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On the Existence of the Graded Exponent for Finite Dimensional $\mathbb{Z}_p$-graded Algebras

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Abstract. Let F be an algebraically closed field of characteristic zero, and let A be an associative unitary F-algebra graded by a group of prime order. We prove that if A is finite dimensional then the graded exponent of A exists and is an integer.

1 Introduction

If A is an algebra, possibly nonassociative, it is acceptable to translate its complexity considering how many polynomial relations are satisfied by its elements: less relations mean more difficult computations with algebra elements. If the base field has characteristic zero, the so called multi-linear relations are enough to describe the complete situation, thanks to a typical process of multi-linearization of the polynomial relations satisfied by the algebra. Within this general setting, one can attach to A a sequence of nonnegative integers $\big(c_n(A)\big)_{n\in\mathbb{N}}$, the codimension sequence of A. This in some sense provides a measure of how many relations hold in A. A classic result states that if an associative algebra A does satisfy at least a nontrivial polynomial relation (a *PI-algebra*) then there exists a constant $d \in \mathbb{N}$ such that $c_n(A) \leqslant d^n$ for all $n \in \mathbb{N}$ [Re1]. The existence of an exponential bound for the codimension sequence easily fails not only for nonassociative algebras, but also for the associative ones. For example, for the free associative algebra $A = F\langle x, y\rangle$, one has $c_n(A) = n!$.

If the codimension sequence of an algebra A is exponentially bounded, it makes sense to ask whether the limit $\lim_n\big(\sqrt[n]{c_n(A)}\big)$ exists, and to try to compute it. This was achieved by Giambruno and Zaicev for associative PI-algebras [GZ1], [GZ2]. They proved not only that the limit does exist, but also that it is a nonnegative integer, called the *exponent* of A, or the *PI-exponent* of A, confirming a conjecture posed by Amitsur. Similar attempts were made for other classes of algebras; what is known is that the *Lie-exponent* of any finite dimensional Lie algebra does exist and is also an integer [Za]. The same conclusions hold for certain simple Jordan algebras of small dimension [GRZ]. These positive answers go together with the partial or negative ones: it has been proved [Vo1], [Vo2] that if L is an infinite dimensional Lie algebra then the Lie codimension sequence may have over-exponential growth. Even when exponentially bounded, the limit $\lim_n\big(\sqrt[n]{c_n^L(A)}\big)$ may be not an integer [MZ]. A new scale to measure the rate of growth of the codimensions of Lie algebras is contained

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in [Pe]. Among the other results, it was shown in [BD] that the finite dimensional condition implies an exponentially bounded codimension sequence, also for nonassociative algebras. Nevertheless it was shown in [GMZ] that for any real number $\alpha > 1$ there exists a nonassociative algebra of exponent α.

In PI-theory a topic of increasing interest is the study of group graded algebras. Apart from their own interesting features, group graded algebras may provide significative information on quite general questions. This is the case, for instance, for the fundamental results of Kemer on PI-algebras, in which $\mathbb{Z}_2$-graded algebras were involved [Ke]. In certain circumstances, actually, this situation is equivalent to have a group acting as a group of algebra automorphisms. A G-codimension sequence $\left(c_n^G(A)\right)$ may be defined in this case, too, and with light assumptions it has exponential growth [GR], so questions about the existence of $\lim_n\left(\sqrt[n]{c_n^G(A)}\right)$ make sense. Indeed, it has been proved in [GZ3], [BGP] that this exponent does exist when $G \cong \mathbb{Z}_2$, and G acts on a finite dimensional associative algebra A either as a group of automorphisms ($\mathbb{Z}_2$-graded algebras) either of anti-automorphisms (algebras with an involution). In the present paper, we shall prove that the exponent of an associative finite dimensional algebra graded by a group of prime order exists, as well, and it is an integer.

2 Basics

Throughout the rest of the paper, let F denote an algebraically closed field of characteristic zero, and let the word *algebra* mean an associative unitary F-algebra. If G is any finite group, we say that A is G-graded if there exist F-subspaces A_g (for each $g \in G$) such that $A = \bigoplus_{g\in G} A_g$ and for all $g, h \in G$ it holds $A_g A_h \subseteq A_{gh}$. Each A_g is called a G-homogeneous component of A, and if $a \in A_g$ we say that g is the degree of a, and write $|a| = g$. The G-grading is trivial if $A_e = A$ (e being the identity element of G). If A, B are G-graded algebras, the structure-preserving homomorphisms are the so called G-homomorphisms: an algebra homomorphism $\varphi\colon A \to B$ is a G-homomorphism (or a graded homomorphism) if for all $g \in G$ it is $\varphi(A_g) \subseteq B_g$.

It is possible to define a free object in the class of G-graded algebras, for an assigned group G: consider a countable set of indeterminates $X := \{x_i^g \mid 1 \leqslant i \in \mathbb{N}, g \in G\}$. We shall call i the *name* and g the *G-degree* of x_i^g. Then the map $|\cdot|: X \to G$ defined by $\left|x_i^g\right| = g$ induces a G-grading on the free associative algebra $F\langle X\rangle$ simply setting $|x_1 \cdots x_m| := |x_1| \cdots |x_m|$ for any monomial in the (not necessarily distinct) indeterminates $x_1, \ldots, x_m \in X$. It is customary to denote by $F\langle X \mid G\rangle$ this G-graded algebra, and call it the *free G-graded algebra*. The freeness property is the following: for any G-graded algebra A and any map $\varphi_0\colon X \to A$ such that $|\varphi_0(x)| = |x|$ for all $x \in X$ there exists a unique G-homomorphism φ extending φ_0. We shall often say that such a map φ_0 is an *admissible G-substitution* for A.

The set $T_G(A) \subseteq F\langle X \mid G\rangle$ of polynomials vanishing under any admissible G-substitution is an ideal of $F\langle X \mid G\rangle$, stable under any graded endomorphism of the free graded algebra. We call G-identities its elements.

As usual, we say that a monomial $x_1 \cdots x_n \in F\langle X, G\rangle$ is *G-graded multi-linear of length n* if $\{name\,(x_i) \mid i = 1, \ldots, n\} = \{1, \ldots, n\}$. Explicitly, if $x_i = x_{j_i}^{g_i}$, then

$x_1 \cdots x_n = x_{j_1}^{g_1} \cdots x_{j_n}^{g_n}$ and $(g_1, \dots, g_n) \in G^n$, $\{j_1, \dots, j_n\} = \{1, \dots, n\}$. Let us denote by V_n^G the F-subspace spanned by all the G-graded multi-linear monomials of length n. That is, the span of all monomials $x_{\sigma(1)}^{g_1} \cdots x_{\sigma(n)}^{g_n}$ for $(g_1, \dots, g_n) \in G^n$ and σ running through the symmetric group S_n.

Since char $F = 0$, a standard Vandermonde argument shows that the knowledge of $T_G(A)$ can be reduced to the knowledge of the sets $V_n^G \cap T_G(A)$ for all $n \geqslant 1$.

The symmetric group S_n acts on V_n^G by renaming the indeterminates, and this turns V_n^G into a left S_n-module. Since any $\sigma \in S_n$ sends x_i^g to $x_{\sigma(i)}^g$, σ can be extended diagonally providing a G-endomorphism φ_σ of $F\langle X \mid G\rangle$. Therefore $V_n^G \cap T_G(A)$ is an S_n-submodule of V_n^G, and we are allowed to consider the S_n-module $V_n^G(A) := V_n^G / \big(V_n^G \cap T_G(A)\big)$. The number $c_n^G(A) := \dim_F V_n^G(A)$ is called the *n-th G-codimension* of A, and its character $\chi_n^G(A)$ is called the *n-th graded cocharacter*.

The study of the S_n-structure of $V_n^G(A)$ can be reduced to the study of smaller spaces of multi-linear polynomials. Let us denote $[n] := \{1, \dots, n\}$. We say that a family $\mathcal{G} := \{\mathcal{G}_g \subseteq [n] \mid g \in G\}$ is a *G-partition* of n if

$$\mathcal{G}_g \cap \mathcal{G}_h = \varnothing \quad \text{if } g \neq h \quad \text{and} \quad \bigcup_{g \in G} \mathcal{G}_g = [n];$$

in this case we write $\mathcal{G} \vdash_G n$. Every monomial $m \in V_n^G$ uniquely defines a G-partition of n, namely $\mathcal{G}(m)$, setting

$$\mathcal{G}_g(m) := \{ j \in [n] \mid x_j^g \text{ appears in } m\} \text{ for } g \in G$$

and, for a G-partition $\mathcal{G}$ of n, we define

$$V_n^G(\mathcal{G}) := \operatorname{span}_F \langle m \in V_n^G \mid m \text{ monomial and } \mathcal{G}(m) = \mathcal{G}\rangle$$

Then (see [DV, Lemma 1])

$$V_n^G = \bigoplus_{\mathcal{G} \vdash_G n} V_n^G(\mathcal{G})$$

and

$$V_n^G(A) := \frac{V_n^G}{V_n^G \cap T_G(A)} \cong_{S_n} \bigoplus_{\mathcal{G} \vdash_G n} \frac{V_n^G(\mathcal{G})}{V_n^G(\mathcal{G}) \cap T_G(A)}.$$

Note that any G-partition of n defines a subgroup of $\operatorname{Sym}(n)$, namely

$$H(\mathcal{G}) := \prod_{g \in G} \operatorname{Sym}(\mathcal{G}_g).$$

The action of this subgroup on $V_n^G(\mathcal{G})$ determines a module structure on it. Moreover, if $\mathcal{G}$ and $\mathcal{S}$ are G-partitions of n such that $|\mathcal{G}_g| = |\mathcal{S}_g|$ for each $g \in G$, then $H(\mathcal{G})$ and $H(\mathcal{S})$ define equivalent actions. Therefore, fixing an order in G, say $G = \{g_1, \dots, g_r\}$, we may denote by $V_{n_1,\dots,n_r}^G$ the $H(\mathcal{G})$-module $V_n^G(\mathcal{G})$ where $|\mathcal{G}_{g_i}| = n_i$, and the indeterminates are labeled in the standard way as $x_1^{g_1}, \dots, x_{n_1}^{g_1}$, then $x_{n_1+1}^{g_2}, \dots, x_{n_1+n_2}^{g_2}$ and so on. We shall denote by $\chi_{n_1,\dots,n_r}^G$ the $H(\mathcal{G})$-character of the factor module $V_{n_1,\dots,n_r}^G / \big(V_{n_1,\dots,n_r}^G \cap T_G(A)\big)$ (see [DV]). Recall that the irreducible

$H(\mathcal{G})$-modules are the tensor products $M(\lambda^1)\otimes\cdots\otimes M(\lambda^r)$, where $\lambda^i \vdash n_i$ and $M(\lambda^i)$ is a chosen representative for the isomorphism class of irreducible S_{n_i}-modules associated to λ^i.

By [DV, Theorem 2], given the cocharacter $\chi^G_{n_1,\dots,n_r}(A)$, the graded cocharacter $\chi_n(A,G)$ is known as well. In particular, the degree $c^G_n(A) = \chi_n(A,G)(1)$ and the degrees $c_{n_1,\dots,n_r}(A) = \chi_{n_1,\dots,n_r}(1)$ are related through the formula

$$c^G_n(A) = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1 \cdots n_r} c_{n_1,\dots,n_r}(A). \tag{2.1}$$

In case the number $\exp^G(A) := \lim_n \sqrt[n]{c^G_n(A)}$ does exist, it is called the *G-graded exponent* of A. The aim of this paper is to show that if $G \cong \mathbb{Z}_p$, p prime, then for any finite dimensional G-graded algebra the G-graded exponent of A does exist and it is possible to compute it.

3 Notation

From now on let A denote a finite dimensional G-graded algebra, where G is a cyclic group of order p prime. There is a well understood duality between G-gradings of A and G-actions on A; notice that a G-action, in our hypotheses, is either faithful either trivial. This duality is a general fact for any finite abelian group. We shall recall it for groups of prime order. So, let ζ be a primitive p-th root of the unity in F and assume $G = \langle\gamma\rangle \cong \mathbb{Z}_p$.

- If G acts on A, then define $A_{\gamma^k} := \{a \in A \mid \gamma \cdot a = \zeta^k a\}$. The sets $(A_{\gamma^k})_{k=0,\dots,p-1}$ define a G-grading on A, which is trivial if and only if the action is trivial.
- If a G-grading is given on A, any element of A can be uniquely written in the form $a = \sum_{k=0}^{p-1} a_{\gamma^k}$, with $a_{\gamma^k} \in A_{\gamma^k}$. Then define the action

$$\gamma \cdot a := \sum_{k=0}^{p-1} \zeta^k a_{\gamma^k}.$$

Notice that this action is trivial if and only if the G-grading is so; otherwise, the action is nontrivial and G can be embedded into $\mathrm{Aut}(A)$. Then, viewing $\gamma \in \mathrm{Aut}(A)$, it is an automorphism of A of order p.

In what follows we shall freely pass from one point of view to the other.

An F-subspace $V \subseteq A$ is G-graded or *homogeneous* if $V = \bigoplus_g (V \cap A_g)$. That is: if $V \ni v = \sum_g v_g$ is the (unique) decomposition of v (viewed in A), then all the summands v_g belong to V, as well. Equivalently, V is G-homogenous if and only if V is stable for the G-action on A. This definition applies to subalgebras of A, and to left, right or two-sided ideals of A. The algebra A is G-*simple* if it has no G-homogenous two-sided ideals apart from 0 and A. Of course, a simple algebra is G-simple.

Thinking G as a group acting on A, it is clear that $J = J(A)$, the Jacobson radical of A, is always a G-graded ideal. As a consequence of [SVO, Corollary 2.10 and

Remark 2.11], it has a G-graded semi-direct complement in A, which is a completely reducible (unitary) G-graded subalgebra B. That is, $A = B + J$, a direct sum as F-vector spaces, and B is a maximal semisimple G-homogeneous subalgebra (not an ideal, however).

Let I be a summand occurring in the decomposition of B into a direct sum of minimal two-sided ideals; then $\gamma(I)$ is a minimal two-sided ideal of B too. If $\gamma(I) = I$ then I is G-homogenous, so it is a G-graded and simple algebra. If $\gamma(I) \neq I$ then $\gamma(I) \cap I = 0$; then, since $G \cong \mathbb{Z}_p$, the orbit $\{\gamma^k(I) \mid k = 0, \dots, p-1\}$ has p elements, and each of them is a minimal two-sided ideal of B, hence occurring in the decomposition of B. Hence $\bigoplus_{k=0}^{p-1} \gamma^k(I)$ is a G-simple subalgebra of B and $B = C_1 \oplus \cdots \oplus C_s$ is a direct sum of G-simple graded subalgebras C_i of B, for some $s \geqslant 1$.

We are going to show the candidate number to be $\exp^G(A)$. Let $\mathcal{S}$ be the set of all possible sequences $(D_1, \dots, D_k)$ for $k \in [s]$, $D_1, \dots, D_k \in \{C_1, \dots, C_s\}$ pairwise distinct, satisfying

$$D_1 J D_2 J \cdots J D_k \neq 0,$$

and let

$$d := \max\{\dim_F(D_1 + \cdots + D_k) \mid (D_1, \dots, D_k) \in \mathcal{S}\}.$$

We shall prove that d is actually the G-exponent of A. From now on, we assume $(C_1, \dots, C_k) \in \mathcal{S}$ and $\dim(C_1 + \cdots + C_k) = d$. Moreover, since A is finite dimensional, J is nilpotent. We fix $l \in \mathbb{N}$ such that $J^l \neq 0$ and $J^{l+1} = 0$. Finally, we fix $B := C_1 \oplus \cdots \oplus C_s$ and $B_k := C_1 \oplus \cdots \oplus C_k$. Then $A = B + J$ and $\dim_F B_k = d$.

The notion of *alternating* polynomial is needed thoroughly. Recall that the symmetric group $\mathrm{Sym}(X)$ acts on X by *renaming* the indeterminates, and we shall denote by $\sigma(f)$ the image of the polynomial $f \in F\langle X\rangle$ under the renaming action of $\sigma \in \mathrm{Sym}(X)$.

Definition 3.1 Let $f = f(\mathcal{X})$ be a polynomial in the indeterminates of $\mathcal{X} \subseteq X$. If $\mathcal{Y} \subseteq \mathcal{X}$ and f is multi-linear on $\mathcal{Y}$, we say that

- *f is alternating* on the set $\mathcal{Y}$ if for any $\sigma \in \mathrm{Sym}(\mathcal{Y})$ it is $\sigma f = (-1)^\sigma f$;
- *f is symmetric* on $\mathcal{Y}$ if for any $\sigma \in \mathrm{Sym}(\mathcal{Y})$ it is $\sigma f = f$.

A polynomial may be alternating or symmetric on several sets of variables. The direct generalization is that if $\mathcal{Y}_i$ $(i \leqslant t)$ are disjoint subsets of $\mathcal{X}$ then f is alternating (resp. symmetric) on the sets $\mathcal{Y}_i$ if for any $\sigma \in \mathrm{Sym}(\mathcal{Y}_1) \cdots \mathrm{Sym}(\mathcal{Y}_k) \leqslant \mathrm{Sym}(\bigcup_{i=1^k} \mathcal{Y}_i)$ it holds $\sigma f = (-1)^\sigma f$ (resp. $\sigma f = f$).

The most basic property of an alternating polynomial is the following remark.

Remark 3.2 Let $f = f(\mathcal{X})$ be alternating on $\mathcal{Y} \subseteq \mathcal{X}$. If $\dim_F(A) = m < |\mathcal{Y}|$ then $f \in T(A)$. In our settings ($A = B + J$, $J^{l+1} = 0$, $B = C_1 \oplus \cdots \oplus C_s$ and $d = \dim(C_1 \oplus \cdots \oplus C_k)$), we record the following direct consequence.

Lemma 3.3 *Let $\mathcal{Y}^g \subseteq \mathcal{X}^g \subseteq X^g$ for $g \in G$, $\mathcal{X} = \bigcup_g \mathcal{X}^g$ and let $f = f(\mathcal{X})$ be a multi-linear G-graded polynomial, alternating on the sets $\mathcal{Y}^g$ $(g \in G)$. If $\sum_g |\mathcal{Y}^g| > d$ then for any A-valued G-substitution φ such that $\varphi(\bigcup_g \mathcal{Y}^g) \subseteq B$ it is $\varphi(f) = 0$.*

Proof Since f is multi-linear it is enough to check the standard G-substitutions. So, let $\mathcal{B}$ and $\mathcal{J}$ be G-homogeneous bases for B and J. If $\varphi(\mathcal{X}) \subseteq \mathcal{B}$ then $\varphi(f) = 0$ unless $\varphi(\mathcal{X}) \subseteq C_i$ for some i, because $C_iC_j = 0$ if $i \neq j$. But even in this case, since

$$\sum_g \left|\mathcal{B} \cap C_i^g\right| = \dim_F C_i \leqslant d < \sum_g |\mathcal{Y}^g|$$

there must be $g \in G$ such that $\left|\mathcal{B} \cap C_i^g\right| < |\mathcal{Y}^g|$. Since $\left|\mathcal{B} \cap C_i^g\right| = \dim_F C_i^g$ and f is alternating on $\mathcal{Y}^g$ it is $\varphi(f) = 0$. Therefore at least one element of $\mathcal{X}$ has to be mapped in $\mathcal{J}$. Then the maximality of d provides $\varphi(f) = 0$ (further details can be found in [GZ1, Lemma 3]). ■

Alternating and symmetric polynomials arise naturally when studying the irreducible characters of the symmetric groups. In our settings, we are interested in the character $\chi^G_{n_1,\dots,n_p}(A)$, so let $(n_1, \dots, n_p) \in \mathbb{N}^p$ satisfy $n_1 + \cdots + n_p = n$. In order to keep the notation as simple as possible, we denote the (weak) composition $(n_1, \dots, n_p)$ of n by $\underline{n}$. If $\lambda^i \vdash n_i$ $(i = 1, \dots, p)$, the sequence $(\lambda^1, \dots, \lambda^p)$ will be denoted by $\underline{\lambda}$. We write also $\underline{\lambda} \vdash \underline{n}$. Moreover, we shall denote by $H_{\underline{\lambda}}$ the subgroup of S_n determined by $\underline{n}$. The irreducible $H_{\underline{n}}$-module associated to $\underline{\lambda}$ will be denoted by

$$M_{\underline{\lambda}} := M(\lambda^1) \otimes \cdots \otimes M(\lambda^p)$$

and $\chi_{\underline{\lambda}}$ will denote its character. Coherently, a multi-tableau on $\underline{\lambda}$ is a filling of λ^1 with the names $\{1, \dots, n_1\}$, of λ^2 with names $\{n_1 + 1, \dots, n_1 + n_2\}$ and so on. Hence we shall denote by $T_{\underline{\lambda}}$ such a multi-tableau, that is $T_{\underline{\lambda}} = (T_{\lambda^1}, \dots, T_{\lambda^p})$. Let $e_{T_{\lambda^i}} = r_{T_{\lambda^i}} c_{T_{\lambda^i}}$ be the essential idempotent generating a minimal left module S_{n^i}-isomorphic to $M(\lambda^i)$, where $r_{T_{\lambda^i}}$ is the sum of the elements of row stabilizer of T_{λ^i} and $c_{T_{\lambda^i}}$ is the signed sum of the elements of the column stabilizer of T_{λ^i}. Then the element $e_{T_{\underline{\lambda}}} = e_{T_{\lambda^1}} e_{T_{\lambda^2}} \cdots e_{T_{\lambda^p}}$ is an essential idempotent associated to $T_{\underline{\lambda}}$. Recall that $\chi_{\underline{\lambda}}$ occurs in the decomposition of $\chi^G_{n_1,\dots,n_p}(A)$ if and only if there exist a multi-tableau $T_{\underline{\lambda}}$ and a polynomial $f \in V^G_{n_1,\dots,n_p}$ such that $f \notin T_G(A)$ and $e_{T_{\underline{\lambda}}} f \notin T_G(A)$.

When $T_{\underline{\lambda}}$ is assigned, the entries of T_{λ^g} are names of variables, all of them having G-degree g. We shall often identify those names with the variables themselves; for instance we shall shortly say that *the variables* $y^g_{j_1}, \dots, y^g_{j_t}$ *occur in the first column of* T_{λ^g} meaning *the set of variables of* G*-degree* g *whose names occur in the first column of* T_{λ^g}. Also, we shall write $\mathcal{Y}^g_j$ to denote the set of variables occurring in the j-th column of T_{λ^g}.

4 The Upper Bound

In this small section we shall display an upper bound for the graded codimension sequence $c^G_n(A)$.

Remark 4.1 Let $\underline{n} = (n_1, \dots, n_p)$, $\underline{\lambda} \vdash \underline{n}$ and let $U_{\underline{\lambda}} \subseteq V_{\underline{n}}$ be an irreducible $H_{\underline{n}}$-module isomorphic to $M(\underline{\lambda})$. If $0 \neq f \in U_{\underline{\lambda}}$ then there exists a multi-tableau $T = T_{\underline{\lambda}}$ such that $0 \neq f' := c_T e_T f \in U_{\underline{\lambda}}$ is alternating in each set of variables $\mathcal{Y}^i_j$, those occurring in the j-th column of T_{λ^i}.

Remark 4.2 As a notational fact, if $\underline{\lambda} \vdash \underline{n}$ is assigned, we shall denote by $h_j^g(\underline{\lambda})$, or simply by h_j^g, the *length* of the j-th column of λ^g. In particular, h_1^g denotes the height of the partition $\lambda^g \vdash n_g$.

Lemma 4.3 *Let $M(\underline{\lambda}) \cong W_{\underline{\lambda}} \nsubseteq T_G(A)$. Then $h_1^g \leqslant \dim_F A^g$ for all $g \in G$ and $\sum_g h_{l+1}^g \leqslant d$. Further,*

$$\dim_F W_{\underline{\lambda}} \leqslant n^{l\cdot \dim_F A} \prod_{\substack{g\in G\\ h_{l+1}^g>0}} (h_{l+1}^g)^{n_g}.$$

Proof Let $0 \neq f_0 \in W_{\underline{\lambda}}$. Then there is a multi-tableau $T = T_{\underline{\lambda}}$ such that $f = c_T e_T f_0 \neq 0$, hence $W_{\underline{\lambda}} = FH_{\underline{n}} f$. Let us denote for short by $\mathcal{Y}_j^g$ the set of variables occurring in the j-th column of T_{λ^g}. The polynomial f does not belong to $T_G(A)$ but is alternating on each $\mathcal{Y}_j^g$. So, in particular, it is $\left|\mathcal{Y}_1^g\right| = h_1^g \leqslant \dim_F A^g$ for all $g \in G$.

Now let us suppose $\sum_g h_{l+1}^g > d$. If $j \leqslant l+1$ it is $h_j^g \geqslant h_{l+1}^g$. Let φ be a standard G-substitution and fix $j \leqslant l+1$. The polynomial f is alternating on each $\mathcal{Y}_j^g$, and the set $\mathcal{X}_j := \bigcup_{g\in G} \mathcal{Y}_j^g$ has size $> d$. If $\varphi(\mathcal{X}_j) \subseteq B$ then $\varphi(f) = 0$ by Lemma 3.3, so at least one variable among those in $\mathcal{X}_j$ should be mapped in J, in order to get a nonzero value $\varphi(f)$. But then at least $l+1$ variables among those in $\bigcup_{j\leqslant l+1} \mathcal{X}_j$ are substituted by elements of J. Therefore $\varphi(f) \in J^{l+1} = 0$, which is absurd.

Finally, let $\delta_g := \dim_F A^g$. Recall $\delta_g \geqslant h_1^g$. By standard combinatorial computations if $h_{l+1}^g > 0$ then $d_{\lambda^g} \leqslant n^{l\delta_g} (h_{l+1}^g)^{n_g}$. Otherwise, $d_{\lambda^g} \leqslant n^{l\delta_g}$. Therefore

$$\dim_F W_{\underline{\lambda}} = \prod_{g\in G} d_{\lambda^g} \leqslant \prod_{g\in G} n^{l\delta_g} \prod_{\substack{g\in G\\ h_{l+1}^g>0}} (h_{l+1}^g)^{n_g} = n^{l\cdot\dim_F A} \prod_{\substack{g\in G\\ h_{l+1}^g>0}} (h_{l+1}^g)^{n_g}. \qquad \blacksquare$$

As a corollary, let us record an upper bound for $c_n^G(A)$.

Theorem 4.4 *There exist α, t such that $c_n^G(A) \leqslant \alpha n^t d^n$, for all $n \in \mathbb{N}$.*

Proof Let $n \in \mathbb{N}$. By equation (2.1) it is

$$c_n^G(A) = \sum_{\underline{n}} \binom{n}{\underline{n}} c_{\underline{n}}(A) = \sum_{\underline{n}} \binom{n}{\underline{n}} \sum_{\underline{\lambda}\vdash\underline{n}} m_{\underline{\lambda}} \dim_F W_{\underline{\lambda}}$$

for some multiplicities $m_{\underline{\lambda}}$. Notice that if $\sum_g h_{l+1}^g(\underline{\lambda}) > d$ then $m_{\underline{\lambda}} = 0$. Now let us choose for each weak composition $\underline{n}$ of n a maximal dimensional module $W_{\underline{\mu}}$ with nonzero multiplicity. All its composing partitions μ^g must lie in a strip of height $\delta_g := \dim_F A^g$, and $d_{\mu^g} \leqslant n^{l\delta_g}$ or $d_{\mu^g} \leqslant n^{l\delta_g}\left(h_{l+1}^g(\underline{\mu})\right)^{n_g}$ according to the cases when $h_{l+1}^g(\underline{\mu}) = 0$ or is > 0, respectively. Let $\mathrm{G}(\underline{n}) := \{g \in G \mid h_{l+1}^g(\underline{\mu}) > 0\}$. Then

$$d_{\underline{\mu}} = \prod_{g\in G} d_{\mu^g} \leqslant \prod_{g\in G} n^{l\delta_g} \prod_{g\in G(\underline{n})} \left(h_{l+1}^g(\underline{\mu})\right)^{n_g} = n^{l\dim A} \prod_{g\in G(\underline{n})} \left(h_{l+1}^g(\underline{\mu})\right)^{n_g}.$$

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Let us denote $t_g := h^g_{l+1}(\underline{\mu})$ for $g \in G(\underline{n})$; notice that t_g depends on the composition $\underline{n}$ only. Then for all $\underline{\lambda} \vdash \underline{n}$ the fact $d_{\underline{\lambda}} \leqslant d_{\underline{\mu}}$ yields

$$c^G_{\underline{n}}(A) \leqslant n^{l\cdot \dim_F A} \sum_{\underline{n}} \Big(\sum_{\underline{\lambda}\vdash \underline{n}} m_{\underline{\lambda}} \Big) \binom{n}{\underline{n}} \prod_{g\in G(\underline{n})} t_g^{n_g}.$$

Further, from the fact that the sum of multiplicities is polynomially bounded (see [Be]), there exist constants $\bar{\alpha}, b \in \mathbb{N}$ such that

$$c^G_n(A) \leqslant \bar{\alpha} n^{l\cdot \dim_F A} n^b \sum_{\underline{n}} \binom{n}{\underline{n}} \prod_{g\in G(\underline{n})} t_g^{n_g}.$$

Now let us consider the summand $\binom{n}{\underline{n}} \prod_{g\in G(\underline{n})} t_g^{n_g}$. Let $|G(\underline{n})| = r$; for sufficiently large n it is $r \geqslant 1$. Moreover, without loss of generality we may assume $G(\underline{n}) = \{\gamma, \gamma^2, \dots, \gamma^r\}$. Write $n_i := n_{\gamma^i}$, $u := \sum_{i=1}^r n_i$ and $v := \sum_{i=r+1}^p n_i$. Notice that $n_i \leqslant l\delta_{\gamma^i}$ when $i \geqslant r+1$ and so $v \leqslant l \dim A$. Then

$$\begin{aligned}\binom{n}{\underline{n}} &= \frac{n!}{n_1!\cdots n_r!\, n_{r+1}!\cdots n_p!} = \frac{u!}{n_1!\cdots n_r!} \frac{v!}{n_{r+1}!\cdots n_p!} \frac{n!}{u!v!} \\ &= \binom{u}{n_1 \cdots n_r} \binom{v}{n_{r+1}\cdots n_p} \binom{n}{u}.\end{aligned}$$

Since the number $\binom{v}{n_{r+1}\cdots n_p}$ occurs in the expansion of $(p-r)^v$, it is

$$\binom{v}{n_{r+1}\cdots n_p} \leqslant (p-r)^v \leqslant p^{l \dim A}.$$

The number $\binom{n}{u} = \binom{n}{v}$ is a polynomial expression in n of degree v for sufficiently large n, hence it is $\leqslant n^{l \dim A}$ as well.

Finally, the number $\binom{u}{n_1\cdots n_r} t_\gamma^{n_1} \cdots t_{\gamma^r}^{n_r}$ occurs in the expansion of $(t_\gamma + \cdots + t_{\gamma^r})^u$; since $t_\gamma + \cdots + t_{\gamma^r} \leqslant d$ and $u \leqslant n$, that number is $\leqslant d^n$.

Therefore we get the inequality

$$c^G_n \leqslant \bar{\alpha} n^{3l \dim_F A + b} \sum_{\underline{n}} d^n.$$

Since there are $\binom{n+p-1}{p-1}$ compositions of n in no more than p parts (see for instance [St, Section 1.2]), we get the inequality

$$c^G_n(A) \leqslant \bar{\alpha} n^{3l \dim_F A + b} \binom{n+p-1}{p-1} d^n.$$

Finally, notice that $\binom{n+p-1}{p-1}$ is a polynomial expression in n of degree $p-1$, so it is definitively $c^G_n(A) \leqslant \alpha n^t d^n$ for certain numbers α, t. ■

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5 The Lower Bound

Now we are going to find a lower bound for the codimension sequence $c_n^G(A)$. In order to obtain it, we shall find polynomials which are of sufficiently high degree and are not graded polynomial identities for A. We shall make use of the following theorem.

Theorem 5.1 ([Fo]) *Let C be a central simple algebra, and let $\mathcal{X}_1, \mathcal{X}_2$ be disjoint sets of variables of size $\dim_F C$. Then there exists an explicit central multi-linear polynomial $f = f(\mathcal{X}_1, \mathcal{X}_2)$, alternating on $\mathcal{X}_i$, and a substitution φ such that $\varphi(f) = 1_C$.*

Actually, if $m \geqslant 1$, the product $\widehat{f}$ of m such polynomials on disjoint sets of variables is still multi-linear, alternating on $2m$ sets of variables $\mathcal{X}_i$ of size $\dim_F C$, and the obvious extension $\widehat{\varphi}$ of φ provides a substitution with $\widehat{\varphi}(\widehat{f}) = 1_C$. So the following corollary holds.

Corollary 5.2 *Let C be a central simple F-algebra, $m \geqslant 1$ and let $\mathcal{X}$ be the disjoint union of $2m$ sets of variables $\mathcal{X}_j$ ($j = 1, \dots, 2m$), each of size $|\mathcal{X}_j| = \dim_F C$. Then there exist a central multi-linear polynomial $f = f(\mathcal{X})$, alternating on each $\mathcal{X}_j$, and a substitution φ such that $\varphi(f) = 1_C$.*

We are indeed mostly interested in Corollary 5.2. The first step to our aims is the following lemma.

Lemma 5.3 *Let G be a finite group, $m \geqslant 1$ and let C be a simple algebra graded by G. For any $g \in G$ let $\mathcal{Y}_1^g, \dots, \mathcal{Y}_{2m}^g$ be disjoint subsets of X^g, each of size $\dim C_g$. Let $\mathcal{Y} := \bigcup_{j,g} \mathcal{Y}_j^g$. Then there exists a central multi-linear polynomial $f = f(\mathcal{Y})$ alternating on each $\mathcal{Y}_j^g$ and a G-substitution φ such that $\varphi(f) = 1_C$.*

Proof Let $\mathcal{B}$ be a G-homogeneous F-basis of C, set $\delta := \dim C$ and let $\mathcal{X}_1, \dots, \mathcal{X}_{2m}$ be $2m$ disjoint sets of (nongraded) variables, each of size δ. Let $\mathcal{X}$ be their disjoint union. So, by Corollary 5.2, there exist $f = f(\mathcal{X})$ multi-linear, alternating on each $\mathcal{X}_j$, and a substitution φ such that $\varphi(f) = 1_C$. Actually, there is no harm in assuming that $\varphi(\mathcal{X}) \subseteq \mathcal{B}$.

Notice that for any $j = 1, \dots, 2m$ it must happen that $\varphi(\mathcal{X}_j) = \mathcal{B}$, because $|\mathcal{X}_j| = \delta$ and f is alternating on $\mathcal{X}_j$. Therefore, if we decompose $\mathcal{X}_j$ into the (disjoint) union of the sets $\mathcal{X}_{j,g} := \{x \in \mathcal{X}_j \mid \varphi(x) \in \mathcal{B} \cap C_g\}$, it must happen $|\mathcal{X}_{j,g}| = \dim C_g$. Indeed, f is alternating on each $\mathcal{X}_{j,g}$ because it is so on the whole $\mathcal{X}_j$; if $|\mathcal{X}_{j,g}| > \dim C_g$ this leads to $\varphi(f) = 0$, a contradiction. Also, notice that

$$\delta = |\mathcal{X}_j| = \sum_{g \in G} |\mathcal{X}_{j,g}| \leqslant \sum_{g \in G} \dim C_g = \delta$$

implies $|\mathcal{X}_{j,g}| = \dim C_g$ for all $g \in G$ and $j \in [2m]$.

Now just replace in f the set of nongraded indeterminates $\mathcal{X}_{j,g}$ by the set of *graded* indeterminates $\mathcal{Y}_j^g$, for all j and g. Notice that $|\mathcal{X}_{j,g}| = \dim C_g = |\mathcal{Y}_j^g|$, $f(\mathcal{Y})$ is alternating on each $\mathcal{Y}_j^g$ (actually on the whole $\mathcal{Y}_j := \bigcup_g \mathcal{Y}_j^g$), the substitution φ gives rise to a natural G-substitution φ_G and $\varphi_G(f) = 1_C$. ■

The next step is when C is G-simple but not simple. While in the previous lemma just "*G finite*" was needed, in the following one we must assume that $G \cong \mathbb{Z}_p$ in order to use our arguments.

Lemma 5.4 *Let $G \cong \mathbb{Z}_p$, $m \geqslant 1$ and let C be a G-simple G-graded algebra. For each $g \in G$ let $\mathcal{Y}_1^g, \dots, \mathcal{Y}_{2m}^g$ be disjoint subsets of X^g, each of size $\left|\mathcal{Y}_i^g\right| = \dim C_g$. Let $\mathcal{Y}^g$ be their union and $\mathcal{Y} = \bigcup \mathcal{Y}^g$. Then there exists a multi-linear polynomial $f = f(\mathcal{Y})$, alternating on any $\mathcal{Y}_j^g$, and a G-substitution φ such that $\varphi(f) = 1$.*

Proof $J = J(C)$ is a G-ideal of C and C is G-simple, hence C is semiprimitive. Since it is finite dimensional, it is completely reducible, so it is a direct sum of minimal two-sided ideals $C = I_1 \oplus \cdots \oplus I_k$. If $k = 1$ then C is simple, and the result follows by the previous lemma. Notice that if the grading is trivial, then any ideal is G-homogeneous, so C must be a simple algebra. Hence assume that C is not simple, that is $k > 1$, and let the grading be nontrivial. Then $G \cong \langle \gamma \rangle = \widehat{G} \leqslant \mathrm{Aut}(A)$ acts on A; therefore a G-homogeneous subspace is precisely a $\widehat{G}$-invariant subspace. Let $I := I_1$. Then it cannot be $\gamma(I) = I$, otherwise I should be a *graded* ideal of C, so $I = C$, a contradiction. Therefore $\gamma(I) \neq I$ and $\gamma(I)$ is a minimal two-sided ideal of C, disjoint with I. Actually, it is easy to see that $\widehat{G}$ acts on the set of minimal two-sided ideals of C by permuting them. Since the stabilizer of this action has to be a subgroup of $\widehat{G}$ properly contained in $\widehat{G}$ (because $\gamma(I) \neq I$), it has to be $\{e\}$. But then

$$L := \sum_{i=1}^{p} \gamma^i(I) = \bigoplus_{i=1}^{p} \gamma^i(I)$$

is a two-sided ideal of C which is stable under the action of $\widehat{G}$, hence it is a nonzero G-ideal of C, so $L = C$. This forces $k = p$.

Therefore, if C is G-simple but not simple, then C is a direct sum of p minimal two-sided ideals, none of them G-homogeneous, constituting a full orbit under $\widehat{G}$ and all of them isomorphic as F-algebras. More precisely, C is the direct sum of p copies of a simple algebra I, and $\widehat{G}$ acts on them by cyclically permuting them.

Now, let $\dim_F I = l$ and let $\mathcal{B}$ be a linear basis for I. Let us define, for any $b \in \mathcal{B}$ and any $g = \gamma^j \in \widehat{G}$,

$$b_g := b + \zeta^{j(p-1)}\gamma(b) + \zeta^{j(p-2)}\gamma^2(b) + \cdots + \zeta^j \gamma^{p-1}(b).$$

Then b_g is a G-homogeneous element of G-degree $g = \gamma^j$, and the set $\mathcal{C} := \{b_g \mid b \in \mathcal{B}\ g \in G\}$ is a G-homogeneous basis for C. More precisely, for a fixed $g = \gamma^j$, the set $\{b_g \mid b \in \mathcal{B}\}$ is a linear basis for C_g. Hence all the graded components of C have the same dimension l.

Then let $f_0 = f_0(\mathcal{X})$ be the polynomial of Corollary 5.2, corresponding to an l-dimensional central simple algebra and to the fixed $m \geqslant 1$, and let φ_0 be a standard substitution on $\mathcal{B}$ such that $\varphi_0(f_0) = 1_I$. Since $\left|\mathcal{Y}_i^g\right| = \dim C_g = \dim I = |\mathcal{X}_i|$, we may pass to the *graded* variables $\mathcal{Y}^g$ (for $g \in G$) and consider $f_0(\mathcal{Y}^g)$. Hence the polynomial

$$f = f(\mathcal{Y}) := \prod_{g \in G} f_0(\mathcal{Y}^g)$$

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(whatever order we choose for the factors) is multi-linear and alternating on each $\mathcal{Y}_i^g$. Moreover, let φ be the following G-substitution: if $x \in \mathcal{X}$ and $\varphi_0(x) = b \in \mathcal{B}$, then for any $y \in \mathcal{Y}^g$ set $\varphi(y) := b_g \in C_g$. By a straightforward computation, it turns out that $\varphi(f) = 1_C$. ■

Now let us go back to the assigned finite dimensional G-graded algebra. Recall that $A = B + J$ as a direct sum of F-vector spaces of the completely reducible G-graded subalgebra B and the Jacobson radical J. We decomposed B into the sum of G-simple subalgebras $B = C_1 \oplus \cdots \oplus C_m$, but we are mostly interested in the sum of the first k summands, namely the subalgebra $B_k = C_1 \oplus \cdots \oplus C_k$, having the following properties:

- $\dim B_k = d$, the candidate integer to be the exponent;
- $C_1 J C_2 \cdots C_{k-1} J C_k \neq 0$.

Each C_i has a central polynomial f_i, by Lemma 5.4. The idea in what follows is to glue these polynomials in order to get a nonvanishing polynomial for A of sufficiently high degree.

Lemma 5.5 *Let $m \geqslant 1$. Then there exists a multi-linear polynomial f in the disjoint union of variables $\mathcal{Y} \cup \mathcal{K} \cup \mathcal{J}$ and an A-valued G-substitution φ satisfying $\varphi(\mathcal{Y}) \subseteq B_k$ such that*

- *$\mathcal{Y}$ is partitioned as $\mathcal{Y} = \bigcup_{g \in G} \mathcal{Y}^g$. Each $\mathcal{Y}^g$ is a disjoint union $\mathcal{Y}^g = \bigcup_{1 \leqslant i \leqslant 2m} \mathcal{Y}_i^g$ of sets $\mathcal{Y}_i^g \subseteq X^g$ with same size $\left|\mathcal{Y}_i^g\right| = \dim_F B_k^g$;*
- *$|\mathcal{K}| = k$, $|\mathcal{J}| = k - 1$;*
- *f is alternating on each set $\mathcal{Y}_i^g$;*
- *$\varphi(f) \neq 0$.*

Proof For each fixed $j \in [k]$ as in Lemma 5.4 let $f_j = f_j(\mathcal{X}_j)$ be a central multi-linear polynomial in the graded variables of $\mathcal{X}_j = \bigcup_{i=1}^{2m} \bigcup_{g \in G} \mathcal{X}_{ij}^g$, alternating on any $\mathcal{X}_{ij}^g \subseteq X^g$, of size $\left|\mathcal{X}_{ij}^g\right| = \dim C_j^g$. By Lemma 5.4 there exists a G-substitution φ_j such that $\varphi_j(f_j) = 1_{C_j}$. There is no loss in generality in assuming that $\varphi_j(\mathcal{X}_j) = \mathcal{B}_j$, a G-homogenous linear basis for C_j.

Choosing disjoint sets $\mathcal{X}_j$ as j runs through $1, \ldots, k$, we may consider for any $i \in [2m]$ and any $g \in G$ the sets

$$\mathcal{Y}_i^g := \bigcup_{j=1}^{k} \mathcal{X}_{ij}^g.$$

Then

$$\left|\mathcal{Y}_i^g\right| = \sum_{j=1}^{k} \left|\mathcal{X}_{ij}^g\right| = \sum_{j=1}^{k} \dim C_j^g = \dim B_k^g.$$

Denote $\mathcal{K} = \{x_1, \ldots, x_k\}$ and $\mathcal{J} = \{x'_1, \ldots, x'_{k-1}\}$. Since $C_1 J \cdots J C_k \neq 0$ there exist $b_j \in C_j$ and $c_1, \ldots, c_{k-1} \in J$ such that $b_1 c_1 \cdots c_{k-1} b_k \neq 0$. Once again, we may

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assume $b_j \in \mathcal{B}_j$ and let c_j be G-homogeneous elements in J. Then consider the G-substitution $\varphi\colon \mathcal{Y} \cup \mathcal{K} \cup \mathcal{J} \to A$ defined by

$$\varphi_{|_{\mathcal{Y}_j^g}} = \varphi_j, \quad \varphi(x_j) := b_j, \quad \varphi(x_j') := c_j.$$

Then let g be the following multi-linear graded polynomial

$$g := x_1 f_1(\mathcal{X}_1) x_1' x_2 f_2(\mathcal{X}_2) x_2' \cdots x_{k-1} f(\mathcal{X}_{k-1}) x_{k-1}' x_k f_k(\mathcal{X}_k).$$

Notice that $\varphi(g) = b_1 c_1 \cdots c_{k-1} b_k \neq 0$; yet g has not the requested alternating properties. In order to get them, let us identify for a moment the names of the involved graded variables with the variables themselves; let $\operatorname{Sym}(\mathcal{Y}_i^g)$ be the symmetric group on the set $\mathcal{Y}_i^g$ and let

$$\mathcal{S} := \prod_{g \in G} \prod_{i=1}^{2m} \operatorname{Sym}(\mathcal{Y}_i^g).$$

Notice that since the $\mathcal{Y}_i^g$ are pairwise disjoint the direct product of these symmetric groups is well defined; for $\sigma \in \mathcal{S}$ let $(-1)^\sigma$ denote the sign of the permutation σ.

Then consider the multi-linear polynomial

$$f(\mathcal{Y} \cup \mathcal{K} \cup \mathcal{J}) := \sum_{\sigma \in \mathcal{S}} (-1)^\sigma \sigma(g).$$

The polynomial f is clearly alternating on any $\mathcal{Y}_i^g$. Moreover, since $C_i C_{i'} = 0$ if $i \neq i'$, it is also clear that in $\varphi(f)$ a summand $\varphi\big(\sigma(g)\big)$ is not vanishing if and only if $\sigma(\mathcal{X}_{ij}^g) = \mathcal{X}_{ij}^g$ for all $i = 1, \dots, 2m$, $j = 1, \dots, k$ and all $g \in G$. Therefore it must happen

$$\sigma \in \prod_{g \in G} \prod_{i=1}^{2m} \prod_{j=1}^{k} \operatorname{Sym}(\mathcal{X}_{ij}^g).$$

In this case, it is $\sigma(g) = (-1)^\sigma g$ hence

$$\varphi\big(\sigma(g)\big) = (-1)^\sigma \varphi(g) = (-1)^\sigma b_1 \varphi_1(f_1) c_1 \cdots c_{k-1} b_k \varphi_k(f_k) = (-1)^\sigma b_1 c_1 \cdots c_{k-1} b_k$$

and therefore, since $\left|\mathcal{X}_{ij}^g\right| = \dim C_j^g$ for all $i = 1, \dots, 2m$, it is

$$\varphi(f) = \Big(\prod_{j=1}^{k} \prod_{g \in G} \dim C_j^g !\Big)^{2m} \cdot b_1 c_1 \cdots c_{k-1} b_k \neq 0. \qquad \blacksquare$$

Theorem 5.6 *There exist constants a, b such that $c_n^G(A) \geqslant a n^b d^n$.*

Proof We continue to adopt the same notation used in the preceding lemmas. We are going to show the inequality by exhibiting a multi-linear polynomial which is not a graded polynomial identity for A of sufficiently high degree. So, take any $n \geqslant 2d + 2k - 1$, and let $n - (2k - 1) = 2dm + t$ for some $0 \leqslant t < 2d$. Corresponding to the number $m \geqslant 1$ let $f = f(\mathcal{Y} \cup \mathcal{K} \cup \mathcal{J})$ be the polynomial and φ the G-graded substitution of the previous lemma. Consider a set of *new* graded variables $\mathcal{Y}' := \{y'_1, \ldots, y'_t\} \subseteq X^e$, let

$$g' := f y'_1 \ldots y'_t$$

then extend φ by sending y'_j in 1_{C_k} for all $j = 1, \ldots, t$. Let us denote this G-substitution by the same symbol φ. Of course, $\varphi(f') \neq 0$, and f' is multi-linear, with the same alternating sets as f, of total degree n. So $f' \in V_n^G \setminus T_G(A)$.

Let H be the permutation group $\prod_{g\in G} \mathrm{Sym}(\mathcal{Y}^g)$, and M be the H-module generated by f' inside V_n^G. Namely, setting $\delta_g := |\mathcal{Y}^g|$, $m_g := |(\mathcal{K} \cup \mathcal{J}) \cap X^g|$ and $n_g := \delta_g + m_g$ for $e \neq g \in G$, $n_e := \delta_e + m_e + t$ and $\underline{n} = (n_e, n_{g_1}, \ldots, n_{g_{p-1}})$, we can view $M \subseteq V_{\underline{n}}$. By identifying the *names* of the graded variables with the variables themselves, let us consider a multi-tableau $T_{\underline{\lambda}}$ of shape $\underline{\lambda} = (\lambda_e, \ldots, \lambda_{g_{p-1}})$ where for each $g \in G$ it is $\lambda_g = \left((2m)^{\delta_g}\right)$, a rectangular diagram of height δ_g, and each column is filled by the variables of $\mathcal{Y}^g$. More precisely, following notation of the previous lemma, if $\mathcal{Y}^g = \mathcal{Y}_1^g \cup \cdots \cup \mathcal{Y}_k^g$ and each set $\mathcal{Y}_j^g$, for $j = 1, \ldots, k$ is the disjoint union $\mathcal{Y}_j^g = \bigcup_{i=1}^{2m} \mathcal{Y}_{ij}^g$, let us fill the i-th column of λ^g downward by the variables $\mathcal{Y}_{i1}^g$, then $\mathcal{Y}_{i2}^g, \ldots, \mathcal{Y}_{ik}^g$.

Let $\mathbf{e}_{T_{\underline{\lambda}}}$ be the essential idempotent of the multi-tableau $T_{\underline{\lambda}}$. Then it is straightforward to check that $\varphi(\mathbf{e}_{T_{\underline{\lambda}}} f') = \varphi(f) \neq 0$, and hence $\mathbf{e}_{T_{\underline{\lambda}}} f' \in M$ and generates an irreducible H-module $W_{\underline{\lambda}}$ which is contained in $V_n^G \setminus T_G(A)$, of dimension $\dim W_{\underline{\lambda}} = \prod_{g\in G} \chi_{\lambda^g}(1)$.

Then the following inequalities hold:

$$c_{\underline{n}}(A) \geqslant \dim W_{\underline{\lambda}} = \prod_{g\in G} \chi_{\lambda^g}(1) \geqslant n^u \prod_{g\in G} \delta_g^{2m\delta_g},$$

for a constant u, by [Re2, Lemma 3.1]. By formula (2.1) it is

$$c_n^G(A) = \sum_{\underline{m}} \binom{n}{\underline{m}} c_{\underline{m}}(A) \geqslant \binom{n}{\underline{n}} c_{\underline{n}}(A) \geqslant n^u \binom{n}{\underline{n}} \prod_{g\in G} \delta_g^{2m\delta_g}.$$

Now recall that $n_e = \delta_e + m_e + t$ and for $g \neq e$ it is $n_g = \delta_g + m_g$. Then for the multinomial coefficient $\binom{n}{\underline{n}}$ it is

$$\binom{n}{\underline{n}} \geqslant \frac{(\sum_g 2m\delta_g)!}{\prod_g (2m\delta_g)!} = \frac{(2md)!}{\prod_g (2m\delta_g)!},$$

therefore

$$c_n^G(A) \geqslant n^u \frac{(2md)!}{\prod_g (2m\delta_g)!} \prod_g n_g^{2m\delta_g}$$

and finally, by Stirling's formula, we get

$$c_n^G(A) \geqslant n^u \frac{(2md)!}{\prod_g (2m\delta_g)!} \geqslant n^\alpha \frac{(2md)^{2md}}{\prod_g (2m\delta_g)^{2m\delta_g}} \prod_g (2m\delta_g)^{2m\delta_g} = n^\alpha d^{2md} = \frac{n^\alpha}{d^{2k-1+t}} d^n.$$

■

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Corollary 5.7 *The limit* $\lim_n \sqrt[n]{c_n^G(A)}$ *does exist and is an integer* $\leqslant \dim_F A$.

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References

[BD] Yu. P. Bahturin and V. Drensky, *Graded polynomial identities of matrices.* Linear Algebra Appl. **357**(2002), 15–34. http://dx.doi.org/10.1016/S0024-3795(02)00356-7

[Be] A. Berele, *Cocharacter sequences for algebras with Hopf algebra actions.* J. Algebra **185**(1996), 869–885. http://dx.doi.org/10.1006/jabr.1996.0354

[BGP] F. Benanti, A. Giambruno and M. Pipitone, *Polynomial identities on superalgebras and exponential growth.* J. Algebra **269**(2003), 422–438. http://dx.doi.org/10.1016/S0021-8693(03)00528-3

[DV] O. M. Di Vincenzo, *Cocharacters of G-graded algebras.* Comm. Algebra **24**(1996), 3293–3310. http://dx.doi.org/10.1080/00927879608825751

[Fo] E. Formanek, *A conjecture of Regev about the Capelli polynomial.* J. Algebra **109**(1987), 93–114. http://dx.doi.org/10.1016/0021-8693(87)90166-9

[GMZ] A. Giambruno, S. Mishchenko and M. Zaicev, *Codimensions of algebras and growth functions.* Adv. Math. **217**(2008), 1027–1052. http://dx.doi.org/10.1016/j.aim.2007.07.008

[GR] A. Giambruno and A. Regev, *Wreath products and P.I. algebras.* J. Pure Appl. Algebra **35**(1985), 133–149. http://dx.doi.org/10.1016/0022-4049(85)90036-2

[GZ1] A. Giambruno and M. Zaicev, *On codimension growth of finitely generated associative algebras.* Adv. Math. **140**(1998), 145–155. http://dx.doi.org/10.1006/aima.1998.1766

[GZ2] ———, *Exponential codimension growth of P.I. algebras: an exact estimate.* Adv. Math. **142**(1999), 221–243. http://dx.doi.org/10.1006/aima.1998.1790

[GZ3] ———, *Involutions codimensions of finite dimensional algebras and exponential growth.* J. Algebra **222**(1999), 471–484. http://dx.doi.org/10.1006/jabr.1999.8016

[GRZ] A. Giambruno, A. Regev and M. Zaicev, *Simple and semisimple Lie algebras and codimension growth.* Trans. Amer. Math. Soc. **352**(2000), 1935–1946. http://dx.doi.org/10.1090/S0002-9947-99-02419-8

[Ke] A. R. Kemer, *Varieties and* $\mathbb{Z}_2$*-graded algebras.* Izv. Akad. Nauk SSSR Ser. Mat. **48**(1984), 1042–1059.

[MZ] S. Mishchenko and M. Zaicev, *An example of a variety of Lie algebras with a fractional exponent.* Algebra, 11. J. Math. Sci. (New York) **93**(1999), 977–982. http://dx.doi.org/10.1007/BF02366352

[Pe] V. Petrogradsky, *Growth of polynilpotent varieties of Lie algebras and rapidly growing entire functions.* Sb. Math. **188**(1997), 913–931.

[Re1] A. Regev, *Existence of identities in* $A \otimes B$. Israel J. Math. **11**(1972), 131–152. http://dx.doi.org/10.1007/BF02762615

[Re2] ———, *The Representations of* S_n *and Explicit Identities for P.I. Algebras*. J. Algebra **51**(1978), 25–40. http://dx.doi.org/10.1016/0021-8693(78)90133-3

[SVO] D. Ştefan and F. Van Oystaeyen, *The Wedderburn–Malcev theorem for comodule algebras.* Comm. Algebra **27**(1999), 3569–3581. http://dx.doi.org/10.1080/00927879908826648

[St] R. P. Stanley, *Enumerative Combinatorics*. Vol. 1. Cambridge Studies in Advanced Mathematics **49**, Cambridge, Cambridge University Press, 1997.

[Vo1] I. B. Volichenko, *On the bases of a free Lie algebra modulo some T-ideals.* Dokl. Akad. Nauk BSSR **24**(1980), 400–403.

[Vo2] ———, *Varieties of Lie algebras with the identity* $\big[[x_1, x_2, x_3], [x_4, x_5, x_6]\big] = 0$ *over a field of characteristic zero.* (Russian) Sibirsk. Mat. Zh. **25**(1984), 40–54.

[Za] M. Zaicev, *Integrality of exponents of growth of identities of finite-dimensional Lie algebras.* Izv. Ross. Akad. Nauk Ser. Mat. **66**(2002), 23–48.

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