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Spherical Space Forms: Homotopy Types and Self-Equivalences for the Group $(\mathbb{Z}/a \rtimes \mathbb{Z}/b) \times SL_2(\mathbb{F}_p)$

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Abstract. Let $G = (\mathbb{Z}/a \rtimes \mathbb{Z}/b) \times \mathrm{SL}_2(\mathbb{F}_p)$, and let $X(n)$ be an n-dimensional CW-complex of the homotopy type of an n-sphere. We study the automorphism group $\mathrm{Aut}(G)$ in order to compute the number of distinct homotopy types of spherical space forms with respect to free and cellular G-actions on all CW-complexes $X(2dn-1)$, where $2d$ is the period of G. The groups $\mathcal{E}(X(2dn-1)/\mu)$ of self homotopy equivalences of space forms $X(2dn-1)/\mu$ associated with free and cellular G-actions μ on $X(2dn-1)$ are determined as well.

Introduction

The study of group actions has been, and continues to be, of considerable interest among mathematicians; see [2, 16]. Useful cohomological and geometrical aspects are presented in [2], where a list of basic conjectures is provided as well. Swan [16] has proved that any finite group with periodic cohomology of period $2d$ acts freely and cellularly on a $(2d-1)$-dimensional CW-complex of the homotopy type of the $(2d-1)$-sphere. By means of results from [16], it is shown in [17] that the set of homotopy types of spherical space forms of all free cellular G-actions on $(2n-1)$-dimensional CW-complexes with the homotopy type of the $(2n-1)$-sphere is in one-to-one correspondence with those orbits which contain a generator of the cyclic group $H^{2n}(G) = \mathbb{Z}/|G|$ under the action of $\pm\,\mathrm{Aut}(G)$ (see [5] for a different approach).

The structure of the group $\mathcal{E}(X)$ of homotopy classes of self homotopy equivalences for certain spaces X has been extensively studied (*e.g.*, [12]). The work of Smallen [13] suggested that the case of spherical space forms is of special interest.

The present paper continues the project of [5–8] and has two goals. The first is to calculate the number of homotopy types of spherical space forms for the groups of the form $(\mathbb{Z}/a \rtimes \mathbb{Z}/b) \times \mathrm{SL}_2(\mathbb{F}_p)$, corresponding to the family V from the table [1, Ch. IV; Theorem 6.15] with the Suzuki–Zassenhaus classification of finite periodic groups. The second is to determine the group of homotopy classes of self-equivalences for space forms given by free $(\mathbb{Z}/a \rtimes \mathbb{Z}/b) \times \mathrm{SL}_2(\mathbb{F}_p)$-actions. The results of [5–8], handling those groups from families I, II, III and IV of the table, are crucial to proving the main results stated in Theorem 2.2 and Theorem 2.4.

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Section 1 makes use of the Lyndon–Hochschild–Serre spectral sequence to produce induced maps on cohomology by automorphisms of the group in question. The main result is Proposition 1.5 and it is of interest in its own right. In Section 2, we compute the number of homotopy types of spherical space forms for free and cellular actions μ of the groups $(\mathbb{Z}/a \rtimes \mathbb{Z}/b) \times \mathrm{SL}_2(\mathbb{F}_p)$ on $X(2dn-1)$, and describe the structure of the groups $\mathcal{E}(X(2dn-1)/\mu)$ of self homotopy equivalences for spherical space forms $X(2dn-1)/\mu$ with respect to μ.

The study of the homotopy types and the self homotopy equivalences of spherical space forms for the last family of groups from the table in [1, Ch. IV; Theorem 6.15] is in progress.

1 Algebraic Background

Let $\mathbb{F}_p$ be the simple field of characteristic p and $\mathbb{F}_p^*$ its multiplicative group. We consider two groups of 2×2-matrices over the field $\mathbb{F}_p$: the *general linear group* $\mathrm{GL}_2(\mathbb{F}_p)$ of order $p(p-1)(p^2-1)$ and its subgroup, the *special linear group* $\mathrm{SL}_2(\mathbb{F}_p)$ of order $p(p^2-1)$ (see [11, Ch. III]).

Because $\mathrm{SL}_2(\mathbb{F}_2) = \mathrm{GL}_2(\mathbb{F}_2) = S_3$, the symmetric group on three letters, *a fortiori* the automorphism group $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_2)) \cong \mathrm{Inn}(S_3) \cong S_3$, where Inn stands for the group of inner automorphisms. Hence, every automorphism of $\mathrm{SL}_2(\mathbb{F}_2)$ is determined by conjugation with a 2×2-matrix in the group $\mathrm{GL}_2(\mathbb{F}_2)$.

For p odd, from [9, 14] every automorphism of $\mathrm{SL}_2(\mathbb{F}_p)$ is also determined by such a conjugation. If det is the determinant function and $\det(A) = \det(B)$ for $A, B \in \mathrm{GL}_2(\mathbb{F}_p)$, then $A^{-1}B \in \mathrm{SL}_2(\mathbb{F}_p)$. In particular, $A^{-1}\left(\begin{smallmatrix} \det(A) & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{F}_p)$, and elements of the outer automorphism group $\mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$ are represented by matrices $\left(\begin{smallmatrix} \omega & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for any $\omega \in \mathbb{F}_p^*$. But $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \omega^{-1} & 0 \\ 0 & \omega^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} \omega^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so any non-singular matrix of the form $\left(\begin{smallmatrix} \omega^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ yields an inner automorphism of $\mathrm{SL}_2(\mathbb{F}_p)$. If $\omega, \omega' \in \mathbb{F}_p^*$ are non-squares, then $\omega^{-1}\omega'$ is a square, and all squares in $\mathbb{F}_p^*$ form a subgroup with index two, *a fortiori* $\mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p)) \cong \mathbb{Z}/2$. Furthermore, -1 is a non-square in $\mathbb{F}_p^*$ if and only if $4 \nmid p-1$ and the center $\mathcal{Z}(\mathrm{SL}_2(\mathbb{F}_p)) \cong \mathbb{Z}/2$ (see [11, Ch. III]). Hence, $\mathrm{Inn}(\mathrm{SL}_2(\mathbb{F}_p)) \cong \mathrm{SL}_2(\mathbb{F}_p)/(\mathbb{Z}/2) = \mathrm{PSL}_2(\mathbb{F}_p)$, the *projective special linear group* of 2×2-matrices over $\mathbb{F}_p$, and we can state the following.

Proposition 1.1 *Let p be an odd prime number. Then there is a short exact sequence*

$$1 \to \mathrm{PSL}_2(\mathbb{F}_p) \longrightarrow \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p)) \longrightarrow \mathbb{Z}/2 \to 1.$$

This sequence splits if and only if $4 \nmid p-1$.

Given a group G, write $H^n(G)$ for its n-th cohomology group with constant coefficients in the integers $\mathbb{Z}$ for $n \geq 0$. Then, any automorphism $\varphi \in \mathrm{Aut}(G)$ gives rise to the induced automorphism $\varphi^* \in \mathrm{Aut}(H^n(G))$.

By a *period* of a group G we mean an integer d such that $H^n(G) = H^{n+d}(G)$ for all $n > 0$ and a group G with this property is called *periodic*. Among all periods of a group G there is the least one which we call the *period of G*. All other periods are multiple of that one and, by [3, §11], a period of any periodic group is even.

If A is now an abelian group and q a prime number, let $A_{(q)}$ denote the q-primary component of A. Replacing $H^*(G)$ by $H^*(G)_{(q)}$, we can define a *q-period* for the finite group G. Then, (see [18, p. 35]), the period of G is the least common multiple of all its least q-periods. Let $\mathbb{Z}[1/p]$ be the localization of $\mathbb{Z}$ with respect to a prime p. By [18, Theorem 9.1], we have:

Theorem 1.2 *The cohomology algebra $H^*(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z}[1/p])$ is isomorphic to the polynomial ring over $\mathbb{Z}/(p^2-1)$ on a 4-dimensional generator, and $H^*(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z})_{(p)}$ is isomorphic to the polynomial ring over $\mathbb{Z}/p$ on a $(p-1)$-dimensional generator.*

From Theorem 1.2 we easily derive that the least common multiple $[4, p-1]$ is the period of $\mathrm{SL}_2(\mathbb{F}_p)$. If $\varphi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p))$ represents a generator of $\mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$, then by Proposition 1.1, we get $(\varphi^*)^2 = \mathrm{id}_{H^*(\mathrm{SL}_2(\mathbb{F}_p))}$ for the induced map φ^* on the cohomology $H^*(\mathrm{SL}_2(\mathbb{F}_p))$. Moreover, again by means of Proposition 1.1, it holds that $\varphi^* = \psi^*$ for any $\varphi, \psi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p))$ representing generators of $\mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$. To find φ^*, we study its restrictions to the q-primary component of $H^*(\mathrm{SL}_2(\mathbb{F}_2))$; by means of the following lemma we can restrict our investigations to the cohomology groups $H^*(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z}/q)$.

Lemma 1.3 *If $\tau\colon \mathbb{Z}/q^n \to \mathbb{Z}/q^n$ is an automorphism such that $\tau^2 = \mathrm{id}_{\mathbb{Z}/q^n}$ and $\tau \otimes \mathrm{id}_{\mathbb{Z}/q}\colon \mathbb{Z}/q^n \otimes \mathbb{Z}/q \to \mathbb{Z}/q^n \otimes \mathbb{Z}/q$ is the identity, then so is τ.*

Proof Let the automorphism τ be determined by an integer l relatively prime with q^n. Because $\tau \otimes \mathrm{id}_{\mathbb{Z}/q}$ is the identity map, *a fortiori* $l \equiv 1 (\mathrm{mod}\, q)$. But $l^2 \equiv 1 (\mathrm{mod}\, q^n)$, so $l \equiv 1 (\mathrm{mod}\, q^n)$, and the result follows. ■

Next, we make use of the following well-known fact (see [3]). If S is a q-Sylow subgroup of a finite group G, then the restriction map $\mathrm{res}^*\colon H^*(G)_{(q)} \to H^*(S)$ is a monomorphism. Moreover, by [15, Theorem 3], the following holds.

Theorem 1.4 *If G is a q-normal finite group and S one of its q-Sylow subgroup, then $\mathrm{res}^*\colon H^*(G)_{(q)} \to H^*(S)$ determines an isomorphism $H^*(G)_{(q)} \xrightarrow{\cong} H^*(S)^{N_G(\mathcal{Z}(S))}$, where $N_G(\mathcal{Z}(S))$ is the normalizer of the center $\mathcal{Z}(S)$ of the q-Sylow subgroup S in G.*

In view of [1, Corollary 6.6], any finite period group is q-normal for all primes q. In particular, that is the case for $\mathrm{SL}_2(\mathbb{F}_p)$, where p is an arbitrary prime. Furthermore, $\mathrm{GL}_2(\mathbb{F}_p)$ is p-normal because any of its p-Sylow subgroups has order p.

To find the induced homomorphism $\varphi^*\colon H^*(\mathrm{SL}_2(\mathbb{F}_p)) \to H^*(\mathrm{SL}_2(\mathbb{F}_p))$, where $\varphi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p))$ represents a generator of $\mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$, we apply Theorem 1.4 and consider the following cases separately.

Case 1 If $q = p$, then $S = \left\{ \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) ; t \in \mathbb{F}_p \right\}$ is an abelian p-Sylow subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$. Because $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & \alpha t \\ 0 & 1 \end{smallmatrix}\right)$ for any $\alpha \in \mathbb{F}_p^*$, $t \in \mathbb{F}_p$ *a fortiori*

$$\left\{ \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) ; \alpha \in \mathbb{F}_p^*, t \in \mathbb{F}_p \right\} \subseteq N_{\mathrm{GL}_2(\mathbb{F}_p)}(S).$$

The isomorphism $H^n(\mathrm{GL}_2(\mathbb{F}_p))_{(p)} \cong H^n(S)^{N_{\mathrm{GL}_2(\mathbb{F}_p)}(S)}$ implied by Theorem 1.4 then yields

$$H^n(\mathrm{GL}_2(\mathbb{F}_p), \mathbb{Z}/p) = \begin{cases} 0 & \text{if } n \neq 2k(p-1), \\ \mathbb{Z}/p & \text{if } n = 2k(p-1), \end{cases}$$

for any $k \geq 0$. On the other hand, the split exact sequence

$$1 \to \mathrm{SL}_2(\mathbb{F}_p) \longrightarrow \mathrm{GL}_2(\mathbb{F}_p) \longrightarrow \mathbb{F}_p^* \to 0$$

leads to an epimorphism $\mathbb{F}_p^* \to \mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p)) \cong \mathbb{Z}/2$. Furthermore, the Lyndon–Hochschild–Serre spectral sequence applied to that exact sequence yields

$$E_2^{r,s} = \begin{cases} 0 & \text{if } r, s > 0, \\ H^0(\mathbb{F}_p^*, \mathbb{Z}/p) & \text{if } r, s = 0, \\ H^0(\mathbb{F}_p^*, H^s(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z}/p) = H^s(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z}/p)^{\mathbb{F}_p^*} & \text{if } r = 0 \text{ and } s > 0. \end{cases}$$

Because $H^{p-1}(\mathrm{GL}_2(\mathbb{F}_p), \mathbb{Z}/p) = 0$ and $H^{p-1}(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z}/p) = \mathbb{Z}/p$ (by Theorem 1.2), *a fortiori* there is a non-trivial action of $\mathbb{F}_p^*$ on $H^{p-1}(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z}/p) = \mathbb{Z}/p$. Consequently, in virtue of Theorem 1.2, the induced homomorphism φ^* restricts to the multiplication by $(-1)^*$ on the p-primary component of $H^{(p-1)*}(\mathrm{SL}_2(\mathbb{F}_p))$; this is because -1 is the only element of $\mathbb{F}_p^*$ with order two.

Case 2 Let q be an odd prime number and $q \mid p-1$. Because of the isomorphism

$$\left\{ \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix} \right) ; \omega \in \mathbb{F}_p^* \right\} \cong \mathbb{Z}/p-1,$$

any q-Sylow subgroup $\mathbb{Z}/q^r$ of $\mathbb{Z}/p-1$ is a q-Sylow subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ as well. Since the greatest common divisor $(q^r, \frac{p-1}{q^r}) = 1$, from the isomorphism $\mathbb{Z}/(p-1) \cong \mathbb{Z}/q^r \oplus \mathbb{Z}/(\frac{p-1}{q^r})$ we see that $H^*(\mathbb{Z}/(p-1)) \cong H^*(\mathbb{Z}/q^r) \oplus H^*(\mathbb{Z}/(\frac{p-1}{q^r}))$. Thus, the map $\mathrm{res}^*\colon H^*(\mathbb{Z}/(p-1))_{(q)} \to H^*(\mathbb{Z}/q^r)$ is an isomorphism. By Theorem 1.2, $H^*(\mathrm{SL}_2(\mathbb{F}_p))_{(q)} = 0$ for $4 \nmid *$ and the map $\mathrm{res}^{4*}\colon H^{4*}(\mathrm{SL}_2(\mathbb{F}_p))_{(q)} \to H^{4*}(\mathbb{Z}/q^r)$ is an isomorphism so $\mathrm{res}^{4*}\colon H^{4*}(\mathrm{SL}_2(\mathbb{F}_p))_{(q)} \to H^{4*}(\mathbb{Z}/(p-1))_{(q)}$ is an isomorphism as well. Furthermore, $\left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix} \right) \left(\begin{smallmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{smallmatrix} \right) = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix} \right)$ for any $\gamma, \omega \in \mathbb{F}_p^*$. Consequently, the induced homomorphism φ^* restricts to the identity on the q-primary component of $H^*(\mathrm{SL}_2(\mathbb{F}_p))$ for any $\varphi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p))$ which represents a generator of $\mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$.

Case 3 Now if q is an odd prime with $q \mid p+1$, then $p^2 \equiv 1 (\operatorname{mod} q)$. Hence, by [1, Corollary 4.8, Ch. VII] there exists an isomorphism

$$H^*(\mathrm{GL}_2(\mathbb{F}_p), \mathbb{Z}/q) \cong H^*(\mathbb{Z}/q^r, \mathbb{Z}/q)^{\mathbb{Z}/2} \cong \mathbb{F}_q[b^2] \otimes E(be)$$

such that $|b| = 2$ and $|e| = 1$. Consequently, $H^n(\mathrm{GL}_2(\mathbb{F}_p), \mathbb{Z}/q) \neq 0$ for $n \equiv 0, 3 \,(\operatorname{mod} 4)$. But, $H^n(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z}/q) \neq 0$ (by Theorem 1.2) for $n \equiv 0, 3 \,(\operatorname{mod} 4)$ as well. Therefore, by applying the Lyndon–Hochschild–Serre spectral sequence to the exact sequence

$$1 \to \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{GL}_2(\mathbb{F}_p) \to \mathbb{F}_p^* \to 0,$$

it follows that the action of $\mathbb{F}_p^*$ on $H^n(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z}/q)$ is trivial.

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Case 4 For $q = 2$ we consider two subcases.

Case 4.1 If $4 \mid p-1$ then, by [1, p. 151], the group

$$Q_{2(p-1)} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} ; \alpha \in \mathbb{F}_p^* \right\}$$

is a generalized quaternion subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ of order $2(p-1)$. Given a matrix $\left(\begin{smallmatrix} \omega & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\omega \in \mathbb{F}_p^*$, we get

$$\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$
$$\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix}$$

for any $\alpha \in \mathbb{F}_p^*$. Therefore, for a generator α_0 of the cyclic group $\mathbb{F}_p^*$, the matrices $X = \left(\begin{smallmatrix} \alpha_0 & 0 \\ 0 & \alpha_0^{-1} \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ are generators of the group $Q_{2(p-1)}$. Now if $\varphi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p))$ is determined by the matrix $\left(\begin{smallmatrix} \omega & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then it restricts to an automorphism of the subgroup $Q_{2(p-1)}$ and $\varphi(X) = X$. Consequently, by [6] the induced homomorphism φ^{4*} restricted to $H^{4*}(Q_{2(p-1)})$ is the identity.

Let $Q_{2^{t+1}}$ be a 2-Sylow subgroup of $Q_{2(p-1)}$. Then, by the description of the cohomology $H^*(Q_{2(p-1)})$ presented in [3, Ch. XII], the map $\mathrm{res}^*\colon H^*(Q_{2(p-1)})_{(2)} \to H^*(Q_{2^{t+1}})$ is an isomorphism. Because $Q_{2^{t+1}}$ is also a 2-Sylow subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ while, by Theorem 1.2, the map $\mathrm{res}^{4*}\colon H^{4*}(\mathrm{SL}_2(\mathbb{F}_p))_{(2)} \to H^{4*}(Q_{2^{t+1}})$ is an isomorphism, *a fortiori* $\mathrm{res}^{4*}\colon H^{4*}(\mathrm{SL}_2(\mathbb{F}_p))_{(2)} \to H^{4*}(Q_{2(p-1)})_{(2)}$ is an isomorphism as well. Consequently, the map φ^{4*} restricts to the identity on the 2-primary component of $H^{4*}(\mathrm{SL}_2(\mathbb{F}_p))$, for any $\varphi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p))$ which represents a generator of $\mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$.

Case 4.2 If $4 \nmid p-1$ then, following [4, Lemma 5.1, Ch. VI], we consider the special orthogonal group $SO_2(\mathbb{F}_p) = \left\{ \left(\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix}\right) ; \alpha^2 + \beta^2 = 1 \right\}$ of 2×2-matrices over $\mathbb{F}_p$. According to [4, lemma 4.1, Ch. VI], there is an isomorphism $SO_2(\mathbb{F}_p) \cong \mathbb{Z}/(p+1)$. Let $X = \left(\begin{smallmatrix} \alpha_0 & -\beta_0 \\ \beta_0 & \alpha_0 \end{smallmatrix}\right)$ be a generator of $SO_2(\mathbb{F}_p)$. By [4, Lemma 3.11, Ch. II] there exist $a, b \in \mathbb{F}_p$ such that $a^2 + b^2 = -1$. Then, the matrix $Y = \left(\begin{smallmatrix} -a & b \\ b & a \end{smallmatrix}\right)$ lies in $\mathrm{SL}_2(\mathbb{F}_p)$ and $Y^2 = X^{\frac{p+1}{2}}$, $Y^{-1}XY = X^{-1}$. Consequently, the subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ generated by X, Y is the generalized quaternion group $Q_{2(p+1)}$. But for $4 \nmid p-1$, the element -1 is not a square in $\mathbb{F}_p$, so the automorphism $\varphi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p))$ determined by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ represents the generator of $\mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$. Because $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) X \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = X^{-1}$, while $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) Y \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = Y \left(\begin{smallmatrix} -a^2+b^2 & -2ab \\ 2ab & -a^2+b^2 \end{smallmatrix}\right)$, *a fortiori* φ restricts to an automorphism of the subgroup $Q_{2(p+1)}$ with $\varphi(X) = X^{-1}$. Hence, using [6], the induced homomorphism φ^* restricts to the identity map on $H^*(Q_{2(p-1)})$. Finally, we derive as above that the induced homomorphism φ^{4*} also restricts to the identity on the 2-primary component of $H^{4*}(\mathrm{SL}_2(\mathbb{F}_p))$.

In the case $p = 2$, then $\mathrm{SL}_2(\mathbb{F}_2) = S_3$ and $\mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_2)) = \mathrm{Inn}(\mathrm{SL}_2(\mathbb{F}_2)) = S_3$. Consequently, the induced homomorphism φ^* is the identity for any $\varphi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_2))$. Summing up the discussion above and using Lemma 1.3, we can now state the following.

Proposition 1.5 *Let p be a prime, φ an automorphism of the group $\mathrm{SL}_2(\mathbb{F}_p)$ representing the generator of the group $\mathrm{Out}(\mathrm{SL}_2(\mathbb{F}_p))$ and φ^* the induced homomorphism on the cohomology groups $H^*(\mathrm{SL}_2(\mathbb{F}_p))$. Then*

(i) *φ^* is the identity map for $p = 2$;*
(ii) *φ^{4*} restricts to the identity map on the q-primary component of $H^{4*}(\mathrm{SL}_2(\mathbb{F}_p))$ for $q \neq p$ and $\varphi^{(p-1)*}$ restricted to the p-primary component of $H^{(p-1)*}(\mathrm{SL}_2(\mathbb{F}_p))$ is given by the multiplication by $(-1)^*$, provided p is an odd prime.*

We point out that, from the above, the induced homomorphism $\varphi^{[4,p-1]*}$ on the cohomology group $H^{[4,p-1]*}(\mathrm{SL}_2(\mathbb{F}_p)) \cong \mathbb{Z}/p(p^2-1)$ is given by the multiplication by $2p^2-1$ provided $4 \nmid p-1$, while it is the identity otherwise.

2 Space Forms and Self-Equivalences

Throughout the rest of the paper $X(n)$ denotes an n-dimensional CW-complex with the homotopy type of the n-sphere and the group $\mathrm{Aut}(\mathbb{Z}/a)$ is identified with the unit group $(\mathbb{Z}/a)^*$ of the mod a ring $\mathbb{Z}/a$. Given a free cellular action μ of a finite group G with order $|G|$ on a CW-complex $X(2n-1)$, let us write $X(2n-1)/\mu$ for the corresponding orbit space; this is known as a $(2n-1)$-*spherical space form* or a *Swan* $(2n-1)$-*complex* (see [2]). The group G is then periodic with period $2d$ dividing $2n$, and by [3, Ch. XVI, §9] there is an isomorphism $H^{2n}(G) \cong \mathbb{Z}/|G|$. Two spherical space forms $X(2n-1)/\mu$ and $X'(2n-1)/\mu'$ are called equivalent if they are homeomorphic; let $\mathcal{K}_G^{2n-1}$ denote the set of all such classes. We say that two such classes $[X(2n-1)/\mu]$ and $[X'(2n-1)/\mu']$ are homotopic if and only if the space forms $X(2n-1)/\mu$ and $X'(2n-1)/\mu'$ are homotopy equivalent. Write $\mathcal{K}_G^{2n-1}/_{\simeq}$ for the associated quotient set of $\mathcal{K}_G^{2n-1}$ and write card $\mathcal{K}_G^{2n-1}/_{\simeq}$ for its cardinality. It is shown in [17], using [16], that elements of the set $\mathcal{K}_G^{2n-1}/_{\simeq}$ are in one-to-one correspondence with orbits which contain a generator in $H^{2n}(G) = \mathbb{Z}/|G|$ under the action of $\pm\,\mathrm{Aut}(G)$ (see [5] for a different approach). But generators of the group $\mathbb{Z}/|G|$ are given by the unit group $(\mathbb{Z}/|G|)^*$ of the ring $\mathbb{Z}/|G|$. Thus, those homotopy types are in one-to-one correspondence with the quotient $(\mathbb{Z}/|G|)^*/\{\pm\varphi^*;\ \varphi \in \mathrm{Aut}(G)\}$, where φ^* is the automorphism on the cohomology $H^{2n}(G) = \mathbb{Z}/|G|$ induced from $\varphi \in \mathrm{Aut}(G)$.

Let G_1 and G_2 be finite groups with relatively prime orders. If G_1 and G_2 are periodic with periods $2d_1$ and $2d_2$, respectively, then by [7] the least common multiple $[2d_1, 2d_2]$ is the period of the product $G_1 \times G_2$. Following *mutatis mutandis* the proof of [5, Proposition 2.1], we observe that one gets this very useful result in the sequel.

Proposition 2.1 *Let $G_1, \dots, G_s$ be finite periodic groups with relatively prime orders.*

If $G = G_1 \times \cdots \times G_s$ then there is an extension

$$0 \to (\mathbb{Z}/2)^t \to (\mathbb{Z}/|G|)^*/\{\pm\varphi^* \,;\, \varphi \in \mathrm{Aut}(G)\}$$
$$\to \prod_{i=1}^{s} (\mathbb{Z}/|G_i|)^*/\{\pm\varphi_i^* \,;\, \varphi_i \in \mathrm{Aut}(G_i)\} \to 0,$$

where φ_i is the restriction of φ to G_i for $i = 1, \ldots, s$, and where t is defined as follows:

(i) $t = 0$, *if* $-1 \in \{\varphi_i^* \,;\, \varphi \in \mathrm{Aut}(G_i)\}$ *for any* $i = 1, \ldots, s$;
(ii) $t = s - 1 - \#\{i \,;\, -1 \in \{\varphi_i^* \,;\, \varphi_i \in \mathrm{Aut}\,(G_i)\}\}$, *otherwise.*

If now $(a, b) = (ab, p(p^2 - 1)) = 1$, where p is a prime p and where $\alpha\colon \mathbb{Z}/b \to (\mathbb{Z}/a)^*$ is an action, then $(\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b) \times \mathrm{SL}_2(\mathbb{F}_p)$ is one of the finite periodic groups in the table from [1, Theorem 6.15, Ch. IV]. By means of [7], $\mathrm{Aut}(\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b) \times \mathrm{SL}_2(\mathbb{F}_p)) = \mathrm{Aut}(\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b) \times \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p))$. We write $|\alpha(g)^*|$ for the order of $\alpha(g)^*$ with $g \in \mathbb{Z}/b$ and $\ell(\alpha) = \big[\, |\alpha(g)^*| : g \in \mathbb{Z}/b \,\big]$ for the least common multiple of those orders. But, by [7, Proposition 2.1], $2\ell(\alpha)$ is the least period of $\mathbb{Z}/b \rtimes_\alpha \mathbb{Z}/a$. Consequently, $2d = [4, p-1, 2\ell(\alpha)]$ is the least period of the group $(\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b) \times \mathrm{SL}_2(\mathbb{F}_p)$.

Now elements of the set $\mathcal{K}^{2dn-1}_{\mathrm{SL}_2(\mathbb{F}_p))}/_{\cong}$ are in one-to-one correspondence with the quotient $(\mathbb{Z}/p(p^2-1))^*/\{\pm\varphi^* \,;\, \varphi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p)\}$, where φ^* is the induced homomorphism by $\varphi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{F}_p))$ on the cohomology $H^{2dn}(\mathrm{SL}_2(\mathbb{F}_p)) \cong \mathbb{Z}/p(p^2-1)$. Because $\mathrm{card}\, \mathcal{K}^{2nd-1}_{\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b}/_{\cong}$ has been determined in [7, Theorem 2.6], the discussion above and Proposition 2.1 give rise to our main result.

Theorem 2.2 *Let $\alpha\colon \mathbb{Z}/b \to (\mathbb{Z}/a)^*$ be an action, $(a, b) = (ab, p(p^2-1)) = 1$ where p is a prime and $G = (\mathbb{Z}/b \rtimes_\alpha \mathbb{Z}/a) \times \mathrm{SL}_2(\mathbb{F}_p)$. Then $2d = [4, p-1, 2\ell(\alpha)]$ is the least period of G and*

$$\mathrm{card}\, \mathcal{K}^{2nd-1}_{G}/_{\cong} = \begin{cases} 2^{t-1}(p-1)\phi(p^2-1)\,\mathrm{card}\, \mathcal{K}^{2nd-1}_{\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b}/_{\cong}, & \text{if } 4 \nmid p-1, \\ 2^{t-2}(p-1)\phi(p^2-1)\,\mathrm{card}\, \mathcal{K}^{2nd-1}_{\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b}/_{\cong}, & \text{if } 4 \mid p-1, \end{cases}$$

where $t \in \{0, 1\}$ is given by Proposition 2.1 *and where ϕ is the Euler function.*

Remark 2.3 Madsen, Thomas and Wall [10] have proved that if a finite group G has periodic cohomology and at most one element of order 2, then G acts freely on a sphere. This applies to the group $\mathrm{SL}_2(\mathbb{F}_p)$ for an odd p, since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the only element of order 2 in $\mathrm{SL}_2(\mathbb{F}_p)$. Furthermore, the groups $\mathrm{SL}_2(\mathbb{F}_3)$, the *binary tetrahedral group* and $\mathrm{SL}_2(\mathbb{F}_5)$, the *binary icosahedral group* admit a free orthogonal action on the sphere $\mathbb{S}^3$.

Given a periodic group G, write $\eta\colon \mathrm{Aut}(G) \to \mathrm{Aut}(H^n(G))$ for the corresponding anti-homomorphism. Then, by [3, Ch. XVI, §9], we have $H^{2n}(G) \cong \mathbb{Z}/|G|$, provided that $2n$ is divisible by the least period of G. We write $\tilde{\eta}\colon \mathrm{Aut}(G) \to (\mathbb{Z}/|G|)^*/\{\pm 1\}$ for the composition of the anti-homomorphism $\eta\colon \mathrm{Aut}(G) \to (\mathbb{Z}/|G|)^*$ with the quotient map $(\mathbb{Z}/|G|)^* \to (\mathbb{Z}/|G|)^*/\{\pm 1\}$.

Now let μ be a free and cellular action of a finite group G on a CW-complex $X(2n-1)$. By [5, Proposition 3.1] (see also [13, Theorem 1.4]) the group $\mathcal{E}(X(2n-1)/\mu)$ of homotopy classes of self homotopy equivalences for the space forms $X(2n-1)/\mu$ is isomorphic to the kernel of the map

$$\tilde{\eta}\colon \operatorname{Aut}(G) \to (\mathbb{Z}/|G|)^*/\{\pm 1\}$$

for all $n \geq 1$, provided $|G| > 2$. It follows that $\mathcal{E}(X(2n-1)/\mu)$ is independent of the action μ of the group G, whence we simply write $\mathcal{E}(X(2n-1)/G)$ for this group.

By Proposition 1.5, we get that $\mathcal{E}(X([4,p-1]n-1)/\operatorname{SL}_2(\mathbb{F}_p)) = \operatorname{Inn}(\operatorname{SL}_2(\mathbb{F}_p)) = \operatorname{PSL}_2(\mathbb{F}_p)$, and certainly

$$\mathcal{E}(X(2dn-1)/G) \cong \mathcal{E}(X(2dn-1)/(\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b)) \times \mathcal{E}(X(2dn-1)/\operatorname{SL}_2\mathbb{F}_p),$$

for $2d = [4, p-1, 2\ell(\alpha)]$ and $G = (\mathbb{Z}/b \rtimes_\alpha \mathbb{Z}/a) \times \operatorname{SL}_2(\mathbb{F}_p)$. But the group $\mathcal{E}(X(2dn-1)/(\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b))$ has been fully described in [7], so we conclude with the following.

Theorem 2.4 *Let $\alpha\colon \mathbb{Z}/b \to (\mathbb{Z}/a)^*$ be an action and $(a,b) = (ab, p(p^2-1)) = 1$ where p is a prime. If the group $G = (\mathbb{Z}/b \rtimes_\alpha \mathbb{Z}/a) \times \operatorname{SL}_2(\mathbb{F}_p)$ acts freely on a CW-complex $X(2dn-1)$, then there is a group isomorphism*

$$\mathcal{E}(X(2dn-1)/G) \cong \mathcal{E}(X(2dn-1)/(\mathbb{Z}/a \rtimes_\alpha \mathbb{Z}/b)) \times \operatorname{PSL}_2(\mathbb{F}_p),$$

where $2d = [4, p-1, 2\ell(\alpha)]$ is the least period of G.

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