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INEQUALITIES FOR RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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ABSTRACT. This paper considers the rational system $P_n(a_1, a_2, \ldots, a_n) := \left\{ \frac{P(x)}{\prod_{k=1}^n (x-a_k)}, P \in P_n \right\}$ with nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ paired by complex conjugation. It gives a sharp (to constant) Markov-type inequality for real rational functions in $P_n(a_1, a_2, \ldots, a_n)$. The corresponding Markov-type inequality for high derivatives is established, as well as Nikolskii-type inequalities. Some sharp Markov- and Bernstein-type inequalities with curved majorants for rational functions in $P_n(a_1, a_2, \ldots, a_n)$ are obtained, which generalize some results for the classical polynomials. A sharp Schur-type inequality is also proved and plays a key role in the proofs of our main results.

1. **Introduction.** Let P_n be the set of all real algebraic polynomials of degree at most n, and let T_n be the set of all real trigonometric polynomials of degree at most n. The following two inequalities are fundamental to the proofs of many inverse theorems in polynomial approximation theory and of course have their own intrinsic interest, see, for example, Borwein and Erdélyi [3, Chapter 5], Cheney [6], Lorentz [9], Milovanović, Mitrinović and Rassias [10, Chapter 6], Natanson [11], Rivlin [15].

MARKOV INEQUALITY. The inequality

$$\|P_n'\|_{[-1,1]} \le n^2 \|P_n\|_{[-1,1]}$$

holds for $P_n \in P_n$.

BERNSTEIN INEQUALITY. The inequality

$$|P_n'(x)| \le \frac{n}{\sqrt{1-x^2}} \|P_n\|_{[-1,1]}, \quad x \in (-1,1)$$

holds for $P_n \in P_n$.

In the above theorems and throughout this paper, $\|.\|_A$ denotes the supremum norm on $A \subset \mathbb{R}$. There are many results on the Bernstein's and Markov's inequalities and their generalization. For the interested readers, see, for example, Borwein and Erdélyi [3], Milovanović, Mitrinović and Rassias [10, Chapter 6] and Rahman and Schmeisser [14] and references therein. Here we just mention that, in 1970, at a conference on "Constructive Function Theory" held in Varna, Bulgaria, P. Turán raised the following problem:

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PROBLEM. Determine $\max_{-1\le x\le 1}|P_n'(x)|$ for all polynomials $P_n(x)$ of degree at most n satisfying the restriction that

$$\sup_{-1<x<1}\frac{|P_n(x)|}{\sqrt{1-x^2}}=1. \tag{1.1}$$

Rahman [13, Theorem 1] completely solved the above problem by

THEOREM A. *Let $P_n\in\mathcal{P}_n$ satisfy $|P_n(x)|\le\sqrt{1-x^2}$ for $x\in[-1,1]$. Then*

$$\|P_n'\|\le 2(n-1), \tag{1.2}$$

and it is sharp by $P_n(x)=(1-x^2)U_{n-2}(x)$, where $U_{n-2}(x)$ is the classical Chebyshev polynomial of the second kind.

For the case of the restriction

$$|P_n(x)|\le(1-x^2)^{-1/2}, \tag{1.3}$$

Lachance [7] obtained the following Bernstein- and Markov-type inequalities

THEOREM B. *Let $P_n\in\mathcal{P}_n$ and P_n satisfy (1.3). Then*

$$|P_n'(x)|\le 2(n+1)(1-x^2)^{-1},\quad -1<x<1, \tag{1.4}$$

and

$$\|P_n\|_{[-1,1]}\le n(n+1)^2, \tag{1.5}$$

and these inequalities are sharp to constant respectively.

Rahman and his associates have extensively investigated these kinds of inequalities for classical polynomials. For more details, see, for example, [10, Section 6.1.4] and references therein.

On the other hand, the Bernstein-Markov type inequality does not exist for the arbitrary rational function, for example, considering $r(x)=-\frac{\delta^2}{x^2+\delta^2}$, then $\|r\|_{[-1,1]}\le 1$ but $r'(\delta)=\frac{1}{2\delta}$ (*cf.* Lorentz [9]).

However, we can develop Bernstein-Markov type inequalities for rational functions with restricted denominators (*cf.* Borwein [2]). Recently, Borwein and Erdélyi and Zhang [5] considered the inequalities of rational functions with prescribed poles. We first introduce some notations in order to state their main results.

We denote

$$\mathcal{P}_m(a_1,a_2,\ldots,a_n):=\left\{\frac{P(x)}{\prod_{k=1}^n(x-a_k)},P\in\mathcal{P}_m\right\} \tag{1.6}$$

and

$$\mathcal{T}_m(a_1,a_2,\ldots,a_n):=\left\{\frac{P(t)}{\prod_{k=1}^n(\cos t-a_k)},P\in\mathcal{T}_m\right\}, \tag{1.7}$$

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where $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ is a fixed set of poles such that $\prod_{k=1}^n (x-a_k) \in \mathcal{P}_n$. In other words, the nonreal poles form complex conjugate pairs. We define the numbers $\{c_k\}_{k=1}^n$ by

$$a_k := \frac{c_k + c_k^{-1}}{2}, \quad |c_k| < 1,$$

that is,

$$c_k = a_k - \sqrt{a_k^2 - 1}, \quad |c_k| < 1. \tag{1.8}$$

Note that $(a_k + \sqrt{a_k^2-1})(a_k - \sqrt{a_k^2-1}) = 1$, throughout this paper, $\sqrt{a_k^2-1}$ will be always defined by (1.8) except special statement.

We denote

$$B_n(x) := \Re\Big(\sum_{k=1}^n \frac{\sqrt{a_k^2-1}}{a_k - x}\Big) (> 0), \quad \tilde{B}_n(t) := B_n(\cos t), \tag{1.9}$$

which are called the *Bernstein factors* and they play important roles in [5], where $\sqrt{a_k^2-1}$ $(k = 1, \ldots, n)$ is defined by (1.8). Throughout this paper, $B_n(x)$ and $\tilde{B}_n(x)$ are always defined by (1.9).

Borwein, Erdélyi and Zhang (*cf.* [5, Theorem 3.1]) obtained a remarkable extension of the well-known Bernstein-Szegő inequality for system $\mathcal{T}_n(a_1, a_2, \ldots, a_n)$, that is,

THEOREM C (BERNSTEIN-SZEGŐ-TYPE INEQUALITY). *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ *be paired by complex conjugation. Then*

$$P'(t)^2 + \tilde{B}_n^2(t)P^2(t) \le \tilde{B}_n^2(t) \max_{\tau\in\mathbb{R}} |P(\tau)|^2, \quad t \in \mathbb{R} \tag{1.10}$$

for every P in $\mathcal{T}_n(a_1, a_2, \ldots, a_n)$, and this inequality is best possible.

They [5] also got a Markov-type inequality for rational system $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ with *real poles* and a Bernstein-type inequality respectively (*cf.* [5, Corollary 3.4] and [5, Theorem 3.5]):

THEOREM D. *(i) (Bernstein-type Inequality) Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ *be paired by complex conjugation. Then*

$$|P'(x)| \le \frac{1}{\sqrt{1-x^2}} |B_n(x)| \|P\|_{[-1,1]}, \quad x \in (-1,1) \tag{1.11}$$

holds for every $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$, and it is best possible.

(ii) (Markov-type Inequality) Let $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$ *real poles. Then*

$$\begin{aligned} \|P'\|_{[-1,1]} &\le \frac{n}{n-1}\Big(\sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|}\Big)^2 \|P\|_{[-1,1]} \\ &\le 2\Big(\sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|}\Big)^2 \|P\|_{[-1,1]} \end{aligned} \tag{1.12}$$

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hold for every $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$, $n = 1, 2, \ldots$.

For more information about inequalities of rational functions with prescribed poles on the unit disk or on the whole real axis, see, for example, Borwein and Erdélyi [3, Section 7.1], [4], Li, Mohapatra and Rodriguez [8], Petrushev and Popov [12]. This is an area of current research activity.

It's natural to ask if we can extend (1.12) to the case of complex poles (with complex conjugation) outside of $[-1, 1]$: $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1, 1]$?

In the present paper, we first consider this question. We obtain a sharp (to constant) Markov-type inequality and our Markov-type inequality is more compact (*cf.* Theorem 2.1). Then, we deduce two Nikolskii-type inequalities for rational system $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ with $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1, 1]$ (*cf.* Theorem 2.4 and Theorem 2.5). We also get a corresponding Markov-type inequality for high derivatives for rational system $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ with $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1, 1]$ (*cf.* Theorem 2.2). Moreover, we get some sharp (to constant) Markov-type and Bernstein-type inequalities for the rational functions with some curved majorants in rational system $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ with $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1, 1]$ (*cf.* Theorem 2.6 and Theorem 2.7), which extend Theorem A and Theorem B in some sense. A sharp Schur-type inequality is also proved and plays a key role in the proofs of our main results.

This paper is organized as follows. In Section 2 we formulate the main results. Section 3 gives two lemmas which will be used to prove our main results. Section 4 contains proofs of Theorems 2.1–2.5. The proofs of Theorems 2.6–2.7 are given in Section 5. In the last section, some remarks are given.

2. Main results.

THEOREM 2.1 (MARKOV-TYPE INEQUALITY). *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1, 1]$ *be paired by complex conjugation. Then*

$$\|P'\|_{[-1,1]} \le 2\|B_n\|_{[-1,1]}^2 \|P\|_{[-1,1]} \tag{2.1}$$

holds for every $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$. *Furthermore, if* $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1, 1]$ *real poles, then*

$$\|B_n\|_{[-1,1]}^2 \le \sup_{0 \ne P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)} \frac{\|P'\|_{[-1,1]}}{\|P\|_{[-1,1]}} \le 2\|B_n\|_{[-1,1]}^2, \tag{2.2}$$

and

$$\|B_n\|_{[-1,1]} = \max\Big\{\sum_{k=1}^n \frac{1 - c_k}{1 + c_k}, \sum_{k=1}^n \frac{1 + c_k}{1 - c_k}\Big\} \ge n. \tag{2.3}$$

REMARK. By the definition of $B_n(x)$, one can easily show that

$$\sum_{k=1}^n \frac{1 - |c_k|}{1 + |c_k|} \le B_n(x) \le \sum_{k=1}^n \frac{1 + |c_k|}{1 - |c_k|}. \tag{2.4}$$

Hence it follows from (2.2)

$$\|P'\|_{[-1,1]} \le 2\Big(\sum_{k=1}^n \frac{1 + |c_k|}{1 - |c_k|}\Big)^2 \|P\|_{[-1,1]}.$$

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It would be interesting to close the gap in (2.2).

We may get a corresponding Markov-type inequality for high derivatives by applying Theorem 2.1 and induction on m, that is,

THEOREM 2.2 (MARKOV-TYPE INEQUALITY FOR HIGH DERIVATIVES). *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ *be paired by complex conjugation. Then*

$$\|P^{(m)}\|_{[-1,1]} \leq m!\,(m+1)!\,\|B_n\|_{[-1,1]}^{2m} \|P\|_{[-1,1]} \tag{2.5}$$

holds for every $P \in P_n(a_1, a_2, \ldots, a_n)$ *and* $m = 1, 2, \ldots$.

COROLLARY 2.3. *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ *be paired by complex conjugation. Then*

$$\|P'\|_{[-1,1]} \leq \|B_n\|_{[-1,1]}^2 \Big(\max_{-1\leq x\leq 1} P(x) - \min_{-1\leq x\leq 1} P(x) \Big) \tag{2.6}$$

for $p \in P_n(a_1, a_2, \ldots, a_n)$. *Particularly, for* $0 \leq P(x) \leq 1$ *for* $-1 \leq x \leq 1$, *we have*

$$\|P'\|_{[-1,1]} \leq \|B_n\|_{[-1,1]}^2, \tag{2.7}$$

and they are sharp to constant for $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$.

The following Nikolskii-type inequality for rational system $P_n(a_1, a_2, \ldots, a_n)$ follows from Theorem 2.1 quite simply.

THEOREM 2.4 (NIKOLSKII-TYPE INEQUALITY). *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ *be paired by complex conjugation. Then*

$$\|P\|_p \leq 2\{2\|B_n\|_{[-1,1]}\}^{2(1/q-1/p)} \|P\|_q \tag{2.8}$$

holds for every $P \in P_n(a_1, a_2, \ldots, a_n)$, *where* $\|P\|_p := \left(\int_{-1}^1 |P(x)|^p\, dx\right)^{1/p}$ *and* $0 < q < p \leq \infty$.

In a certain weighted L_2-norm, we can get an exact Nikolskii-type inequality which has a smaller Nikolskii constant under some conditions. Precisely, we have

THEOREM 2.5. *Let* $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$ *be distinct. Then*

$$\|P\|_{[-1,1]} \leq \left(\frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|} \right)^{1/2} \|P\|_{2,v} \tag{2.9}$$

for $P \in P_n(a_1, a_2, \ldots, a_n)$. *Moreover, if* $\{a_k\}_{k=1}^n$ *keep constant sign, then it is exact, where*

$$\|P\|_{2,v} := \left(\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |P(x)|^2\, dx \right)^{1/2}.$$

For the simplicity of the statements of inequalities of rational functions with some curved majorants, we first introduce an assumption:

ASSUMPTION (A). Let $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$. If there exists some constant α such that

$$|a_k| \geq \alpha > 1, \tag{2.10}$$

i.e., the poles must stay outside an interval which contains $[-1,1]$ in its interior, we say that $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$ satisfy Assumption (A).

It is easy to see that Assumption (A) is equivalent to

$$|c_k| \leq \gamma, \quad k = 1, \ldots, n,$$

where $0 \leq \gamma = \alpha - \sqrt{\alpha^2 - 1} < 1$. If this condition is satisfied, we say that $\{c_k\}_{k=1}^n$ satisfy Assumption (C). For convenience, we often use Assumption (C) later, instead of Assumption (A).

THEOREM 2.6. *Let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ be paired by complex conjugation. Then, for $P \in P_n^*(a_1, a_2, \ldots, a_n)$, we have*

$$\|P'\|_{[-1,1]} \leq 2\|B_n\|_{[-1,1]} \tag{2.11}$$

and

$$|P'(x)| \leq \left(x^2(1-x^2)^{-1} + (\|B_n\|_{[-1,1]} - 1)^2\right)^{1/2} \tag{2.12}$$

for $-1 < x < 1$. Furthermore, if $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$ satisfy Assumption (A), then

$$2\Big(\frac{1-\gamma}{1+\gamma}\Big)^5 (n-2) \leq \sup_{P \in P_n^*(a_1,a_2,\ldots,a_n)} \|P'\|_{[-1,1]} \leq 2\frac{1+\gamma}{1-\gamma} n, \tag{2.13}$$

for $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$, $n = 2, 3 \ldots$, where

$$P_n^*(a_1, a_2, \ldots, a_n) := \left\{P \in P_n(a_1, a_2, \ldots, a_n) : |P(x)| \leq \sqrt{1-x^2}, x \in [-1,1]\right\}.$$

THEOREM 2.7. *Let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ be paired by complex conjugation. Then, for $P \in P_{n-1}^{**}(a_1, a_2, \ldots, a_n)$, we have*

$$(1-x^2)|P'(x)| \leq 2\|B_n\|_{[-1,1]}, \quad x \in [-1,1] \tag{2.14}$$

and

$$\|P'\|_{[-1,1]} \leq 2\|B_n\|_{[-1,1]}^3. \tag{2.15}$$

Furthermore, if $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$ satisfy Assumption (A), then (2.14) is sharp to constant and

$$\frac{1}{3}\left(\Big(\frac{1-\gamma}{1+\gamma}\Big)^3 n^3 - \frac{(1+\gamma)^2 + 2\gamma}{1-\gamma^2} n\right) \leq \sup_P \|P'\|_{[-1,1]} \leq 2\Big(\frac{1+\gamma}{1-\gamma}\Big)^3 n^3, \tag{2.16}$$

where the supremum is taken for $P \in P_{n-1}^{**}(a_1, a_2, \ldots, a_n)$, $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$, and

$$P_{n-1}^{**}(a_1, a_2, \ldots, a_n) := \left\{P \in P_{n-1}(a_1, a_2, \ldots, a_n) : \sqrt{1-x^2}|P(x)| \leq 1, x \in [-1,1]\right\}.$$

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REMARK. When $a_k \longrightarrow \pm\infty$ (that means $\gamma = 0$), it would be interesting to compare our Theorems 2.6–2.7 with Theorems A–B.

3**. Lemmas.** First we modify Rahman's argument (*cf.* [13]) to prove a Schur-type inequality.

LEMMA 3.1 (SCHUR-TYPE INEQUALITY). *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ *be paired by complex conjugation. Then*

$$\|P\|_{[-1,1]} \le \|B_n\|_{[-1,1]} \left\|\sqrt{1-x^2}P(x)\right\|_{[-1,1]} \tag{3.1}$$

holds for every $P \in \mathcal{P}_{n-1}(a_1, a_2, \dots, a_n)$*.*

PROOF. We may assume that $\sqrt{1-x^2}|P(x)| \le 1$, we will prove $\|P\| \le \|B_n\|_{[-1,1]}$.

It is easy to see that our hypothesis implies that $\sin t\, P(\cos t) \in \mathcal{T}_n(a_1, a_2, \dots, a_n)$ and $|\sin t\ P(\cos t)| \le 1$. Applying Bernstein-Szegő inequality (1.10) for $\sin t\ P(\cos t)$, we then have

$$\tilde{B}_n^2(t)\sin^2 tP^2(\cos t) + \left(\cos tP(\cos t) + \sin t\left\{\frac{d}{dt}P(\cos t)\right\}\right)^2 \le \tilde{B}_n^2(t_0).$$

Let t_0 be a maximum point of $|P(\cos t)|$, that is, $|P(\cos t_0)| = \|P(\cos t)\|$. We then have that $\frac{d}{dt}\{P(\cos t)\}|_{t=t_0} = 0$. Therefore,

$$\tilde{B}_n^2(t_0)\sin^2 t_0P^2(\cos t_0) + \cos^2 t_0P^2(\cos t_0) \le \tilde{B}_n^2(t_0), \tag{3.2}$$

or

$$\left(\tilde{B}_n^2(t_0) - 1\right)\sin^2 t_0P^2(\cos t_0) + P^2(\cos t_0) \le \tilde{B}_n^2(t_0). \tag{3.3}$$

We distinguish two cases: (i) $\tilde{B}_n(t_0) \ge 1$ and (ii) $\tilde{B}_n(t_0) < 1$. In the first case, (3.3) implies that $|P(\cos t_0)| \le \tilde{B}_n(t_0) \le \|B_n\|_{[-1,1]}$.

In the second case, (3.2) implies that

$$P^2(\cos t_0) + \left(\frac{1}{\tilde{B}_n^2(t_0)} - 1\right)\cos^2 t_0P^2(\cos t_0) < 1,$$

hence, $|P(\cos t_0)| < 1$. Also, it is easy to show that $\|B_n\|_{[-1,1]} \ge 1$. Thus, we still have $|P(\cos t_0)| \le \|B_n\|_{[-1,1]}$. Therefore, combining cases (i) and (ii), we complete the proof Lemma 3.1. ∎

REMARK. For real poles case $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$, [3, E.8, p. 337] also showed (3.1) by using an entirely different way.

We now make an observation about the Bernstein factor $B_n(x)$.

LEMMA 3.2. *Let* $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$ *and* $B_n(x)$ *be defined by (1.9). Then* $B_n(x)$ *is a convex function on* $[-1,1]$ *and its maximum on* $[-1,1]$ *is always attained at* ± 1*:*

$$\begin{aligned}\|B_n\|_{[-1,1]} &= \max\{B_n(-1), B_n(1)\} \\ &= \max\left\{\sum_{k=1}^n \frac{1-c_k}{1+c_k}, \sum_{k=1}^n \frac{1+c_k}{1-c_k}\right\} \ge n.\end{aligned} \tag{3.4}$$

PROOF. Since

$$B_n(x) = \sum_{a_k>0} \frac{\sqrt{a_k^2-1}}{a_k - x} + \sum_{a_k<0} \frac{\sqrt{a_k^2-1}}{-a_k + x},$$

where $\sqrt{a_k^2-1}$ denotes the principal square root of a_k^2-1. Then we can quickly show that $B_n''(x) > 0$ on $[-1,1]$, this implies that $B_n(x)$ is a convex function on $[-1,1]$. Note that $B_n(x) > 0$, so the first equality follows. Note that

$$n^2 = \Big(\sum_{k=1}^n \sqrt{d_k}\frac{1}{\sqrt{d_k}}\Big)^2 \le \sum_{k=1}^n d_k \sum_{k=1}^n \frac{1}{d_k}$$

for any $d_k > 0$. Hence, we may also prove the last inequality in (3.4). ∎

REMARK. In general, Lemma 3.2 does not hold for $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$. For example, taking $a_1 = i, a_2 = -i$, then it is easy to show that

$$B_2(x) = \frac{2\sqrt{2}}{x^2+1},$$

which is not a convex function and $\|B_2\|_{[-1,1]} = B_2(0)$.

Next lemma gives a sufficient condition which guarantees $B_n(x)$ to be asymptotic to n.

LEMMA 3.3. *Let $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ satisfy Assumption (A). Then*

$$\frac{1-\gamma}{1+\gamma}n \le B_n(x) \le \frac{1+\gamma}{1-\gamma}n, \quad x \in [-1,1], \tag{3.5}$$

PROOF. It is directly from (2.4). ∎

4. Proofs of Theorems 2.1–2.5.

PROOF OF THEOREM 2.1. Since the repeated poles in $P_n(a_1, a_2, \ldots, a_n)$ are allowed, and if we denote $a_{n+1} := a_1, \ldots, a_{2n} := a_n$, then we can consider

$$P' \in P_{2n-1}(a_1, a_2, \ldots, a_{2n}).$$

Thus, by Lemma 3.1 we have

$$\|P'\|_{[-1,1]} \le \|B_{2n}\|_{[-1,1]} \Big\|P'(x)\sqrt{1-x^2}\Big\|_{[-1,1]}. \tag{4.1}$$

But, in this case, one can check that

$$B_{2n}(x) = 2B_n(x).$$

Hence, combining (4.1) and (1.11) we conclude that

$$\|P'\|_{[-1,1]} \le 2\|B_n\|_{[-1,1]}^2 \|P(x)\|_{[-1,1]}.$$

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Next we will show the left-side inequality in (2.2). We use T_n and U_n to denote the Chebyshev polynomials of the first and second kinds associated with $\mathcal{P}_n(a_1, \ldots, a_n)$, respectively. Note that (*cf.* [5])

$$U_n^2(x) = \frac{1 - T_n^2(x)}{1 - x^2},$$

then we can quickly get that

$$U_n^2(\pm 1) = |T_n'(\pm 1)| = |B_n(\pm 1) U_n(\pm 1)|,$$

this implies

$$T_n'(1) = \bigl(B_n(1)\bigr)^2, \quad |T_n'(-1)| = \bigl(B_n(-1)\bigr)^2, \tag{4.2}$$

Hence, by taking $P := T_n \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$ and using Lemma 3.2, we can easily show the left-side inequality in (2.2). (2.3) follows Lemma 3.2. ∎

REMARK. If $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1, 1]$, then (2.2) can also be expressed as

$$\max\{|T_n'(-1)|, |T_n'(1)|\} \le \sup_{0 \ne P} \frac{\|P'\|_{[-1,1]}}{\|P\|_{[-1,1]}} \le 2 \max\{|T_n'(-1)|, |T_n'(1)|\}, \tag{4.3}$$

where T_n is the Chebyshev polynomial of the first kind associated with $\mathcal{P}_n(a_1, \ldots, a_n)$, and the supremum is taken for $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$.

PROOF OF THEOREM 2.2. We prove it by induction on m. The case of $m = 1$ is from Theorem 2.1. Suppose that (2.5) is true for $m = k$, that is

$$\|P^{(k)}\|_{[-1,1]} \le k!\,(k+1)!\,\|B_n\|_{[-1,1]}^{2k} \|P\|_{[-1,1]} \tag{4.4}$$

for every $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$.

Let $a_{in+1} = a_1, \ldots, a_{(i+1)n} = a_n$, $i = 1, \ldots, k+2$, then we can consider

$$P^{(k+1)} \in \mathcal{P}_{(k+2)n-1}(a_1, a_2, \ldots, a_{(k+2)n})$$

as in the proof of Theorem 2.1. Similarly, we have

$$B_{(k+2)n}(x) = (k+2) B_n(x),$$

where $B_{(k+2)n}(x)$ is the corresponding Bernstein factor with respect to $\mathcal{P}_{(k+2)n}(a_1, a_2, \ldots, a_{(k+2)n})$. Now using (4.4) and applying the Schur-type inequality (3.1) and the Bernstein-type inequality (1.11) for $P^{(k+1)}$, we have

$$\begin{aligned}
\|P^{(k+1)}\|_{[-1,1]} &\le \|B_{(k+2)n}\|_{[-1,1]} \Bigl\|\sqrt{1-x^2} P^{(k+1)}(x)\Bigr\|_{[-1,1]} \\
&\le (k+2) \|B_n\|_{[-1,1]} \|B_{(k+1)n}\|_{[-1,1]} \|P^{(k)}\|_{[-1,1]} \\
&\le (k+1)!\,(k+2)!\,(\|B_n\|_{[-1,1]})^{2(k+1)} \|P\|_{[-1,1]},
\end{aligned}$$

hence (2.5) holds for $m = k+1$ and we complete its proof. ∎

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PROOF OF COROLLARY 2.3. Since

$$R(x) := 2P(x) - \Big(\max_{-1\le x\le 1} P(x) - \min_{-1\le x\le 1} P(x)\Big) \in \mathcal{P}_n(a_1, a_2, \ldots, a_n),$$

then by Theorem 2.1 we have

$$\|P'\|_{[-1,1]} = \frac{1}{2}\|R'\|_{[-1,1]} \le \|B_n\|^2_{[-1,1]}\Big(\max_{-1\le x\le 1} P(x) - \min_{-1\le x\le 1} P(x)\Big).$$

It is easy to show that (2.6) and (2.7) are sharp to constant by taking $P(x) = T_n(x)$ and $P(x) = \frac{1+T_n(x)}{2}$ respectively. ∎

PROOF OF THEOREM 2.4. First we prove it for $p = \infty$. For given $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$, we may suppose that $|P(y)| = \|P\|_{[-1,1]}$, where $y \in [-1,1]$. Also we denote $\lambda_n := 2\|B_n\|^2_{[-1,1]}$. Then by Theorem 2.1 and the Mean Value Theorem we can quickly get that

$$|P(x)| > \frac{1}{2}P(y) = \frac{1}{2}\|P\|_{[-1,1]}$$

for every $x \in I := \{t : |t-y| \le \frac{1}{2\lambda_n}, t \in [-1,1]\}$. Thus

$$\|P\|_q^q \ge \int_I |P(t)|^q\,dt \ge \frac{1}{2^q}\|P\|^q_{[-1,1]}\frac{1}{2\lambda_n},$$

it follows that

$$\|P\|_{[-1,1]} \le 2\{2\lambda_n\}^{1/q}\|P\|_q. \tag{4.5}$$

On the other hand, for $0 < q < p < \infty$, by (4.5) we deduce that

$$\begin{aligned}\|P\|_p^p &= \int_{-1}^{1} |P(t)|^{p-q+q}\,dt \le \|P\|^{p-q}_{[-1,1]}\|P\|_q^q\\ &\le \{2\lambda_n\}^{p-q/q}\|P\|_q^{p-q}\|P\|_q^q\end{aligned}$$

yields (2.8). ∎

PROOF OF THEOREM 2.5. Let $\{R_k^*\}_{k=0}^n$ be the orthonormal system with respect to the rational system (2.2) in the norm $\|\cdot\|_{2,v}$, then we know that (*cf.* [5, Theorem 4.7])

$$R_0^* = \frac{1}{\sqrt{\pi}}, \quad R_n^* = \sqrt{\frac{2}{\pi(1-c_n^2)}}(T_n + c_n T_{n-1}).$$

We may denote $P := \sum_{k=0}^n \alpha_k R_k^*$ and assume that $\|P\|_{2,v} = 1$, which implies $\sum_{k=0}^n \alpha_k^2 = 1$. Moreover, note that (*cf.* [3]) $\|T_n\|_{[-1,1]} = 1$ and by the Cauchy's inequality, we then have

$$\begin{aligned}P^2 &\le \sum_{k=0}^n \alpha_k^2 \sum_{k=0}^n (R_k^*)^2 \le \frac{1}{\pi} + \sum_{k=1}^n \left(\sqrt{\frac{2}{\pi(1-c_k^2)}}(1+|c_k|)\right)^2\\ &= \frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|},\end{aligned}$$

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it implies (2.9). Taking

$$P=\frac{1}{\left(\frac{1}{\pi}+\frac{2}{\pi}\sum_{k=1}^{n}\frac{1+\operatorname{sgn}(c_k)c_k}{1-\operatorname{sgn}(c_k)c_k}\right)^{1/2}}\left\{\frac{1}{\sqrt{\pi}}R_0^*+\sum_{k=1}^{n}\left(\frac{2}{\pi}\frac{1+\operatorname{sgn}(c_k)c_k}{1-\operatorname{sgn}(c_k)c_k}\right)^{1/2}R_k^*\right\},$$

It is easy to show that (2.9) is best possible under the hypotheses. ■

5. Proofs of Theorems 2.6–2.7.

PROOF OF THEOREM 2.6. Our hypothesis implies that

$$P(x)=\frac{(1-x^2)P_{n-2}(x)}{\prod_{k=1}^{n}(x-a_k)}=\sqrt{1-x^2}\frac{\sqrt{1-x^2}P_{n-2}(x)}{\prod_{k=1}^{n}(x-a_k)}:=\sqrt{1-x^2}Q(x),$$

and $|Q(x)|\le 1$.

Since

$$\begin{aligned}|P'(x)| &= \left|-x(1-x^2)^{-1/2}Q(x)+\sqrt{1-x^2}Q'(x)\right| \\ &\le |x|(1-x^2)^{-1/2}|Q(x)|+\left|\sqrt{1-x^2}Q'(x)\right|\end{aligned} \tag{5.1}$$

and $Q(\cos t)\in \mathcal{T}_n(a_1,a_2,\ldots,a_n)$, by the Bernstein-Szegő-type inequality (1.10) for $\mathcal{T}_n(a_1,a_2,\ldots,a_n)$ we have

$$\left|\frac{d}{dt}\{Q(\cos t)\}\right|\le \tilde{B}_n(t),$$

that is,

$$\left|\sqrt{1-x^2}Q'(x)\right|\le B_n(x). \tag{5.2}$$

Note that $(1-x^2)^{-1/2}Q(x)\in \mathcal{P}_{n-1}(a_1,a_2,\ldots,a_n)$, moreover,

$$|(1-x^2)^{1/2}(1-x^2)^{-1/2}Q(x)|=|Q(x)|\le 1,\quad -1\le x\le 1.$$

Thus, Lemma 3.1 yields

$$|(1-x^2)^{-1/2}Q(x)|\le \|B_n\|_{[-1,1]} \tag{5.3}$$

for $-1<x<1$. Combining (5.1), (5.3) and (1.11), we obtain (2.11).

Next we prove (2.13). Obviously, the right-side inequality in (2.13) follows from (2.10) and Lemma 3.3.

We let

$$P(x):=\frac{(1-x^2)U_{n-2}(x)}{(x-a_{n-1})(x-a_n)}\big(\operatorname{sgn}(a_{n-1})-a_{n-1}\big)\big(\operatorname{sgn}(a_n)-a_n\big)$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind associated with $\mathcal{P}_n(a_1,\ldots,a_n)$. Since (*cf.* [5]) $\sqrt{1-x^2}|U_{n-2}(x)|\le 1$ for $-1\le x\le 1$, thus, $P\in\mathcal{P}_n(a_1,a_2,\ldots,a_n)$ and $|P(x)|\le\sqrt{1-x^2}$. Note that (*cf.* [5, Theorem 2.1])

$$\tilde{T}_n'(t)=-\tilde{B}_n(t)\tilde{U}_n(t),\ \tilde{U}_n'(t)=\tilde{B}_n(t)\tilde{T}_n(t),\quad t\in\mathbb{R}$$

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where $\tilde{B}_n(t) := B_n(\cos t)$. Hence,

$$T_n'(x) = B_n(x)U_n(x), \tag{5.4}$$

and

$$U_n'(x) = \frac{xU_n(x) - B_n(x)T_n(x)}{1 - x^2}. \tag{5.5}$$

Further, in this case, we can quickly get from (2.4) that

$$\begin{aligned} |P'(1)| &\geq |U_{n-2}(1) + B_{n-2}(1)| \frac{(|a_{n-1}| - 1)(|a_n| - 1)}{(|a_{n-1}| + 1)(|a_n| + 1)} \\ &= 2B_{n-2}(1)\Big(\frac{1 - |c_{n-1}|}{1 + |c_{n-1}|}\Big)^2 \Big(\frac{1 - |c_n|}{1 + |c_n|}\Big)^2 \\ &\geq 2\Big(\frac{1-\gamma}{1+\gamma}\Big)^5 (n-2). \end{aligned} \tag{5.6}$$

Similarly, we have

$$|P'(-1)| \geq 2\Big(\frac{1-\gamma}{1+\gamma}\Big)^5 (n-2). \tag{5.7}$$

Hence, we have shown the left-side inequality in (2.13).

Also, since $Q(\cos t) \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$, hence using Rahman's argument (*cf.* [13]), Bernstein-Szegő inequality (1.10) and (5.1) we have

$$\begin{aligned} |P'(x)| &\leq |x|(1-x^2)^{-1/2}|Q(x)| + \Big|\sqrt{1-x^2}\,Q'(x)\Big| \\ &\leq |x|(1-x^2)^{-1/2}|Q(x)| + (\|B_n\|_{[-1,1]} - 1)\big(1 - |Q(x)|^2\big)^{1/2} \\ &\leq \max_{-1\leq y\leq 1}\{|x|(1-x^2)^{-1/2}y + (\|B_n\|_{[-1,1]} - 1)(1-y^2)^{1/2}\} \\ &\leq \big(x^2(1-x^2)^{-1} + (\|B_n\|_{[-1,1]} - 1)^2\big)^{1/2}, \end{aligned}$$

this implies (2.12). ∎

PROOF OF THEOREM 2.7. From our hypothesis, we know that $\sin tP(\cos t) \in \mathcal{T}_n(a_1, a_2, \ldots, a_n)$ and $|\sin tP(\cos t)| \leq 1$. Then, applying the Bernstein-Szegő inequality (1.10) to $\sin tP(\cos t)$, we have

$$|\cos tP(\cos t) - \sin^2 tP'(\cos t)| \leq \tilde{B}_n(t),$$

and combining Lemma 3.1 we get

$$(1-x^2)|P'(x)| = |\sin^2 tP'(\cos t)| \leq \tilde{B}_n(t) + \|P\|_{[-1,1]} \leq 2\|B_n\|.$$

Next we show that (2.14) is sharp to constant under the hypothesis. Let $P(x) := U_n(x)$, the Chebyshev polynomial of the second kind associated with $\mathcal{P}_n(a_1, \ldots, a_n)$, since (*cf.* [5]) $T_n^2(x) + (1-x^2)U_n^2(x) = 1$, then taking $x = x_k$, of the zeros $U_n(x)$, we then have

$$(1-x_k^2)|U_n'(x_k)| = |B_n(x_k)T_n(x_k)| = B_n(x_k).$$

Hence (2.15) is sharp to constant by Lemma 3.3.

Combining the Markov-type inequality (2.1) and the Schur-type inequality (3.1), we can easily show that

$$\|P'\|_{[-1,1]} \le 2\|B_n\|^2_{[-1,1]}\|P\|_{[-1,1]} \le 2\|B_n\|^3_{[-1,1]},$$

hence, (2.14) follows.

On the other hand, by (5.5) we conclude that

$$U_n'(\pm 1) = -\frac{U_n(\pm 1) + U_n'(\pm 1) - B_n'(\pm 1) - B_n^2(\pm 1)U_n(\pm 1)}{2},$$

this implies

$$|U_n'(1)| = \frac{1}{3}|B_n^3(1) + B_n'(1) - B_n(1)| > \frac{1}{3}\big(B_n^3(1) - |B_n'(1)| - B_n(1)\big).$$

Similarly, we have

$$|U_n'(-1)| > \frac{1}{3}\big(B_n^3(-1) - |B_n'(-1)| - B_n(-1)\big).$$

But, for $|c_k| \le \gamma < 1$, we have

$$|B_n'(x)| \le \sum_{k=1}^{n} \frac{\sqrt{a_k^2 - 1}}{(|a_k| - 1)^2} \le d(\alpha)n, \tag{5.8}$$

where $d(\alpha)$ is some positive constant depending only on α. Hence combining Lemma 3.3 and (5.8) we show that

$$\|U_n'(x)\|_{[-1,1]} \ge \max\{|U_n'(1)|, |U_n'(-1)|\} \ge \frac{1}{3}\left(\left(\frac{1-\gamma}{1+\gamma}\right)^3 n^3 - \frac{(1+\gamma)^2 + 2\gamma}{1-\gamma^2}n\right),$$

but $U_n \in P^{**}_{n-1}(a_1, a_2, \ldots, a_n)$, so the left-side inequality in (2.24) follows. The right-side inequality in (2.16) follows from (2.14) and Lemma 3.3. ■

6. Remarks.

REMARK 1. From the above theorems, the estimate of $\|P'\|_{[-1,1]}$ (Markov-type inequality) and the pointwise estimate of $|P'(x)|$ (Bernstein-type inequality) are dependent on the given poles $\{a_k\}_{k=1}^n$ for $P \in P_n(a_1, a_2, \ldots, a_n)$. However, Borwein, Erdélyi and Zhang [5] observed the following result:

$$|P'(0)| \le n\|P\|_{[-1,1]} \tag{6.1}$$

for $P \in P_n(a_1, a_2, \ldots, a_n)$ and real poles $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1, 1]$. Here we also prove

THEOREM 6.1. *Let* $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1, 1]$. *Then*

$$|P'(x)| \le \frac{n}{1-x^2}\|P\|_{[-1,1]} \tag{6.2}$$

for $P \in P_n(a_1, a_2, \ldots, a_n)$ *and* $x \in (-1, 1)$.

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PROOF. Let $Q(y) := P(\frac{x+y}{1+xy})$ for given $x \in (-1,1)$, then it is easy to see that $Q \in \mathcal{P}_n\big(b_1(x), b_2(x), \ldots, b_n(x)\big)$ and $\|Q\|_{[-1,1]} = \|P\|_{[-1,1]}$, where $b_k(x) = (a_k - x)/(1 - xa_k) \in \mathbb{R} \setminus [-1,1]$. Moreover, by (6.1) we have

$$(1-x^2)|P'(x)| = |Q'(0)| \le n\|Q\|_{[-1,1]} = n\|P\|_{[-1,1]},$$

which is nothing but (6.2). ∎

Theorem 6.1 improves upon a result of Borwein, Erdélyi and Zhang [5, Corollary 3.7] which had $n/(1-|x|)$ instead of $n/(1-x^2)$ on the right-hand side of (6.2).

Hence, it is easy to obtain the following Markov-type inequality by the exactly same way as the proof of [5, Theorem 3.5]:

COROLLARY 6.2. *Let* $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$ *be real poles and* $\{c_k\}_{k=1}^n$ *be defined by (1.8). Then*

$$\|P'\|_{[-1,1]} \le \frac{2\sqrt{n}}{\sqrt{n}+\sqrt{n-1}} \left(\sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|}\right)^2 \|P\|_{[-1,1]} \le \sqrt{\frac{n}{n-1}} \left(\sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|}\right)^2 \|P\|_{[-1,1]} \tag{6.3}$$

hold for every $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$, $n = 1, 2, \ldots$.

REMARK 2. Using the argument of the proof in Theorem 2.2, it is not difficult to show the following Bernstein-type inequality for high derivatives with respect to $\mathcal{T}_n(a_1, a_2, \ldots, a_n)$:

THEOREM 6.3. *Let* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ *with its nonreal elements being complex conjugation, and* $\tilde{B}_n(t)$ *be defined by (1.9). Then*

$$\|P^{(m)}\| \le m!\, \|\tilde{B}_n\|^m \|P\| \tag{6.4}$$

for $P \in \mathcal{T}_n(a_1, a_2, \ldots, a_n)$, *where* $\|P\| = \max_{\tau \in \mathbb{R}} |P(\tau)|$.

REMARK 3. After we wrote this paper, we have shown that Assumption (A) is equivalent to say that $\{R_n^*\}$ are uniformly bounded for all n, where $\{R_n^*\}$ be the orthonormal system with respect to the rational system (2.2) in the norm $\|\cdot\|_{2,\nu}$. We here omit its proof.

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