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SEMISIMPLE ALGEBRAS OF INFINITE VALUED LOGIC AND BOLD FUZZY SET THEORY

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0. Introduction. In classical two-valued logic there is a three way relationship among formal systems, Boolean algebras and set theory. In the case of infinite-valued logic we have a similar relationship among formal systems, MV-algebras and what is called Bold fuzzy set theory. The relationship, in the latter case, between formal systems and MV-algebras has been known for many years while the relationship between MV-algebras and fuzzy set theory has hardly been studied. This is not surprising. MV-algebras were invented by C. C. Chang [**1**] in order to provide an algebraic proof of the completeness theorem of the infinite-valued logic of Lukasiewicz and Tarski. Having served this purpose (see [**2**]), the study of these algebras has been minimal, see for example [**6**], [**7**]. Fuzzy set theory was also being born around the same time and only in recent years has its connection with infinite-valued logic been made, see e.g. [**3**], [**4**], [**5**]. It seems appropriate then, to take a further look at the structure of MV-algebras and their relation to fuzzy set theory. This relation, in itself, is not complicated; those MV-algebras which are semi-simple are precisely the Bold algebras of fuzzy sets. Now, though MV-algebras are a certain generalization of Boolean algebras their structure is much more complicated and one striking difference between them is that not every MV-algebra is semi-simple. Indeed, it is known ([**1**]) that the class of semi-simple algebras is not a universal class. So the wrinkle here is in deciding whether or not a given MV-algebra is semi-simple. To this end a deeper structure theory needs development and it is the purpose of this paper to initiate such an investigation. We end this paper in a representation theorem for $\aleph_0$-complete atomic MV-algebras. Many questions remain unanswered, in particular whether or not a complete MV-algebra is semi-simple.

In Section 1 we give the basic definitions and properties of MV-algebras. Since all this material is in [**1**], [**2**] we only present what is needed later. Section 2 looks at the relationship between MV-algebras and Bold fuzzy set theory presenting an analogue of the Stone representation theorem for Boolean algebras. Section 3 begins the deeper study of our subject and introduces a distributive lattice intimately associated to a

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given MV-algebra. This lattice will provide some insight into the prime ideal space of an MV-algebra. Section 4 studies representations and concludes with a structure theorem for $\aleph_0$-complete and atomic MV-algebras.

1. An MV-*algebra* is a non-empty set A with two binary operations $+, \cdot$, one unary operation $^-$ and two special elements 0, 1 such that

1. $\langle A, +, 0\rangle, \langle A, \cdot, 1\rangle$

are commutative semi-groups with identity.

2. $x + \bar{x} = 1, x \cdot \bar{x} = 0, \bar{0} = 1$

for all $x \in A$.

3. $\overline{(x + y)} = \bar{x} \cdot \bar{y}, \overline{x \cdot y} = \bar{x} + \bar{y}, \bar{\bar{x}} = x$

for all $x, y \in A$.

4. Defining $\vee$, $\wedge$ by

$$x \vee y = x + \bar{x} \cdot y, x \wedge y = (x + \bar{y}) \cdot y$$

we have that

$$\langle A, \vee, 0\rangle, \langle A, \wedge, 1\rangle$$

are to be commutative semi-groups with identity.

5. $x \cdot (y \vee z) = x \cdot y \vee x \cdot z, x + (y \wedge z) = (x + y) \wedge (x + z)$

for all $x, y, z \in A$.

Though these axioms seem cumbersome and are not the most efficient they are convenient. These are the same axioms as in [**1**].

The defined operations $\vee$, $\wedge$ with 0, 1 make A a commutative lattice with least element 0 and greatest element 1 under the ordering $x \leqq y$ if and only if $x \wedge y = x$. We then have (see [**1**]).

THEOREM 1. *Let A be an* MV-*algebra and let $x, y \in A$. Then*

a) $x \cdot y \leqq x \wedge y \leqq x$

b) $x \leqq x \vee y \leqq x + y$

c) $x \leqq y$ *if and only if* $x \cdot \bar{y} = 0$ *if and only if* $\bar{x} + y = 1$

d) $x + y = y$ *if and only if* $x \cdot y = x$

e) $x \cdot (y + z) \leqq y + x \cdot z$

f) $x \cdot \bar{y} \wedge \bar{x} \cdot y = 0$

g) $x \cdot x = x$ *if and only if* $x \wedge \bar{x} = 0$.

One of the most important examples for us of an MV-algebra is the unit interval $A = [0, 1]$ of real numbers. Here we define

$$x + y = \min(1, x + y), x \cdot y = \max(0, x + y - 1) \text{ and } \bar{x} = 1 - x.$$

Then

$$x \vee y = \max(x, y), x \wedge y = \min(x, y).$$

We will refer to this algebra just by [0, 1].

Another example of importance, though not for this paper, is the set of equivalence classes of well-formed formulas of infinite-valued propositional logic under the relation: $\alpha \equiv \beta$ if and only if α, β are provably equivalent.

Recall that given a set X, a fuzzy subset of X is a function $f:X \rightarrow [0, 1]$. Consequently $A = [0, 1]^X$ becomes an MV-algebra under the obviously induced operations. Any subalgebra of A will be called a *Bold algebra* of fuzzy sets. Usually the algebras of fuzzy sets use only $\vee$, $\wedge$. For more of this see [**3**].

Let G be any linearly ordered abelian group, written additively, $a \in G$, $a > 0$. Let

$$A = [0, a] = \{x \in G: 0 \leqq x \leqq a\}.$$

Then A is an MV-algebra under the operations

$$x + y = \min(a, x + y), x \cdot y = \max(0, x + y - a), \bar{x} = a - x.$$

In this algebra the role of "1" is played by a. It is shown in [**2**] that every linearly ordered MV-algebra arises in this manner from some linearly ordered abelian group.

A quick examination of these examples will show the elements of an MV-algebra are not, in general, idempotent nor do they satisfy the distributive law

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

This is another major difference between MV-algebras and Boolean algebras. In fact an MV-algebra in which each element is idempotent is a Boolean algebra; similarly an MV-algebra in which the distributive law holds is also Boolean.

2. In this section we give an analogue of the Stone representation theorem. Our approach is more from the view point of the Gelfand representation theorem since it seems more suitable to the algebra of fuzzy sets. The prime and maximal ideal spaces are herein introduced. In an MV-algebra not all prime ideals are maximal so the corresponding spaces are distinct. We present here only one simple theorem about these spaces since we shall show in the next section each of these spaces is homeomorphic to the corresponding space of some distributive lattice.

Let A be an MV-algebra throughout this section.

Definition. An *ideal* in A is a non-empty subset $I \subseteq A$ such that $x, y \in I$ imply $x + y \in I$, $x \in I$, $y \leqq x$ imply $y \in I$.

We give two easy characterizations of ideals.

THEOREM 2. i) *$I \subseteq A$ is an ideal if and only if $I \neq \emptyset$, I is closed under $+$, and $x \in I$, $y \in A$ imply $x \wedge y \in I$.*

ii) *$I \subseteq A$ is an ideal if and only if $I \neq \emptyset$, I is closed under $+$, and $x \in I$, $y \in A$ imply $x \cdot y \in I$.*

Proof. i) This is obvious since $x \wedge y = y$ if and only if $y \leqq x$. ii) One direction is clear since $x \cdot y \leqq x$. Assume then, $I \neq \emptyset$, is closed under $+$ and $x \in I$ and $y \in A$ imply $x \cdot y \in A$. Now let $x \in I$, $y \leqq x$. Then $x \cdot (\bar{x} + y) \in I$. But $x \cdot (\bar{x} + y) = x \wedge y$ so $x \wedge y = y \in I$.

An ideal M of A is *maximal* if and only if $M \neq A$ and for all ideals $I \subseteq A$, $M \subseteq I \subseteq A$ implies $M = I$ or $M = A$.

As usual, ideals give rise to congruence relations, $x \equiv y$ if and only if $x \cdot \bar{y} + \bar{x} \cdot y \in I$, and consequently quotient algebras A/I, homomorphisms etc. [**1**].

For maximal ideals M we have,

THEOREM 3. *A/M is isomorphic to a subalgebra of* $[0, 1]$.

For the proof of this see the remarks after 3.21 of [**1**] and Lemma 6 of [**2**].

Thus we have for each maximal ideal M a homomorphism

$$\varphi_M : A \rightarrow A/M \rightarrow [0, 1].$$

Now let $\mathscr{M}$ be the set of all maximal ideals of A. With each $x \in A$ we associate a fuzzy subset of $\mathscr{M}$, namely,

$$\hat{x} : \mathscr{M} \rightarrow [0, 1] \quad \text{where } \hat{x}(M) = \varphi_M(x).$$

If $\mathscr{F}(A) = \{\hat{x} : x \in A\}$ then $\mathscr{F}(A)$ is a Bold algebra of fuzzy sets,

$$\hat{x} + \hat{y} = \widehat{(x + y)}, \ \hat{x} \cdot \hat{y} = \widehat{(x \cdot y)}, \ \hat{\bar{x}} = \bar{\hat{x}}, \ \hat{0} = 0, \ \hat{1} = 1.$$

Then the map $x \rightarrow \hat{x}$ is a homomorphism of A into $\mathscr{F}(A)$. In general this map is not an injection.

THEOREM 4. *For each* MV-*algebra A there is a Bold algebra of fuzzy sets $\mathscr{F}(A)$ and a homomorphism of A onto $\mathscr{F}(A)$. The homomorphism $x \rightarrow \hat{x}$ is one-one if and only if* $\mathrm{Rad}(A) = 0$, *where* $\mathrm{Rad}(A)$ *is the intersection of all maximal ideals of A.*

Proof. (This is basically 4.9 of [**1**]). We know $x \rightarrow \hat{x}$ is a homomorphism. $\hat{x} = \hat{y}$ if and only if for each maximal ideal M, $x \cdot \bar{y} + \bar{x} \cdot y \in M$. Thus

$\text{Rad}(A) = 0$ implies $x \cdot \bar{y} + x \cdot \bar{y} = 0$ so $x \cdot \bar{y} + \bar{x} \cdot y = 0$; by Theorem 1, $x = y$. If $\text{Rad}(A) \neq 0$ choose $y \in \text{Rad}(A)$, $y \neq 0$. Then $\hat{y} = 0 = \hat{0}$ and so $x \to \hat{x}$ is not one-one.

Rad(A) is called the *radical* of A and we say A is *semi-simple* if and only if $\text{Rad}(A) = 0$. The above theorem tells us, then, that the semi-simple MV-algebras are Bold algebras of fuzzy sets. As mentioned in the introduction, not all MV-algebras are semi-simple. Later we will give a necessary and sufficient condition for an MV-algebra to be semi-simple, which will imply each Bold algebra of fuzzy sets to be semisimple.

For each $x \in A$ let

$$S(x) = \{M:M \in \mathscr{M} \text{ and } x \notin M\}.$$

It is obvious that $S(x \cdot y) \subseteq S(x) \cap S(y)$, hence the sets $S(x)$ generate a topology $\mathscr{T}$ on $\mathscr{M}$. It is easy to show that

THEOREM 5. *$\mathscr{T}$ is a compact T_1-space and if x is idempotent then $S(x)$ will be open and closed.*

In MV-algebras the prime ideals play a more important role than the maximal ideals. It seems likely that the prime ideal space will be more revealing than the maximal ideal space. Though neither of these spaces is much investigated in this paper we shall prove in the next section a general theorem about them.

An ideal $P \subseteq A$ is called *prime* if and only if $I \neq A$ and for each $x, y \in A$, $x \cdot \bar{y} \in P$ or $\bar{x} \cdot y \in P$.

It is shown in [**2**] that P is a prime ideal if and only if A/P is a linearly ordered MV-algebra. Since [0, 1] is linearly ordered we see that every maximal ideal is prime. Examples are given in [**1**] of linearly ordered MV-algebras not contained in [0, 1], consequently prime ideals need not be maximal.

Letting $\mathscr{P}$ denote the set of all prime ideals of A and defining

$$R(x) = \{P \in \mathscr{P}: x \notin P\}$$

we have as before that the sets $R(x)$, $x \in A$, generate a topology on $\mathscr{P}$. This topology is also a compact T_1-space and for x idempotent $R(x)$ will be open and closed.

3. In this and the following section occurs the principal part of this paper. We construct a lattice associated to a given MV-algebra A whose ideal structure parrots that of A. This will give us another way to get at the prime and maximal ideal spaces of A. The relationship between A and its associated lattice will also enable us to get at certain structural features of A itself. Though undoubtedly the transition from A to this lattice loses some information, enough is retained to make its introduction worthwhile.

Note that this lattice is not $\langle A, \vee, \wedge, 0, 1\rangle$ whose ideal structure is in general quite different from that of A. Just take $[0, 1]$ for example which only has the trivial ideals as an MV-algebra whereas every subinterval $[0, x]$ is a lattice ideal.

Since we shall work primarily with prime ideals in this section we begin with a characterization reminiscent of prime ideals in rings. A will be a fixed MV-algebra throughout this section.

THEOREM 6. *Let P be an ideal of A. Then, P is prime if and only if for any $x, y \in P$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$.*

Proof. Suppose $x \wedge y \in P$ implies $x \in P$ or $y \in P$. Since $x \cdot \bar{y} \wedge \bar{x} \cdot y = 0$ it follows that $x \cdot \bar{y} \in P$ or $\bar{x} \cdot y \in P$ so P is prime. Conversely suppose P is prime and $x \wedge y \in P$. Assume $x \cdot \bar{y} \in P$. Then

$$x \wedge y + x \cdot \bar{y} \in P.$$

So

$$(\bar{x} + y) \cdot x + x \cdot \bar{y} \in P.$$

Now we know from part 1 that

$$x = x \cdot 1 \leqq x \cdot (\bar{x} + x \vee y) \leqq x \cdot (\bar{x} + y + x \cdot \bar{y})$$
$$\leqq (\bar{x} + y) \cdot x + x \cdot \bar{y}$$

and so $x \in P$. Similarly $\bar{x} \cdot y \in P$ implies $y \in P$.

Now consider the set $\mathscr{P}$ of prime ideals of A. Let $\mathscr{S} \subseteq \mathscr{P}$ be any non-empty subset. On the algebra A define $x \equiv y (\mathrm{mod}\ \mathscr{S})$ if and only if for each $P \in \mathscr{S}$, $x \in P$ if and only if $y \in P$. Thus two elements of A are related $(\mathrm{mod}\ \mathscr{S})$ if and only if no prime $P \in \mathscr{S}$ can separate them.

THEOREM 7. $\equiv (\mathrm{mod}\ \mathscr{S})$ *is a congruence relation on A with respect to* $+, \wedge$.

Proof. Clearly $\equiv (\mathrm{mod}\ \mathscr{S})$ is an equivalence relation. Suppose then that $x \equiv y$, $z \equiv w$, (mod $\mathscr{S}$ is to be understood). Let $P \in \mathscr{S}$ and assume $x + z \in P$. Then $x, z \in P$ and so $y, w \in P$ and we have $y + w \in P$. By symmetry, then, $x + z \equiv y + w$. Suppose now that $x \wedge z \in P$. Since P is prime, $x \in P$ or $z \in P$. Thus $y \in P$ or $w \in P$. In either case $y \wedge w \in P$. Again by symmetry, $x \wedge z \equiv y \wedge w$.

Note that $\equiv (\mathrm{mod}\ \mathscr{S})$ is also a congruence with respect to $\vee$ and that $x + z \equiv x \vee z$.

Now for each $x \in A$ let $[x]$ denote the equivalence class of x under $\equiv$ and let $[A]_{\mathscr{S}}$ denote the set of all such equivalence classes. On $[A]_{\mathscr{S}}$ define $+, \cdot$ by

$$[x] + [y] = [x + y], [x] \cdot [y] = [x \wedge y].$$

Let $1 = [1]$, $0 = [0]$ and define $[x] \leqq [y]$ if and only if $[x] \cdot [y] = [x]$; the relation $\leqq$ is obviously well defined.

THEOREM 8. *$\langle [A]_{\mathscr{S}}, +, \cdot, \leqq, 0, 1\rangle$ is a distributive lattice with least element* 0 *and greatest element* 1.

Proof. Clearly $x + x \equiv x$, $x \wedge x \equiv x$. Also obvious is $x + 0 = x$, $x \wedge 1 \equiv x$. So we get that

$$[x] + [x] = [x], [x] \cdot [x] = [x], [x] + 0 = [x], [x] \cdot 1 = [x].$$

Both $+$, $\cdot$ are obviously commutative and associative. To prove distributivity we need to show

$$x \wedge (y + z) \equiv (x \wedge y) + (x \wedge z).$$

So let $P \in \mathscr{S}$ and suppose

$$(x \wedge y) + (x \wedge z) \in P.$$

Then $x \wedge y, x \wedge z \in P$. If $x \in P$ then $x \wedge (y + z) \in P$. If $x \notin P$ then by Theorem 6, $y, z \in P$, so $y + z \in P$. Again, then, $x \wedge (y + z) \in P$. Conversely, suppose $x \wedge (y + z) \in P$. Now

$$x \wedge y, x \wedge z \leqq x \wedge (y + z)$$

so we get $x \wedge y, x \wedge z \in P$; thus

$$(x \wedge y) + (x \wedge z) \in P.$$

Thus $[A]_{\mathscr{S}}$ is an idempotent, commutative semi-ring with 0 and 1. Now

$$[x] \cdot ([x] + [y]) = [x] + [x] \cdot [y] = [x + x \wedge y] = [x]$$

since clearly $x + x \wedge y \equiv x$. Thus $[x] \leqq [x] + [y]$; similarly $[y] \leqq [x] + [y]$. Suppose then that $[x] \leqq [z]$, $[y] \leqq [z]$. Thus

$$[x] \cdot [z] = [x], [y] \cdot [z] = [y],$$

so

$$[z] \cdot ([x] + [y]) = [x] + [y].$$

Hence $[x] + [y] \leqq [z]$; so we see that

$$\text{l.u.b.}([x], [y]) = [x] + [y].$$

Clearly $[x] \cdot [y] \leqq [x], [y]$, so if $[z] \leqq [x], [y]$ we get

$$[z] \cdot ([x] \cdot [y]) = [z]$$

so that

$$\text{g.l.b.}([x], [y]) = [x] \cdot [y].$$

Finally, it is evident that $0 \leqq [x] \leqq 1$ and the theorem is proved.

Note that for each $[x] \in [A]_{\mathscr{S}}$ there is a $[\bar{x}] \in [A]_{\mathscr{S}}$ with $[x] + [\bar{x}] = 1$; in general $[x] \cdot [\bar{x}] \neq 0$ so $[A]_{\mathscr{S}}$ is not a complemented lattice.

Recall that an ideal in a distributive lattice is a non-empty subset I closed under $+$ and such that if $a \in I$, and b is arbitrary, then $ab \in I$. We now relate the ideals of A to those of $[A]_{\mathscr{S}}$.

Let I be an ideal of A. Define

$$I^* = \{[x] \in [A]_{\mathscr{S}}: \text{for some } y \in [x], y \in I\}.$$

THEOREM 9. *I^* is an ideal of $[A]_{\mathscr{S}}$.*

Proof. Since $I \neq \emptyset$ it is evident that $I^* \neq \emptyset$. Let $[x], [y] \in I^*$. Then there are $z \in [x]$, $w \in [y]$ with $z, w \in I$. Hence $z + w \in I$. Thus $[z + w] \in I^*$. But

$$[z + w] = [z] + [w] = [x] + [y]$$

and so I^* is closed under $+$. Now let $[x] \in I^*$, $[y] \in [A]_{\mathscr{S}}$. For some $z \in [x]$ we have $z \in I$. Thus $z \wedge y \in I$; so

$$[x] \cdot [y] = [z] \cdot [y] = [z \wedge y] \in I^*.$$

Now I^* need not be proper; however, if $I \subseteq \cup\mathscr{S}$ then I^* will be proper. For suppose $1 \in I^*$. Then for some $x \in [1]$ we have $x \in I$. So if $I \subseteq \cup\mathscr{S}$ then for some prime $P \in \mathscr{S}$ we would have $x \in P$. But $x \equiv 1 \pmod{\mathscr{S}}$ and so $1 \in P$ which is impossible.

As noted previously, A is a lattice under the operations of $\vee$, $\wedge$; moreover the mapping $x \to [x]:A \to [A]_{\mathscr{S}}$ is a lattice morphism. Since lattice ideals of A are not in general ideals of A this morphism, as a lattice morphism is not of much use to us. As a map however we see it does carry ideals of A to those of $[A]_{\mathscr{S}}$.

We now describe how to go from the ideals of $[A]_{\mathscr{S}}$ to those of A.

Let L be an ideal of $[A]_{\mathscr{S}}$.

THEOREM 10. *Let $L_* = \cup\{[x]:[x] \in L\}$. Then*

i) *L_* is an ideal of A*

ii) $(L_*)^* = L$

iii) *$1 \in L_*$ if and only if $[1] \in L$.*

Proof. i) Let $x, y \in L_*$. Then $[x], [y] \in L$ so $[x + y] \in L$; thus $x + y \in L_*$. If $x \in L_*$, $y \in A$ then $[x \wedge y] \in L$ and so $x \wedge y \in L_*$. Hence L_* is an ideal.

ii) Let $[x] \in (L_*)^*$. Then for some $y \in [x]$ we have $y \in L_*$. Hence $[y] \in L$. But $[y] = [x]$ so $[x] \in L$ and we have $(L_*)^* \subseteq L$. Conversely, let $[x] \in L$. Then $x \in L_*$ so $[x] \in (L_*)^*$. Thus $L \subseteq (L_*)^*$.

iii) If $1 = [1] \in L$ then obviously $1 \in L_*$. On the other hand if $1 \in L_*$ then $1 = [1] \in (L_*)^* = L$.

A proper ideal in a distributive lattice is called *prime* if and only if whenever it contains a product it contains one of the factors.

Let $\mathscr{P}_{\mathscr{S}}$ denote the set of prime ideals of $[A]_{\mathscr{S}}$. Then,

THEOREM 11. *If $P \in \mathscr{S}$ then $P^* \in \mathscr{P}_{\mathscr{S}}$.*

Proof. $P \subseteq \cup\mathscr{S}$ obviously so P^* is proper. Suppose then that $[x] \cdot [y] \in P^*$. Thus, $[x \wedge y] \in P^*$. So there is a $z \in [x \wedge y]$ with $z \in P$. Since $P \in \mathscr{S}$ it follows that $x \wedge y \in P$. Hence $x \in P$ or $y \in P$ so $[x] \in P^*$ or $[y] \in P^*$. Thus $P^* \in \mathscr{P}_{\mathscr{S}}$.

A kind of converse to this theorem is

THEOREM 12. *Let $P \in \mathscr{S}$. Then $(P^*)_* = P$.*

Proof. Let $x \in P$. Then $[x] \in P^*$ so $x \in (P^*)_*$. Conversely, let $x \in (P^*)_*$. Then $x \in [y]$ for some $[y] \in P^*$. Thus there is a $z \in [y]$ with $z \in P$. But $[x] = [z] = [y]$ so $x \in P$. Hence $P = (P^*)_*$.

THEOREM 13. *Suppose $M \in \mathscr{S}$ is a maximal ideal of A. Then M^* is a maximal ideal of $[A]_{\mathscr{S}}$.*

Proof. If $M \in \mathscr{S}$ then M^* is proper in $[A]_{\mathscr{S}}$. Let, then, L be an ideal of $[A]_{\mathscr{S}}$ with $M^* \subseteq L$. Then $(M^*)_* \subseteq L_*$. Now M is maximal, hence prime in A so we have by Theorem 12, $(M^*)_* = M$. Thus $L_* = A$ or $L_* = M$. If $L_* = A$ then $L = [A]_{\mathscr{S}}$ by Theorem 10 iii). If $L_* = M$ then $L = (L_*)^* = M^*$. Thus M^* is maximal.

Thus the map $I \to I_*$ carries primes and maximals in $\mathscr{S}$ to primes and maximals respectively of $[A]_{\mathscr{S}}$. Next we show that primes in $[A]_{\mathscr{S}}$ give rise to primes in A though not necessarily in $\mathscr{S}$.

THEOREM 14. *Let $L \in \mathscr{P}_{\mathscr{S}}$. Then $L_* \in \mathscr{P}$.*

Proof. L is proper, hence so is L_*. Now let $x \wedge y \in L_*$. Then $[x \wedge y] \in (L_*)^* = L$; so $[x] \cdot [y] \in L$ and since $L \in \mathscr{P}$, $[x] \in L$ or $[y] \in L$. So we see that $x \in L_*$ or $y \in L_*$, so $L_* \in \mathscr{P}$.

Let $\mathscr{I}_{\mathscr{S}}$ denote the ideals of A contained in $\cup\mathscr{S}$. Let $\mathscr{I}_{\mathscr{S}}^*$ be the proper ideals of $[A]_{\mathscr{S}}$. We have a mapping $I \to I_*$ of $\mathscr{I}_{\mathscr{S}}$ on $\mathscr{I}_{\mathscr{S}}^*$ which carries the prime and maximal ideals in $\mathscr{S}$ into the prime and maximal ideals respectively of $[A]_{\mathscr{S}}$. We also have a mapping $L \to L_*$ of $\mathscr{I}_{\mathscr{S}}^*$ to the set of ideals of A. This map carries prime ideals to prime ideals. Suppose now that for $x \in A$, $[x] \neq [1] \in [A]_{\mathscr{S}}$. Then $x \not\equiv 1(\text{mod}\,\mathscr{S})$ so for some prime ideal $P \in \mathscr{S}$, $x \in P$. So if L is any proper ideal in $[A]_{\mathscr{S}}$ it follows that $L_* \subseteq \cup\mathscr{S}$. Thus we see that the mapping $L \to L_*$ is a map of $\mathscr{I}_{\mathscr{S}}$ onto $\mathscr{I}$. Moreover it is one-one by Theorem 10 ii).

Let's now consider the case where $\mathscr{S} = \mathscr{P}$. We obtain

THEOREM 15. *$P \to P_*$ is a one-one map of $\mathcal{P}$ to $\mathcal{P}_*$. This mapping, moreover, carries $\mathcal{M}$ onto $\mathcal{M}_*$, the set of maximal ideals of $[A]_{\mathcal{P}}$.*

Proof. Let $P, Q \in \mathcal{P}$. If $P^* = Q^*$ then by Theorem 12 $P = (P^*)_* = (Q^*)_* = Q$. (By Theorem 11 we know $P^*, Q^* \in \mathcal{P}^*$). If $L \in \mathcal{P}^*$ then $L_* \in \mathcal{P}$ and $(L_*)^* = L$. If $M \in \mathcal{M}$ we know by Theorem 13 that $M^* \in \mathcal{M}^*$. Suppose then that $L \in \mathcal{M}^*$. Let I be an ideal of A, $I \neq A$, $L^* \subseteq I$. Then

$$L = (L_*)^* \subseteq I^* \neq [A]_{\mathcal{P}}.$$

Hence $L = I^*$. If $x \in I$ then $[x] \in L$ so $x \in L_*$. Thus $I = L_*$ so $L_* \in \mathcal{M}$, and we see the map carries $\mathcal{M}$ onto $\mathcal{M}^*$.

Also, in the case $\mathcal{S} = \mathcal{P}$, we have that $x \equiv 1 \pmod{\mathcal{P}}$ if and only if $\operatorname{ord} x < \infty$ where $\operatorname{ord} x$ is the least integer n such that $nx = 1$ (see [**1**]). For clearly, $\operatorname{ord} x < \infty$ implies $x \equiv 1 \pmod{\mathcal{P}}$. When no such n exists we say that $\operatorname{ord} x = \infty$. In this case let

$$I = \{y : y \leqq nx \text{ for some } n \geqq 0\}.$$

I will then be a proper ideal in A hence contained in some maximal ideal $M \in \mathcal{P}$. Then $x \in M$, $1 \notin M$ so

$$x \not\equiv 1 \pmod{\mathcal{P}}.$$

We now compare the topologies on $\mathcal{P}$ and $\mathcal{P}^*$. Recall from Section 2 that the sets $R(x)$, $x \in A$, generate a topology on $\mathcal{P}$. Consider also the family of sets

$$T([x]) = \{L : L \in \mathcal{P}^* \text{ and } [x] \notin L\},\ x \in A,$$

which determine a topology on $[A]_{\mathcal{P}}$. Now for any subset $\mathcal{U} \subseteq \mathcal{P}$ let

$$\mathcal{U}^* = \{P^* : P \in \mathcal{U}\}.$$

THEOREM 16. *$R(x)^* = T([x])$, where $R(x)^*$ has the obvious meaning.*

Proof. Let $P^* \in R(x)^*$. Then $x \notin P$ so $[x] \cap P = \emptyset$. Thus $[x] \notin P^*$ so $P^* \in T([x])$. Conversely, let $L \in T([x])$. Then by Theorem 15 $L = P^*$ for some $P \in \mathcal{P}$. So $[x] \notin P^*$, hence $x \notin (P^*)_* = P$. So $P \in R(x)$ and $P^* = L \in R(x)^*$.

Now let

$$\mathcal{T} = \{R(x) : x \in A\},\ \mathcal{T}^* = \{T([x]) : x \in A\}.$$

We have a map $R(x) \to R(x)^*$ of $\mathcal{T}$ to $\mathcal{T}^*$. We will show this map is one-one, onto and preserves arbitrary unions, finite intersections.

THEOREM 17. *$R(x)^* = R(y)^*$ implies $R(x) = R(y)$.*

Proof. Given $R(x)^* = R(y)^*$ we have $T([x]) = T([y])$. So for each $P \in \mathcal{P}^*$ we have $[x] \notin P^*$ if and only if $[y] \notin P^*$. This implies $x \notin P$ if and only if $y \notin P$ since $P = (P^*)_*$. Therefore $R(x) = R(y)$.

THEOREM 18. $(R(x) \cap R(y))^* = R(x)^* \cap R(y)^*$.

Proof. Obviously $R(x) \cap R(y) = R(x \wedge y)$. So

$$(R(x) \cap R(y))^* = T([x \wedge y]) = T([x][y]) = T([x]) \cap T([y])$$

also obviously. Thus

$$(R(x) \cap R(y))^* = R(x)^* \cap R(y)^*.$$

THEOREM 19. *Let J be an index set. Then*

$$(\cup_{x \in J} R(x))^* = \cup_{x \in J} R(x)^*.$$

Proof. Let

$$P^* \in (\cup_{x \in J} R(x))^*.$$

Then by definition $P \in \cup_{x \in J} R(x)$ so for some $x \in J$, $P \in R(x)$. Thus $P^* \in R(x)^*$ so we have inclusion in one direction. Conversely, if

$$P^* \in \cup_{x \in J} R(x)^*$$

then $P^* \in R(x)^*$ for some $x \in J$ so

$$P \in R(x) \subseteq \cup_{x \in J} R(x).$$

Hence

$$P^* \in (\cup_{x \in J} R(x))^*$$

and we have inclusion in the other direction.

Summarizing we have,

THEOREM 20. *For each* MV-*algebra A there is a distributive lattice $[A]_{\mathscr{P}}$ with* 0, 1 *such that the prime ideal spaces of A and $[A]_{\mathscr{P}}$ are homeomorphic.*

We point out that the map $A \rightarrow [A]_{\mathscr{S}}$ is a functor from the cateogry of MV-algebras to the category of distributive lattices with 0, 1. It is not known which distributive lattices are in the range of this functor.

Having now shown the structure of the set of ideals of an MV-algebra is the same as that of some distributive lattice, we wish to take up the relationship between the structure of A and that of $[A]_{\mathscr{S}}$. We will assume here that $\mathscr{S} \subseteq \mathscr{P}, \mathscr{S} \neq \emptyset$ and $\cap \mathscr{S} = \{0\}$. We note that $\cap \mathscr{P} = \{0\}$. $\cap \mathscr{S} = \{0\}$ implies that in $[A]_{\mathscr{S}}$, $[0] = \{0\}$ since if $x \neq 0$ there is a prime $P \in \mathscr{S}$ with $0 \in P$ and $x \notin P$. We use this fact many times.

Recall from [**1**] that an MV-algebra is *locally finite* means every $x \in A$, $x \neq 0$, has *finite order*, i.e., ord $x < \infty$. Now every locally finite A is a subalgebra of [0, 1]. This is just the case where $\{0\}$ is a maximal ideal. Obviously every subalgebra of [0, 1] is locally finite. Locally finite algebras are, then, just the *simple* MV-algebras, i.e., the only ideals are $\{0\}$, A. We have,

THEOREM 21. *Let A be locally finite. Then $[A]_{\mathscr{S}} = \{0, 1\}$, $0 \neq 1$.*

Proof. Let $x \in A$, $x \neq 0$. then $nx = 1$ for some n so we have $x \equiv 1 \pmod{\mathscr{S}}$. Thus $[1] = 1$. Clearly $[x] \neq [0]$.

A kind of converse to this theorem is

THEOREM 22. *If $\mathscr{S} = \mathscr{P}$ and $[A]_{\mathscr{P}} = \{0, 1\}$ then A is locally finite.*

Proof. Let $x \in A$, $x \neq 0$. Then $[x] = 1$ so $x \equiv 1 \pmod{\mathscr{P}}$. Hence x belongs to no prime ideals. It follows from 4.4, 4.6 of [**1**] that $\operatorname{ord} x < \infty$.

Every locally finite algebra is linearly ordered obviously and we can extend the above results to

THEOREM 23. *A is linearly ordered if and only if $[A]_{\mathscr{S}}$ is linearly ordered.*

Proof. If $[A]_{\mathscr{S}}$ is linearly ordered then all its ideals are prime (see ex. 1, pg. 42, Birkhoff's Lattice Theory). Hence $\{[0]\}$ is prime. So $\{[0]\}_* = \{0\}$ is prime in A by Theorem 14. Thus A is linearly ordered (Lemma 1 of [**2**]). Now if A is linearly ordered then for each $x, y \in A$ either $x \wedge y = x$ or $x \wedge y = y$. Hence in $[A]_{\mathscr{S}}$ either $[x] \cdot [y] = [x]$ or $[x] \cdot [y] = [y]$. So we may conclude that $[A]_{\mathscr{S}}$ is linearly ordered.

The next result deals with atoms of $[A]_{\mathscr{S}}$ and is somewhat curious.

LEMMA. *Let $[x]$ be an atom of $[A]_{\mathscr{S}}$. Then as a subset of A, $[x]$ is linearly ordered.*

Proof. Let $a, b \in [x]$; then $[a] = [x] = [b]$. Set $a' = a \cdot \bar{b}$, $b' = \bar{a} \cdot b$. Then

$$a' \wedge b' = a \cdot \bar{b} \wedge \bar{a} \cdot b = 0$$

by Theorem 1, f). Hence $[a'] \cdot [b'] = [0]$. Now $a' = a' \wedge a$ since by Theorem 1, a) $a' \leqq a$. So it follows that $[a'] \leqq [a] = [x]$. Similarly, $[b'] \leqq [x]$. Since $[x]$ is an atom we have $[a'] = [x]$ or $[a'] = 0$ and $[b'] = [x]$ or $[b'] = [0]$. Since $[a'] \cdot [b'] = [0] = 0$ we know $[a'] \neq [x]$ or $[b'] \neq [x]$. If $[a'] \neq [x]$ then $[a'] = 0$ which means $a \cdot \bar{b} = 0$ and so $a \leqq b$. Similarly if $[b'] \neq [x]$ then $b \leqq a$. Thus $[x]$ is linearly ordered.

THEOREM 24. *If $[x]$ is an atom of $[A]_{\mathscr{S}}$ then $\{0\} \cup [x]$ is a linearly ordered ideal of A.*

Proof. Let $L = \{0, [x]\}$; then L is an ideal of $[A]_{\mathscr{S}}$ and so L_* is an ideal of A. $L_* = \{0\} \cup [x]$ and by the lemma is linearly ordered.

For $X \subseteq A$, $X \neq \emptyset$ define

$$X^{\perp} = \{a{:}a \in A \text{ and } a \wedge X = 0\}$$

where

$$a \wedge X = \{a \wedge x{:}x \in X\}.$$

$X^{\perp}$ will be called the *annihilator* of X. As in ring theory the annihilator produces an ideal.

THEOREM 25. *If $X \subseteq A$, $X \neq \emptyset$, $X \neq \{0\}$, then $X^{\perp}$ is a proper ideal in A.*

Proof. Let $a, b \in X^{\perp}$, $x \in X$. Now

$$(a + b) \wedge x = (a + b + \bar{x}) \cdot x \leqq a + (b + \bar{x}) \cdot x$$
$$= a + (b \wedge x) = a.$$

So

$$(a + b) \wedge x = (a + b) \wedge x \wedge x = a \wedge x = 0.$$

Thus $a + b \in X^{\perp}$. If $a \in X^{\perp}$ and $b \leqq a$ it is clear that $b \in X^{\perp}$. So $X^{\perp}$ is an ideal. Since $X \neq \{0\}$ we have $1 \notin X^{\perp}$ so $X^{\perp}$ is proper.

In some cases $X^{\perp}$ is a prime ideal, that is

THEOREM 26. *If I is a linearly ordered ideal of A then $I^{\perp}$ is a prime ideal.*

Proof. We know $I^{\perp}$ is a proper ideal. Suppose then that $a \wedge b \in I^{\perp}$, $a \notin I^{\perp}$, $b \notin I^{\perp}$. Then there are $c', c'' \in I$ with $a \wedge c' \neq 0$, $b \wedge c'' \neq 0$. Let $c = c' \vee c''$. Since I is an ideal and $c \leqq c' + c''$, we see that $c \in I$. Moreover, since

$$a \wedge c' \leqq a \wedge c, \ b \wedge c'' \leqq b \wedge c$$

we have $a \wedge c$, $b \wedge c$ are not 0. Since I is an ideal, $a \wedge c, b \wedge c \in I$. We can assume $a \wedge c \leqq b \wedge c$. Thus

$$a \wedge c = a \wedge a \wedge c \leqq a \wedge b \wedge c = 0.$$

This contradiction proves the theorem.

Notice the above theorem only requires that I be a linearly ordered ideal of the lattice $\langle A, \vee, \wedge, 0, 1\rangle$. It is not known if all prime ideals of A can be obtained as annihilators in this manner.

Now for each $x \in A$ let

$$L_x = \{ [y]{:}[y] \leqq [x] \}.$$

Then L_x is a principal ideal in $[A]_{\mathscr{S}}$ and depends only on the equivalence class of x. Thus $(L_x)_*$ is an ideal of A. Let $J_x = (L_x)_*^{\perp}$ and $B_x = A/J_x$ which also only depends on the equivalence class of x. Combining the previous results gives

THEOREM 27. *If* $[x]$ *is an atom then* B_x *is a linearly ordered* MV-*algebra and* x/J_x, *the image of* x *in* B_x, *is not* 0.

Proof. Since $[x]$ is an atom, $L_x = \{0, [x]\}$. Thus $(L_x)_*$ is a linearly ordered ideal of A. Thus J_x is a prime ideal so B_x is linearly ordered. Now $x \in (L_x)_*$ and $x \wedge x \neq 0$ so $x \notin J_x$; hence $x/J_x \neq 0$.

We will use these results in the next section.

4. In this section we consider a special type of subdirect product representation of a given MV-algebra A which, in some sense, is minimal. Whether or not a given A has this type of representation is crucial to its being semisimple.

Let A be a given MV-algebra. By a *representation* of A we will mean a collection of MV-algebras $\{A_i\}$, $i \in \Gamma$ for some index set Γ, such that A can be embedded isomorphically into the direct product $\prod_i A_i$ subdirectly. The A_i will be referred to as the *components* of the representation.

Given a representation $\{A_i\}$, $i \in \Gamma$, of A and $i_0 \in \Gamma$, the component A_{i_0} will be called *superfluous* if $\{A_i\}$, $i \in \Gamma - \{i_0\}$ is also a representation of A. A component A_i is *essential* if and only if it is not superfluous.

We can characterize essential components by

THEOREM 28. A_i *is essential if and only if there is an* $x \in A$ *such that* $x_i > 0$ *and* $x_j = 0$ *for* $j \neq i$, *where* x_i *is the image of* x *under the mapping*

$$A \to \prod_i A_i \to A_i.$$

Proof. Suppose we have such an x. For each j let P_j be the kernel of the map

$$A \to \prod_i A_i \to A_j.$$

For $j \neq i$ we have $x_j = 0$ so $x \in P_j$. Thus

$$x \in \cap_{j \neq i} P_j;$$

hence A is not contained in $\prod_{j \neq i} A_j$ subdirectly. Thus A_i is essential. Conversely, suppose A_i is an essential component of some representation $\{A_j\}$, $j \in \Gamma$. Then $\{A_j\}$, $j \in \Gamma - \{i\}$ is not a representation. Thus

$$\cap_{j \neq i} P_j \neq 0.$$

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$$x \in \cap_{j \neq i} P_j, \quad x \neq 0.$$

Then $x_j = 0, j \neq i$ and $x_i > 0$.

We will call a representation $\{A_i\}$, $i \in \Gamma$, *irreducible* if and only if each A_i is essential. We then have

THEOREM 29. *An* MV-*algebra* A *has an irreducible representation if and only if there is an* $\mathcal{S} \subseteq \mathcal{P}$ *such that* $\cap\mathcal{S} = \{0\}$ *and* $[A_{\mathcal{S}}]$ *is atomic.*

Proof. Suppose $\{A_i\}$, $i \in \Gamma$, is an irreducible representation. For each $i \in \Gamma$ let P_i be as in the above theorem. Let $\mathcal{S} = \{P_i : i \in \Gamma\}$. Then $\mathcal{S} \subseteq \mathcal{P}$ and $\cap\mathcal{S} = \{0\}$. Now let $[x] \in [A_{\mathcal{S}}]$, $x \neq 0$. Choose $i \in \Gamma$ so that $x_i \neq 0$. Since A_i is essential there is a $y \in A$ with $y_i > 0$, $y_j = 0$, $j \neq i$. Now if $\mathcal{P} \in \mathcal{S}$ then $P = P_j$ for some $j \in \Gamma$. If $x \in P$ then $x_j = 0$ so $j \neq i$. Hence $y_j = 0$ so $y \in P$. This implies

$$x \wedge y \equiv y \ (\mathrm{mod}\ \mathcal{S}),$$

i.e., $[x] \cdot [y] = [y]$ so $[y] \leqq [x]$ and $[y] \neq 0$. If $[z] < [y]$ then $[z \wedge y] = [z]$ and for some $P \in \mathcal{S}$, $z \in P$, and $y \notin P$. So for $j \neq i$, $y \in P_j$ thus $z \wedge y \in P_j$, hence $z \in P_j$. So for i we have $P = P_i$ and $z \in P_i$. Thus

$$z \in \cap_{j \in \Gamma} P_j = \cap\mathcal{S};$$

so $z = 0$. Hence $[y]$ is an atom of $[A_{\mathcal{S}}]$, $[y] \leqq [x]$, so $[A_{\mathcal{S}}]$ is atomic. Conversely, suppose $[A_{\mathcal{S}}]$ is atomic, $\cap\mathcal{S} = \{0\}$. Let AT be the set of atoms of $[A_{\mathcal{S}}]$. Let

$$B = \prod_{[x] \in AT} B_x,$$

where $B_x = A/J_x$ as in Theorem 27. We know by Theorem 27 that each B_x is linearly ordered and is an epimorphic image of A. Let $\varphi : A \to B$ be given by

$$\varphi(a)_{[x]} = a/J_x.$$

φ is clearly a homomorphism. Suppose $a > 0$. Then $[a] > 0$ so there is a $[x] \in AT$ with $[x] \leqq [a]$. Now $x/J_x \neq 0$ in B_x. If $a \in J_x$ then $a \wedge x = 0$ so $[x] \cdot [a] = [0]$; but $[x] \cdot [a] = [x] \neq [0]$. So $a \notin J_x$, and $a/J_x \neq 0$. Thus $\varphi(a) \neq 0$ so we have $\varphi(a) = 0$ if and only if $a = 0$ and this implies φ is an isomorphism of A into B. Since each B_x is an epimorphic image of A, we can infer that $\{B_x\}$, $[x] \in AT$, is a representation of A. Now for $[x] \in AT$ we know $x/J_x \neq 0$. If $AT = \{[x]\}$ then clearly B_x is essential so we can assume there is a $[y] \in AT$, $[y] \neq [x]$. Then $[x] \cdot [y] = 0$ and so $x \wedge y = 0$. In fact, if $[y'] = [y]$ then again, $x \wedge y' = 0$. Thus for $L_y = \{0, [y]\}$ we have $x \in (L_y)^{\perp}_{*}$; that is $x \in J_y$. Hence $x/J_y = 0$. So by Theorem 28 we see that B_x is essential. Therefore the representation $\{B_x\}$, $[x] \in AT$, is irreducible.

Now let x be an atom of A. Then for each $y \in A$, $x \wedge y = x$ or $x \wedge y = 0$. Thus for any $\mathcal{S} \subseteq \mathcal{P}$ with $\cap\mathcal{S} = \{0\}$ we have $[x] \neq 0$, $[x] \cdot [y] = [x]$ or $[x] \cdot [y] = 0$. It follows that $[x]$ is an atom of $[A_{\mathcal{S}}]$. Hence

THEOREM 30. *If A is atomic then A has an irreducible representation.*

Proof. Let $\mathcal{S} = \mathcal{P}$. Choose any $[x] \in [A]_{\mathcal{P}}$, $[x] \neq 0$. then $x \neq 0$ so there's an atom $y \in A$, $y \leqq x$. Thus, by the preceeding remarks, $[y]$ is an atom of $[A]_{\mathcal{P}}$ and $[y] \leqq [x]$. Hence $[A]_{\mathcal{P}}$ is atomic so by the previous theorem A has an irreducible representation.

Suppose now that A is locally finite so that for each $x \in A$, $x \neq 0$ there is an n with $nx = 1$. Now A, which is necessarily linearly ordered (**[1]**), is Archimedean in the sense that for each $x, y \in A$, $x \neq 0$, there is an n with $y \leqq nx$. We are trying, here, to get some understanding of semisimple MV-algebras. So now let A be semisimple with a representation $\{A_i\}$, $i \in \Gamma$, each A_i locally finite. Suppose $x, y \in A$ are such that $nx \leqq y$ for all $n \geqq 0$. If A were locally finite this would imply $x = 0$ or $y = 1$. Thus in the subdirect product,

$$A \subseteq \prod_{i \in \Gamma} A_i$$

we would have $nx_i \leqq y_i$ for all n and each i. Each A_i is locally finite so we have $x_i = 0$ or $y_i = 1$. Consider now the product $x \cdot y$ in A. Then in each A_i we have

$$(x \cdot y)_i = x_i \cdot y_i = \begin{cases} 0, & \text{if } x_i = 0; \\ x_i, & \text{if } x_i \neq 0. \end{cases}$$

Thus for each i, $(x \cdot y)_i = x_i$. It follows then that $x \cdot y = x$. We use this discussion to motivate the following definition.

Definition. An MV-algebra A is *archimedean* if and only if for each $x, y \in A$, if $nx \leqq y$ for all $n \geqq 0$ then $x \cdot y = x$.

Thus we have proved

THEOREM 31. *Every semisimple* MV-*algebra A is archimedean.*

As we have seen, every locally finite MV-algebra A is archimedean. We also have

THEOREM 32. *A linearly ordered archimedean algebra A is locally finite.*

Proof. Suppose not. Let x be such that $x \neq 0$, $nx < 1$ for all n. Since A is linearly ordered we must have $nx < \bar{x}$ for all n as $\bar{x} \leqq nx$ implies $(n + 1)x = 1$. A is archimedean so $x \cdot \bar{x} = x$. But then $x = 0$ contrary to our choice of x.

The property of being archimedean characterizes the semisimple algebras, that is we have the converse of Theorem 31:

THEOREM 33. *Let A be archimedean. Then A is semisimple.*

Proof. Suppose $x \in \mathrm{Rad}(A)$. Then x has infinite order, i.e., $nx < 1$ for all n. Let P be a prime ideal. If $x \notin P$ then P is not maximal. In A/P, x/P, the image of x, must also have infinite order; for P can be extended to a maximal ideal M and $x \in M$, so $x/P \in M/P$ and M/P is a proper ideal in A/P since $M \neq P$. Thus $n(x/P) < \bar{x}/P$ for all n, A/P being linearly ordered. Now A has a representation $\{A/P\}$, $P \in \mathscr{P}$. Consider the element $(nx) \wedge \bar{x}$; its image in the Pth-component A/P is

$$(nx \wedge \bar{x})/P = \begin{cases} 0, & \text{if } x \in P; \\ nx/P, & \text{if } x \notin P. \end{cases}$$

So for each $P \in \mathscr{P}$,

$$(nx \wedge \bar{x})/P = nx/P,$$

hence $nx \wedge \bar{x} = nx$ for all n. Thus $nx \leqq \bar{x}$ for all $n \geqq 0$ and so by the archimedean property $x \cdot \bar{x} = 0$, therefore $x = 0$. This shows that A is semisimple.

It is easy to see that each Bold algebra of fuzzy sets is archimedean; a slight modification of the argument for Theorem 31 will do it.

For each positive integer m let $A(m)$ be the subalgebra of $[0, 1]$ defined by

$$A(m) = \{0, 1/m, 2/m, \ldots, (m - 1)/m, 1\}.$$

We then have

THEOREM 34. *If A is atomic and semisimple then A has a representation $\{A(m)\}$, $m \in \Gamma$, where Γ is a subset of the non-negative integers.*

Proof. Let At be the set of atoms of A. For each $x \in At$ let P_x be a prime ideal with $x \notin P_x$. Let

$$J = \cap_{x \in At} P_x.$$

If $y \in J$, $y \neq 0$ then there is an $x \in At$ with $x \leqq y$. Thus $x \in J$ which implies $x \in P_x$. Thus $J = \{0\}$ and so $\{A/P_x\}$, $x \in At$, is a representation of A. Now x/P_x is an atom in A/P_x. For let $0 < y/P_x \leqq x/P_x$. Then

$$y/P_x \wedge x/P_x = (y \wedge x)/P_x = y/P_x.$$

Since $y \wedge x = 0$ or $y \wedge x = x$ we must have $y \wedge x = x$, so $x \leqq y$. Thus $y/P_x = x/P_x$, and x/P_x is an atom. Suppose x/P_x has infinite order. Then,

$$n(x/P_x) \leqq \bar{x}/P_x \text{ for all } n.$$

Consider then, the element $nx \wedge \bar{x} \in A$. For any $z \in At$, $z \neq x$ we have $x \wedge z = 0 \in P_z$. Since $z \neq P_z$ we have $x \in P_z$. Thus for any n, $(nx)/P_z = 0$ and so

$$(nx \wedge \bar{x})/P_z = 0.$$

Also we have

$$(nx \wedge \bar{x})/P_x = nx/P_x.$$

Thus for each $y \in At$ we have

$$(nx \wedge \bar{x})/P_y = nx/P_y$$

and this implies $nx \wedge \bar{x} = nx$. So for each n, $nx \leqq \bar{x}$ and since, by Theorem 31, A is archimedean we get $x = x \cdot \bar{x} = 0$. This contradiction shows that x/P_x has finite order; by 3.19 of [**1**] we infer that A/P_x is isomorphic to some $A(m)$ for some integer $m > 0$.

One might try to prove the above theorem more directly by using the fact that

$$A \subseteq \prod_{i \in \Gamma} A_i$$

where each A_i is locally finite, then showing each A_i is atomic. However it seems that this approach requires that one show that no maximal ideal contains all of the atoms. In an arbitrary MV-algebra this is not true (see pg. 474 of [**1**]).

We now begin the task of showing that if A is atomic and complete then A is also semisimple. Since in general an atomic MV-algebra need not be semisimple, (the algebra $C \times C$ for example, pg. 474 of [**1**]), some extra condition must be imposed.

Again let A be an MV-algebra $\mathcal{S} \subseteq \mathcal{P}$, $\cap\mathcal{S} = \{0\}$.

THEOREM 35. *Let $a \in A$ be such that $[a]$ is an atom of $[A]_{\mathcal{S}}$. Suppose $x \in [a]$, $x \neq 0$ an idempotent of A. Then $x \geqq [a]$ i.e., $x \geqq y$ for all $y \in [a]$.*

Proof. We have $[x] = [a]$ so for any prime $P \in \mathcal{S}$, $x \in P$ if and only if $a \in P$. For any prime P, $x \wedge \bar{x} \in P$ since for an idempotent x, $x \wedge \bar{x} = 0$. Thus if $x \notin P$ then $\bar{x} \in P$ and so $a \cdot \bar{x} \in P$. If $x \in P \in \mathcal{S}$ then so is a and again $a \cdot \bar{x} \in P$. Hence for all $P \in \mathcal{S}$, $a \cdot \bar{x} \in P$ so $a \cdot \bar{x} = 0$; thus $a \leqq x$. Now if $y \in [a]$ then $[y] = [a]$ and the above argument shows $y \leqq x$.

Clearly then, an atom $[a]$ can contain at most one idempotent. We want to see what happens if an atom $[a]$ contains no idempotent. To this end let I be an arbitrary proper ideal of A and let A_I be the subalgebra generated by I. We describe A_I thusly:

THEOREM 36. *$x \in A_I$ if and only if $x \in I$ or there are $y, z \in I$ and $x = y + \bar{z}$.*

Proof. Clearly if $x \in I$ or $x = y + \bar{z}$ with $y, z \in I$ then $x \in A_I$. Thus, let

$$B = I \cup \{y + \bar{z}: y, z \in I\}.$$

Then $I \subseteq B \subseteq A_I$. Let $u, v \in B$. We have several cases to examine.

i) $u, v \in I$. Then certainly $u + v \in I \subset B$.

ii) $u \in I, v = y + \bar{z}, y, z \in I$. Then $u + v = (u + y) + \bar{z} \in B$.

iii) $u = y_1 + \bar{z}_1, v = y_2 + \bar{z}_2, y_1, y_2, z_1, z_2 \in I$. Then

$$u + v = y_1 + y_2 + \overline{z_1 \cdot z_2} \in B$$

since $y_1 + y_2 \in I, z_1 \cdot z_2 \in I$.

Thus B is closed under $+$.

iv) $u \in I, v \in B$ implies $u \cdot v \in I \subseteq B$.

v) $u = y_1 + \bar{z}_1, v = y_2 + \bar{z}_2, y_1, y_2, z_1, z_2 \in I$. Then

$$u \cdot v = (y_1 + \bar{z}_1) \cdot (y_2 + \bar{z}_2).$$

So

$$u \cdot v = 0 + \overline{\bar{y}_1 \cdot z_1 + \bar{y}_2 \cdot z_2}.$$

Since $\bar{y}_1 \cdot z_1, \bar{y}_2 \cdot z_2 \in I$, $0 \in I$ we see that $u \cdot v \in B$.

Thus B is closed under $+, \cdot$.

vi) If $u \in I$ then clearly $\bar{u} \in B$. If $u = y + \bar{z}$, $y, z \in I$ then $\bar{u} = \bar{y} \cdot z \in I$.

Thus B is closed under $+, \cdot, \bar{}\,$; $0 \in I \subseteq B$ so $1 \in B$. Therefore B is a subalgebra of A containing I and obviously any subalgebra of A that contains I must contain B. Hence $B = A_I$ and the theorem is proved.

THEOREM 37. *Suppose $[a]$ is an atom of $[A]_{\mathcal{S}}$ not containing any idempotent of A. Let $I = (L_a)_*$, $J = I^{\perp}$. Then A_I is isomorphically embeddable in A/J.*

Proof. For $x \in A_I$ let $\varphi(x) = x/J$. Clearly φ is a homomorphism of A_I into A/J. Suppose $\varphi(x) = 0$. Then $x \in J$; since $x \in A_I$, $x \in I$ or $x = y + \bar{z}$ for some $y, z \in I$. If $x \in I$ then $x \in I \cap J$ so $x = x \wedge x = 0$. Suppose then $x = y + \bar{z}, y, z \in I$. $x \in J$ implies $y, \bar{z} \in J$. Now $y \in I \cap J$ so $y = 0$. $z \in I$ so $z \wedge \bar{z} = 0$ thus z is an idempotent of A. Now $z = 0$ implies $x = 1$ which in turn implies $1 \in J$ and that's not possible. $z \neq 0$ implies $z \in [a]$ since $I = \{0\} \cup [a]$. Thus $[a]$ contains an idempotent contrary to the assumption on $[a]$. It follows then that $x = 0$ and so we see that φ is one-one.

The above argument can be modified slightly to arrive at the same conclusion under the assumption that $[a]$ has no maximal element.

Again, let $[a]$ be an atom of $[A]_{\mathscr{S}}$, let $e \in [a]$, e an idempotent. Let $I = (L_a)_*$. Define $1_a = e, \tilde{x} = \bar{x} \cdot e, x \in I$. Let

$$\mathscr{A}_a = \langle I, +, \cdot, \tilde{}, 0, 1_a \rangle.$$

Then

THEOREM 38. *$\mathscr{A}_a$ is a linearly ordered* MV-*algebra.*

Proof. Since $[a]$ is an atom we know from Theorem 24 that I is linearly ordered. Since I is an ideal it is closed under $+, \cdot$ and contains 0. Since $e \in [a]$ we see that $1_a \in I$. Now from Theorem 35 we know $x \leqq 1_a$ for all $x \in I$. So for $x \in I$, $x + 1_a \leqq 1_a$ so $x + 1_a = 1_a$. It now follows from Theorem 1 d) that $x \cdot 1_a = x$. Since $e = 1_a \in I$ we see that for all $x \in I$ that $\tilde{x} = \bar{x} \cdot 1_a \in I$ and clearly $x \cdot \tilde{x} = 0$. Now

$$x + \tilde{x} = x + \bar{x} \cdot 1_a = x \vee 1_a = 1_a,$$

for $x \in I$. For $x, y \in I$,

$$\widetilde{(x + y)} = \overline{x + y} \cdot 1_a = \bar{x} \cdot 1_a \cdot \bar{y} \cdot 1_a = \tilde{x} \cdot \tilde{y}$$

and

$$\tilde{\tilde{x}} = \overline{\bar{x} \cdot 1_a} \cdot 1_a = (x + \bar{1}_a) \cdot 1_a = x \wedge 1_a = x.$$

Thus

$$x \cdot y = \tilde{\tilde{x}} \cdot \tilde{\tilde{y}} = \widetilde{(\tilde{x} + \tilde{y})};$$

hence

$$\widetilde{x \cdot y} = \tilde{x} + \tilde{y}. \tilde{0} = \bar{0} \cdot 1_a = 1 \cdot 1_a = 1_a.$$

Define $x \vee_a y = x \cdot \tilde{y} + y$; then

$$x \vee_a y = x \cdot \bar{y} \cdot 1_a + y.$$

But $x \cdot \bar{y} \in I$ if $x \in I$, so then

$$x \cdot \bar{y} \cdot 1_a = x \cdot \bar{y} \text{ and } x \vee_a y = x \vee y.$$

Similarly, define

$$x \wedge_a y = (x + \tilde{y}) \cdot y = (x + \bar{y} \cdot 1_a) \cdot y.$$

Now since $x \wedge \bar{1}_a \leqq 1_a \wedge \bar{1}_a$ we have $x \wedge \bar{1}_a = 0$; so by 3.2 of [**1**],

$$x + \bar{y} \cdot 1_a = 1_a \cdot (x + \bar{y}).$$

Thus,

$$x \wedge_a y = 1_a \cdot (x + \bar{y}) \cdot y = (x + \bar{y}) \cdot y = x \wedge y.$$

It now follows that all the axioms of an MV-algebra are satisfied and so the theorem is proved.

Note that the above theorem goes through even if $[a]$ is not an atom but then $\mathscr{A}_a$ would not necessarily be linear ordered.

Now by an $\aleph_0$-*complete* algebra we mean an MV-algebra A such that for any countable subset $X \subseteq A$ the least upper bound $\sum X$ exists in A. For $X = \{nx:n \geqq 0\}$ we write $\sum nx$ for $\sum X$. We then obtain

THEOREM 39. *Let A be an $\aleph_0$-complete* MV-*algebra*, $[a]$ *an atom in* $[A]_{\mathscr{S}}$. *Then* $[a]$ *contains an idempotent.*

Proof. If $[A]_{\mathscr{S}} = \{0, 1\}$, then $[a] = 1 = [1]$ so $1 \in [a]$. Assume then that $[a] \neq 1$ and contains no idempotents. Then for $I = (L_a)_*$, $J = I^{\perp}$ we know I is proper and A_I is isomorphically embeddable into A/J, I is linearly ordered, J is prime. Thus A_I is linearly ordered. Suppose for some $x \in A_I$, $x \neq 0$, that x has infinite order. Then $nx \leqq \bar{x}$ for all n. Since $A_I \subseteq A$ and A is $\aleph_0$-complete, we know $\sum nx$ exists in A and $\sum nx \leqq \bar{x}$. Thus

$$x \cdot \sum nx = 0.$$

We claim

$$x + \sum nx = \sum nx = \sum (x + nx).$$

For let $z \geqq x + nx$ for all n. Then

$$z \cdot \bar{x} \geqq (x + nx) \cdot \bar{x} = \bar{x} \wedge nx = nx \text{ for all } n.$$

So $z \cdot \bar{x} \geqq \sum nx$ and therefore

$$x + z \cdot \bar{x} \geqq x + \sum nx.$$

But $x + z \cdot \bar{x} = x \vee z$ and $x \leqq z$ so

$$z = x \vee z \geqq x + \sum nx.$$

It follows that

$$x + \sum nx = \sum (x + nx);$$

since it's evident that

$$\sum (x + nx) = \sum (n + 1)x = \sum nx$$

we then get

$$x + \sum nx = \sum nx.$$

By Theorem 1 d) then

$$x \cdot \sum nx = x.$$

So $x = 0$ which is impossible. Thus A_I is locally finite. Now $a \in [a] \subseteq I$ so $a \in A_I$. Hence for some n, $na = 1$. But then $1 \in I$ which is impossible. Thus $[a]$ must contain an idempotent.

LEMMA. *Suppose* $[a]$ *an atom in* $[A]_{\mathscr{S}}$. *Let* $x \in A$ *and suppose for some* $y \in [a]$ *that* $y \not\leqq x$. *Then there are* $u, v \in A$ *with* $[u] \leqq [a]$, $[v] \cdot [a] = 0$, $x = u + v$.

Proof. Let $u = x \wedge y$, $v = x \cdot \bar{u}$. Then $u \leqq x$ so we have

$$u + v = u + x \cdot \bar{u} = x \vee u = x.$$

Since $u \leqq y \in [a]$ we have $[u] \leqq [a]$. Let $b = a \wedge v$. Then we have

$$b = a \wedge x \cdot \bar{u} \leqq x \cdot \bar{u} \leqq \bar{u}.$$

Now

$$u + b = (u + a) \wedge (u + v) = (u + a) \wedge x,$$

by axiom 5. Thus $u + b \leqq x$. If $b, u \neq 0$ then $b, u \in [a]$ and so $b + u \in [a]$ since $\{0\} \cup [a]$ is an ideal in A. Now $[a]$ is linearly ordered and since $y \in [a]$ we get

$$u + b \leqq y \text{ or } y \leqq u + b.$$

If $u, b = 0$ then certainly $u + b \leqq y$. Now $y \leqq u + b$ implies $y \leqq x$ which is contrary to our assumption. Thus $u + b < y$. Hence

$$u + b \leqq x \wedge y = u$$

so

$$u + b = u = 0 + u.$$

Now $b \leqq v \leqq \bar{u}$ and $0 \leqq \bar{u}$ so by 1.14 of [**1**], $b = 0$. So $a \wedge v = 0$ and we have $[a] \cdot [v] = 0$.

THEOREM 40. *Let* $[a]$ *be an atom in* $[A]_{\mathscr{S}}$, $I = (L_a)_*$, $J = I^{\perp}$. *Suppose* $[a]$ *contains an idempotent. Then* $\mathscr{A}_a$ *is isomorphic to* A/J, *in symbols* $\mathscr{A}_a \cong A/J$.

Proof. Let e be the unique idempotent in $[a]$. For x in $\mathscr{A}_a$ let

$$\varphi(x) = x/J \in A/J.$$

Clearly φ preserves $+, \cdot, ^-, 0$. Moreover $\varphi(x) = 0$ if and only if $x \in J \cap I$ if and only if $x = 0$, so φ is one-one. $e \notin J$ so $\varphi(e) \neq 0$; $e^2 = e$ so $\varphi(e)^2 = \varphi(e)$ so $\varphi(e)$ is an idempotent in A/J. But A/J is linearly ordered and the only idempotents in a linearly ordered MV-algebra are 0, 1. Hence $\varphi(e) = 1$. Now for $x \in I$, i.e., x in $\mathscr{A}_a$, $\tilde{x} = \bar{x} \cdot e$ so

$$\varphi(\tilde{x}) = \varphi(\bar{x}) \cdot \varphi(e) = \overline{\varphi(x)}.$$

Thus φ is an injection of $\mathscr{A}_a$ into A/J. Now let $z \in A$ and consider $z/J \in A/J$. If $z \geqq y$ for all $y \in [a]$ then $e \leqq z$ so

$$1 = e/J \leqq z/J \leqq 1$$

and so

$$\varphi(e) = 1 = z/J.$$

Suppose then there is a $y \in [a]$ with $y \nleqq z$. By the lemma there are $u, v \in A$ with $u + v = z$, $[u] \leqq [a]$, $[v] \cdot [a] = 0$. This means $u \in I$, $v \in J$. Hence

$$\varphi(u) = u/J = u/J + v/J = (u + v)/J = z/J.$$

Thus φ is an epimorphism so $\mathscr{A}_a \cong A/J$.

Assembling the previous theorems we have

THEOREM 41. *Suppose A is $\aleph_0$-complete and $[A]_{\mathscr{S}}$ is atomic. Then A is semisimple.*

Proof. From Theorem 29 we know that A has an irreducible representation $\{B_x\}$, $[x] \in AT$; thus $A \subseteq \prod_{[x]\in AT} B_x$ subdirectly where $B_x = A/J_x$, $J_x = (L_x)^{\perp}_{*}$. By the preceeding theorem $B_x \cong \mathscr{A}_x$. Let $x \in AT$. Now the elements of $\mathscr{A}_x$ are also elements of A; let y belong to $\mathscr{A}_x$, $y \neq 0$, and suppose y has infinite order. Since $\mathscr{A}_x$ is linearly ordered this implies $ny \leqq \bar{y}$ for all n. But $\bar{y} = \bar{y} \cdot e$ where e is the idempotent in $[x]$ guaranteed by Theorem 39. Thus $ny \leqq \bar{y}$ for all n in A. As in the proof of Theorem 39 this implies $y = 0$ which is impossible. Hence $\mathscr{A}_x$ is locally finite and therefore B_x is locally finite. It now follows that A is semisimple.

Our final theorem is

THEOREM 42. *If A is $\aleph_0$-complete and atomic then A has a subdirect product representation of finite linearly ordered* MV-*algebras of the type $A(m)$. Moreover this representation is irreducible.*

Proof. From Theorem 30 we know $[A]_{\mathscr{S}}$ is atomic with $\mathscr{S} = \mathscr{P}$. Hence by the preceeding theorem A is semisimple. The result now follows from Theorems 30 and 34.

5. Throughout this paper several questions concerning MV-algebras presented themselves. Theorem 41 raises the question of whether it's necessary to assume A is atomic, i.e., is any complete MV-algebra semisimple? I haven't been able to show this without some additional assumptions, e.g. something that would imply $\sum nx$ is idempotent. My suspicion is that not every complete MV-algebra is semisimple. Another question of interest is whether or not, in Theorem 42, A is the direct product of the algebras $A(m)$. It is easy to show that A is "dense" in the direct product in the sense that given any $y \in \prod_m A(m)$ and any finite set of indices F, there is an $x \in A$ with $x_i = y_i$ for each $i \in F$. I suspect that A is in fact isomorphic to the direct product of the algebras $A(m)$.

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