired by Sylvester's work [4], Chen in [5] proposed another way to solve the equation $x^3 + px + q = 0$. We set $p = -3rs$ and $q = rs(r + s)$. Then the equation can be solved (if $r \neq s$) via the following new identity

$$x^3 - 3rsx + rs(r + s) = \frac{s}{s - r}(x - r)^3 + \frac{r}{r - s}(x - s)^3. \tag{1}$$

Chen obtains the following results:

(i) If $x^2 - (r + s)x + rs = 0$ has a repeated root $r = s$, the left hand side of (1) reduces to

$$x^3 - 3r^2x + 2r^3 = (x - r)^2(x + 2r).$$

(ii) If $r \neq s$ (where $s \neq 0$ since $p \neq 0$), then the three roots x_1, x_2, x_3 of $x^3 + px + q = 0$ are $x_i = \dfrac{r - su_i}{1 - u_i}$ for $i = 1, 2, 3$, where u_1, u_2, u_3 are the cube roots of $\frac{r}{s}$.

The goal of this paper is to continue Chen's work in [5] (see also Liao and Shiue [6]), studying in detail his method of solving the equation $x^3 + px + q = 0$.

2. *An alternative solution to the general cubic equation*

Although this case has been covered in [5], we begin here, for completeness, with the case $r = s$, in which $x^3 - 3rsx + rs(r + s) = 0$ can be factorised as follows:

$$x^3 - 3r^2x + 2r^3 = \left(x^3 - r^3\right) - 3r^2(x - r)$$

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$$= (x - r)\left(x^2 + rx - 2r^2\right)$$

$$= (x - r)^2 (x + 2r).$$

Hence the roots are r, r, $-2r$ or, equivalently, $\sqrt{rs}$, $\sqrt{rs}$, $-2\sqrt{rs}$. Then we have the following theorem:

Theorem 1: Let $x^3 - 3rsx + rs(r + s) = 0$ be an equation with real coefficients. If $r = s$ then the three roots are

$$\sqrt{rs},\ \sqrt{rs},\ -2\sqrt{rs}.$$

Example 1: Solve $x^3 - 12x + 16 = 0$.

Solution: Because $rs = 4$ and $rs(r + s) = 16$, we have $r + s = 4$. Hence r and s are the two roots of $t^2 - 4t + 4 = 0$. Solving this quadratic equation we obtain $r = s = 2$. By Theorem 1, the three roots of $x^3 - 12x + 16 = 0$ are 2, 2, −4.

Remark 1: It is instructive to see how this relates to Cardano's formula. The formula, as we have given it above, applies to the general cubic $x^3 + px + q = 0$. We work with the general form $x^3 - 3rsx + rs(r + s) = 0$. So we have $p = -3rs$, $q = rs(r + s)$, or $rs = -\frac{1}{3}p$, $r + s = -\frac{3q}{p}$. Hence p and q are roots of the quadratic $t^2 + \frac{3q}{4}t - \frac{p}{3} = 0$. The discriminant of this quadratic equation is

$$\frac{9q^2}{p^2} + \frac{4p}{3} = \frac{4p^3 + 27q^2}{3p^2}.$$

The term $4p^3 + 27q^2$ is exactly the classic discriminant of a cubic equation.

On the other hand, if $r \neq s$, from the identity (1) we get

$$\frac{s}{s - r}(x - r)^3 + \frac{r}{r - s}(x - s)^3 = x^3 - 3rsx + rs(r + s) = 0.$$

Then

$$s(x - r)^3 = r(x - s)^3 \Rightarrow \left(\frac{x - r}{x - s}\right)^3 = \frac{r}{s}. \tag{2}$$

Since both rs and $rs(r + s)$ are real, there are only two cases to be considered: either both r and s are real or r and s are a pair of complex conjugates.

Suppose both r and s are real. Then so is $\frac{r}{s}$. Hence we have

$$\frac{x - r}{x - s} = \sqrt[3]{\frac{r}{s}},\ \sqrt[3]{\frac{r}{s}}\,\omega,\ \sqrt[3]{\frac{r}{s}}\,\omega^2.$$

where $\omega = e^{2\pi i/3} = \frac{1}{2}\left(-1 + \sqrt{3}i\right)$.

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When $\dfrac{x - r}{x - s} = \sqrt[3]{\dfrac{r}{s}}$, then

$$\sqrt[3]{s}(x - r) = \sqrt[3]{r}(x - s) \Rightarrow \left(\sqrt[3]{s} - \sqrt[3]{r}\right)x = \sqrt[3]{sr} - \sqrt[3]{rs}.$$

Therefore we get

$$x = \frac{\sqrt[3]{sr} - \sqrt[3]{rs}}{\sqrt[3]{s} - \sqrt[3]{r}} = \frac{\sqrt[3]{r}\sqrt[3]{s}\left(\sqrt[3]{r^2} - \sqrt[3]{s^2}\right)}{\sqrt[3]{s} - \sqrt[3]{r}} = -\sqrt[3]{s}\sqrt[3]{r}\left(\sqrt[3]{r} + \sqrt[3]{s}\right).$$

Similarly, when $\dfrac{x - r}{x - s} = \sqrt[3]{\dfrac{r}{s}}\omega$, we have

$$\sqrt[3]{s}(x - r) = \sqrt[3]{r}\omega(x - s) \Rightarrow \left(\sqrt[3]{s} - \sqrt[3]{r}\right)x = \sqrt[3]{sr} - \omega\sqrt[3]{rs}.$$

It follows that

$$x = \frac{\sqrt[3]{sr} - \omega\sqrt[3]{rs}}{\sqrt[3]{s} - \omega\sqrt[3]{r}} = \frac{\sqrt[3]{r}\sqrt[3]{s}\left(\sqrt[3]{r^2} - \omega\sqrt[3]{s^2}\right)}{\sqrt[3]{s} - \omega\sqrt[3]{r}} = \frac{\sqrt[3]{r}\sqrt[3]{s}\left(\sqrt[3]{r^2} - \omega\sqrt[3]{s^2}\right)\omega^2}{\left(\sqrt[3]{s} - \omega\sqrt[3]{r}\right)\omega^2}$$

$$= -\frac{\sqrt[3]{r}\sqrt[3]{s}\left(\omega^2\sqrt[3]{r^2} - \sqrt[3]{s^2}\right)\omega^2}{\omega^2\left(\omega\sqrt[3]{r} - \sqrt[3]{s}\right)} = -\omega\sqrt[3]{r}\sqrt[3]{s}\left(\omega\sqrt[3]{r} + \sqrt[3]{s}\right)$$

$$= -\sqrt[3]{r}\sqrt[3]{s}\left(\omega^2\sqrt[3]{r} + \omega\sqrt[3]{s}\right).$$

When $\dfrac{x - r}{x - s} = \sqrt[3]{\dfrac{r}{s}}\,\omega^2$, a similar argument yields

$$x = -\sqrt[3]{r}\sqrt[3]{s}\left(\omega\sqrt[3]{r} + \omega^2\sqrt[3]{s}\right).$$

Summarising the discussion above we obtain the following theorem:

Theorem 2: Let $x^3 - 3rsx + rs(r + s) = 0$ have real coefficients with $r \neq s$. If both r and s are real, then the three roots are

$$-\sqrt[3]{r}\sqrt[3]{s}\left(\sqrt[3]{r} + \sqrt[3]{s}\right), \qquad -\sqrt[3]{r}\sqrt[3]{s}\left(\omega^2\sqrt[3]{r} + \omega\sqrt[3]{s}\right), \qquad -\sqrt[3]{r}\sqrt[3]{s}\left(\omega\sqrt[3]{r} + \omega^2\sqrt[3]{s}\right).$$

The equation has one real root and a pair of complex conjugate roots.

Example 2: Find all solutions to $x^3 - 6x - 9 = 0$.

Solution: The numbers r and s are the roots of the quadratic equation $t^2 + \frac{9}{2}t + 2 = 0$. Solving this equation, we get $r = -\frac{1}{2}$ and $s = -4$, which are two distinct real numbers. Hence by Theorem 2 the three solutions to the original equations are

$$-\sqrt[3]{2}\left(\sqrt[3]{-1/2} + \sqrt[3]{-4}\right) = -\left(\sqrt[3]{-1} + \sqrt[3]{-8}\right) = 3,$$

$$-\sqrt[3]{2}\left(\frac{-1 - \sqrt{3}i}{2}\sqrt[3]{-1/2} + \frac{-1 + \sqrt{3}i}{2}\sqrt[3]{-4}\right) = -\left(\frac{1 + \sqrt{3}i}{2} + \frac{2 - 2\sqrt{3}i}{2}\right) = \frac{-3 + \sqrt{3}i}{2},$$

$$-\sqrt[3]{2}\left(\frac{-1 + \sqrt{3}i}{2}\sqrt[3]{-1/2} + \frac{-1 - \sqrt{3}i}{2}\sqrt[3]{-4}\right) = -\left(\frac{1 - \sqrt{3}i}{2} + \frac{2 + 2\sqrt{3}i}{2}\right) = \frac{-3 - \sqrt{3}i}{2}.$$

Next let us consider the second case, when r, s is a pair of complex conjugates. We first fix some notation.

Any complex number $z = x + iy \neq 0$ can be expressed as $z = |z| e^{i\theta} = |z| (\cos\theta + i \sin\theta)$, where $\theta = \mathrm{Arg}(z)$ is the argument of z in the interval $(-\pi, \pi]$. We write $\bar{z} = x - iy$ for the complex conjugate of z and $\mathrm{Re}(z)$ for the real part of z. For any positive integer m, the complex number z has m m-th roots $\alpha\zeta, \alpha\zeta^2, \alpha\zeta^3, \ldots, \alpha\zeta^{m-1}$, where $\alpha = \sqrt[m]{|z|}e^{i\theta/m}$ and $\zeta = e^{i(2\pi/m)}$. In this paper we write α as $z^{1/m}$. In particular, when $\theta = 0$, $z^{1/m} = \sqrt[m]{z} \in \mathbb{R}$.

From (2), we get $\dfrac{x - r}{x - s} = \sqrt[3]{\dfrac{r}{s}}, \sqrt[3]{\dfrac{r}{s}}\,\omega, \sqrt[3]{\dfrac{r}{s}}\,\omega^2$. After a similar computation, we see that the solutions are given by the same expressions as in the former case:

$$-\sqrt[3]{r}\sqrt[3]{s}\left(\sqrt[3]{r} + \sqrt[3]{s}\right), -\sqrt[3]{r}\sqrt[3]{s}\left(\omega\sqrt[3]{r} + \omega^2\sqrt[3]{s}\right), -\sqrt[3]{r}\sqrt[3]{s}\left(\omega^2\sqrt[3]{r} + \omega\sqrt[3]{s}\right). \quad (3)$$

However, when r, s are complex conjugates, the expressions can be further simplified as follows:

Let $r = |r| e^{i\theta}$ and $s = |s| e^{-i\theta}$ with $|r| = |s|$. Then

$$\sqrt[3]{r} = \sqrt[3]{|r|}e^{i\theta/3} = \sqrt[6]{rs}\, e^{i\theta/3}, \sqrt[3]{s} = \sqrt[6]{rs}\, e^{-i\theta/3} = \overline{\sqrt[3]{r}},$$

and we can rewrite the solutions as

$$-\sqrt[3]{r}\sqrt[3]{s}\left(\sqrt[3]{r} + \sqrt[3]{s}\right) = -(rs)^{1/3 + 1/6}\left(e^{i\theta/3} + e^{-i\theta/3}\right) = -2\sqrt{rs}\cos\frac{\theta}{3},$$

$$-\sqrt[3]{r}\sqrt[3]{s}\left(\omega\sqrt[3]{r} + \omega^2\sqrt[3]{s}\right) = -\sqrt[3]{rs}\left(\omega\sqrt[3]{r} + \overline{\omega\sqrt[3]{r}}\right) = -2\sqrt[3]{rs}\,\mathrm{Re}\left(\omega\sqrt[3]{r}\right)$$

$$= -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right),$$

$$-\sqrt[3]{r}\sqrt[3]{s}\left(\omega^2\sqrt[3]{r} + \omega\sqrt[3]{s}\right) = -\sqrt[3]{rs}\left(\omega^2\sqrt[3]{r} + \overline{\omega^2\sqrt[3]{r}}\right) = -2\sqrt[3]{rs}\,\mathrm{Re}\left(\omega^2\sqrt[3]{r}\right)$$

$$= -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right).$$

Theorem 3: Let $x^3 - 3rsx + rs(r + s) = 0$ have real coefficients with $r \neq s$. If r and s are a pair of complex conjugates and we let $\theta = \mathrm{Arg}(r)$, then the three roots of the equation are

$$-2\sqrt{rs}\cos\frac{\theta}{3}, -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right), -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right).$$

The equation has three real roots.

Remark 2: Note that from the result of Theorem 3 we can derive the following trigonometric identities by Vieta's formula:

$$\cos\frac{\theta}{3} + \cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + \cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) = 0,$$

$$\cos\frac{\theta}{3}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + \cos\frac{\theta}{3}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) + \cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) = -\frac{3}{4},$$

$$\cos\frac{\theta}{3}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) = -\frac{\cos\theta}{4}.$$

The last identity follows from the fact that the product of the three roots is equal to $-rs(r + s)$ and $r = |r|e^{i\theta}$, $s = |r|e^{-i\theta}$.

Remark 3: In Remark 2, we have

$$\cos\frac{\theta}{3} + \cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + \cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) = 0,\ s = 4e^{-i(3\pi/4)}.$$

By direct computation, we know that

$$A^3 + B^3 + C^3 - 3ABC = (A + B + C)\left(A^2 + B^2 + C^2 - AB - BC - CA\right).$$

This factorisation, together with the identity from Remark 2, leads to the following trigonometric identity:

$$\cos^3\frac{\theta}{3} + \cos^3\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + \cos^3\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) = 3\cos\frac{\theta}{3}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)$$
$$= -\frac{3\cos\theta}{4}$$

for arbitrary θ.

Example 3: Find all solutions to $x^3 - 48x - 64\sqrt{2} = 0$.

Solution: Since $rs = 16$, $r + s = -4\sqrt{2}$, r, s are the two roots of $t^2 + 4\sqrt{2}t + 16 = 0$, we have

$$r = \frac{-4\sqrt{2} + \sqrt{32 - 64}}{2} = -2\sqrt{2} + 2\sqrt{2}i = 4e^{-i\frac{3\pi}{4}}.$$

Hence $\theta = \frac{3}{4}\pi$. Applying Theorem 3 we obtain the solutions to the original equations as

$$-2 \cdot 4\cos\frac{\pi}{4} = -8 \cdot \frac{1}{\sqrt{2}} = -4\sqrt{2},$$

$$-2 \cdot 4\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) = 8\cos\frac{\pi}{12} = 8 \cdot \frac{1 + \sqrt{3}}{2\sqrt{2}} = 2\sqrt{2} + 2\sqrt{6},$$

$$-2 \cdot 4\cos\left(\frac{\pi}{4} + \frac{4\pi}{3}\right) = -8\cos\frac{19\pi}{12} = -8 \cdot \frac{-1 + \sqrt{3}}{2\sqrt{2}} = 2\sqrt{2} - 2\sqrt{6}.$$

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Example 4: Find all solutions to $x^3 - \frac{3}{4}x + \frac{\sqrt{3}}{8} = 0$.

Solution: Since $rs = \frac{1}{4}$, $r + s = \frac{1}{2}\sqrt{3}$, r, s are the two roots of $t^2 - \frac{\sqrt{3}}{2}t + \frac{1}{4} = 0$, we have $r = \frac{1}{4}(\sqrt{3} + i) = \frac{1}{2}e^{i\pi/6}$, $s = \frac{1}{4}(\sqrt{3} - i) = \frac{1}{2}e^{-i\pi/6}$. These are complex conjugates and $\theta = \frac{1}{6}\pi$. Applying Theorem 3, we find that the three solutions to the original equations are

$$-2\sqrt{\frac{1}{4}}\cos\left(\frac{1}{3}\cdot\frac{\pi}{6}\right) = -\cos\frac{\pi}{18} = \sin\frac{14\pi}{9},$$

$$-\cos\left(\frac{\pi}{18} + \frac{2\pi}{3}\right) = -\cos\frac{13\pi}{18} = \sin\frac{2\pi}{9},$$

$$-\cos\left(\frac{\pi}{18} + \frac{4\pi}{3}\right) = -\cos\frac{25\pi}{18} = \sin\frac{8\pi}{9}.$$

Example 5: Find all solutions to $x^3 - \frac{3}{4}x + \frac{1}{8} = 0$.

Solution: Since $rs = \frac{1}{4}$, $r + s = \frac{1}{2}$, we know that r, s are the roots of $t^2 - \frac{1}{2}t + \frac{1}{4} = 0$. Then $r = \frac{1}{4}(1 + \sqrt{3}i) = \frac{1}{2}e^{i\pi/3}$, $s = \frac{1}{4}(1 - \sqrt{3}i) = \frac{1}{2}e^{-i\pi/3}$ are complex conjugates and $\theta = \frac{\pi}{3}$. Applying Theorem 3, we find that the three solutions to the original equations are

$$-2\sqrt{\frac{1}{4}}\cos\left(\frac{1}{3}\cdot\frac{\pi}{3}\right) = -\cos\frac{\pi}{9} = \cos\frac{8\pi}{9},$$

$$-\cos\left(\frac{\pi}{9} + \frac{2\pi}{3}\right) = -\cos\frac{7\pi}{9} = \cos\frac{2\pi}{9},$$

$$-\cos\left(\frac{\pi}{9} + \frac{4\pi}{3}\right) = -\cos\frac{13\pi}{9} = \cos\frac{4\pi}{9}.$$

3. *Conclusion*

Considering the results we have so far, one notices that although we divide the solutions to the equation $x^3 - 3rsx + rs(r + s) = 0$ into three cases (r and s are equal; r and s are distinct real numbers; r and s are a pair of complex conjugates), the solutions we obtain in these three cases can actually be expressed uniformly as (3). Therefore we may summarise our results as follows:

Theorem 4: Given a polynomial equation of degree three $x^3 + px + q = 0$ ($p, q \neq 0$) with real coefficients, we can rewrite it as $x^3 - 3rsx + rs(r + s) = 0$. Then the three solutions to this equation are

$$-\sqrt[3]{r}\sqrt[3]{s}\left(\sqrt[3]{r} + \sqrt[3]{s}\right), -\sqrt[3]{r}\sqrt[3]{s}\left(\omega\sqrt[3]{r} + \omega^2\sqrt[3]{s}\right), -\sqrt[3]{r}\sqrt[3]{s}\left(\omega^2\sqrt[3]{r} + \omega\sqrt[3]{s}\right).$$

In particular, when $r = s$ the three solutions can be further simplified as

$$-2\sqrt{rs}, \qquad \sqrt{rs}, \qquad \sqrt{rs}.$$

When $r \neq s$, and r and s are complex conjugates, we can let $\theta = \text{Arg}(r)$. Then the three solutions can be further simplified as

$$-2\sqrt{rs}\cos\frac{\theta}{3},\ -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right),\ -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right).$$

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