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AN INEQUALITY FOR PROBABILITY DENSITY FUNCTIONS ARISING FROM A DISTINGUISHABILITY PROBLEM

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Abstract

An integral inequality is established involving a probability density function on the real line and its first two derivatives. This generalizes an earlier result of Sato and Watari. If f denotes the probability density function concerned, the inequality we prove is that

$$\int_{-\infty}^{+\infty} \frac{[f'(x)^2]^{\gamma\alpha}}{[f(x)]^{\gamma(\beta+1)-1}} dx \le \left(\frac{2\alpha-1}{\beta-1}\right)^{\gamma\alpha} \left(\int_{-\infty}^{+\infty} \frac{|f''(x)|^{\alpha-1}}{[f(x)]^{\beta-\alpha}} dx\right)^{\gamma}$$

under the conditions $\beta > \alpha > 1$ and $1/(\beta+1) < \gamma \le 1$.

1. Introduction

In this article we establish a general integral inequality involving a probability density and its first two derivatives (in the distributional sense). Integrals of the sort involved can arise in probabilistic extremal problems *via* the calculus of variations and optimal control theory and our result has an interest for such applications.

The genesis of the present ideas lies in a striking distinguishability problem whose roots go back half a century to a paper of Kakutani [2]. Suppose $X = (X_i)_1^\infty$ is a sequence of independent and identically distributed random variables and $a = (a_i)_1^\infty$ an associated numerical sequence, a_i representing the error in centering X_i. When are the sample paths X and $X + a$ distinguishable?

A key concept to unlock this question is that of *finite information*. We say that X has finite information if the common distribution of the X_i has an almost surely positive and (locally) absolutely continuous density function f satisfying

$$I_1(f) \equiv \int_{-\infty}^{\infty} [f'(x)]^2/f(x)\, dx < \infty, \qquad (1.1)$$

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where f' is the derivative of f in the distributional sense. For X with finite information we can distinguish if and only if $\sum a_i^2 = \infty$. These results were established by Shepp [6], who made use of the machinery of the Hilbert space L_2 of square–integrable functions and of some work of Kakutani on the equivalence of infinite product measures. This introduces the stronger concept of *total indistinguishability*, which signifies in physical terms that for every observed sequence, there is doubt as to whether it came from X or from $X + a$. A necessary and sufficient condition for total indistinguishability is that the infinite product measures μ_X and μ_{X+a} induced by X and $X + a$ be mutually absolutely continuous or *equivalent*, denoted by $\mu_X \sim \mu_{X+a}$.

Subsequently there was exploration of the more general question where a is replaced by an identically and independently distributed sequence $Y = (Y_i)_1^\infty$ of symmetric random variables. Kitada and Sato [3] have given sufficient conditions for distinguishability under the requirement

$$I_2(f) \equiv \int_{-\infty}^{\infty} [f''(x)]^2/f(x)\,dx < \infty. \tag{1.2}$$

They proved also that (1.2) implies (1.1) if f is monotone for large $|x|$.

More recently Sato and Watari [5] established that

$$I_1(f) \leq \frac{3}{2}[I_2(f)]^{3/2}, \tag{1.3}$$

showing that (1.2) implies (1.1) quite generally. They derive as an application that if (1.2) holds and the distributions of Y are symmetric with $Y \in \ell_4$ almost surely, then $X + Y$ and X induce mutually absolutely continuous probability measures and so are totally indistinguishable. Here ℓ_α (for $\alpha > 1$) denotes the space of all random sequences such that $\sum_{k=1}^{\infty} Y_k^\alpha < \infty$ a.s. The condition $Y \in \ell_4$ a.s. had arisen in early work by Rozanov [4] and Fernique [1] as a necessary and sufficient condition for equivalence of the measures on sequence space in the case when X, Y are centred Gaussian.

In the present paper we provide a generalization of (1.3). To be specific, suppose that f is an a.s. positive density function and write

$$I_{\gamma.\alpha.\beta} = \int_{-\infty}^{+\infty} \frac{[f'(x)^2]^{\gamma\alpha}}{[f(x)]^{\gamma(\beta+1)-1}}\,dx, \qquad J_{\alpha.\beta} = \int_{-\infty}^{+\infty} \frac{|f''(x)|^\alpha}{[f(x)]^{\beta-\alpha}}\,dx.$$

In Section 2 we establish the following result.

THEOREM 1. *If f is such that $J_{\alpha.\beta} < \infty$ for some $\beta > \alpha > 1$, then*

$$I_{\gamma.\alpha.\beta} \leq \left(\frac{2\alpha - 1}{\beta - 1}\right)^{\gamma\alpha} J_{\alpha.\beta}^{\gamma}$$

for $1/(\beta + 1) < \gamma \leq 1$.

This includes the inequalities of Sato and Watari [5] as the special cases $\alpha = 2$, $\beta = 3$, $\gamma = 1$ and $\gamma = 1/2$.

2. Proof of Theorem 1

The proof of Theorem 1 is based on the following lemma.

LEMMA 1. *Let* $[a, b]$, $(-\infty \leq a < b \leq +\infty)$ *be a closed interval and* h *a nonnegative continuously differentiable function on* $[a, b]$. *Suppose the derivative* h' *is an absolutely continuous non-vanishing function on* (a, b), $h'(a+) = h'(b-) = 0$, *and* $\int_a^b |h''(x)|^\alpha/[h(x)]^{\beta-\alpha}dx < \infty$ *for the second derivative* h'' *when* $\beta > \alpha > 0$. *Then*

$$\int_a^b \frac{[h'(x)^2]^\alpha}{[h(x)]^\beta}\,dx = \frac{2\alpha - 1}{\beta - 1}\int_a^b \frac{[h'(x)^2]^{\alpha-1}h''(x)}{[h(x)]^{\beta-1}}\,dx, \tag{2.1}$$

$$\int_a^b \frac{[h'(x)^2]^{\gamma\alpha}}{[h(x)]^{\gamma(\beta+1)-1}}\,dx \leq \left(\frac{2\alpha-1}{\beta-1}\right)^{\gamma\alpha}\left(\int_a^b h(x)dx\right)^{1-\gamma}\left(\int_a^b \frac{|h''(x)|^\alpha}{[h(x)]^{\beta-\alpha}}\,dx\right)^\gamma \tag{2.2}$$

for $1/(\beta + 1) < \gamma \leq 1$.

PROOF OF LEMMA 1. For $\eta > 0$ we define $h_\eta(x) = h(x) + \eta$ as in [5]. For every $\beta - 1 > \varepsilon > 0$, since $h'(a+) = h'(b-) = 0$, we can choose a closed interval $[a', b']$ contained in (a, b) such that

$$\left|\left[\frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^{\beta-1}}\right]_{a'}^{b'}\right| \leq \varepsilon\int_{a'}^{b'} \frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^\beta}\,dx. \tag{2.3}$$

Further we have

$$\begin{aligned}\int_{a'}^{b'} \frac{[h'(x)^2]^{\alpha-1}h''(x)}{[h_\eta(x)]^{\beta-1}}\,dx &= \int_{a'}^{b'} \frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^{\beta-1}}\frac{h''(x)}{[h'(x)^2]}\,dx \\ &= -\left[\frac{[h'(x)]^{(2\alpha-1)}}{[h_\eta(x)]^{\beta-1}}\right]_{a'}^{b'} + 2\alpha\int_{a'}^{b'} \frac{[h'(x)^2]^{\alpha-1}h''(x)}{[h_\eta(x)]^{\beta-1}}\,dx \\ &\quad - \beta\int_{a'}^{b'} \frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^\beta}\,dx,\end{aligned}$$

that is,

$$\int_{a'}^{b'} \frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^\beta}\,dx = \frac{2\alpha-1}{\beta-1}\int_{a'}^{b'} \frac{[h'(x)^2]^{\alpha-1}h''(x)}{[h_\eta(x)]^{\beta-1}}\,dx - \frac{1}{\beta-1}\left[\frac{[h'(x)]^{(2\alpha-1)}}{[h_\eta(x)]^{\beta-1}}\right]_{a'}^{b'}.$$

From (2.3) and Hölder's inequality we have

$$\begin{aligned}
\int_{a'}^{b'} \frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^\beta}\,dx &\le \frac{2\alpha-1}{\beta-1-\varepsilon}\int_{a'}^{b'} \frac{[h'(x)^2]^{\alpha-1}h''(x)}{[h_\eta(x)]^{\beta-1}}\,dx \\
&\le \frac{2\alpha-1}{\beta-1-\varepsilon}\left[\int_{a'}^{b'}\left(\frac{[h'(x)^2]^{\alpha-1}}{[h_\eta(x)]^{(\alpha-1)\beta/\alpha}}\right)^{\alpha/(\alpha-1)} dx\right]^{1-1/\alpha} \\
&\quad\times\left[\int_{a'}^{b'}\left(\frac{|h''(x)|}{[h_\eta(x)]^{(\beta-\alpha)/\alpha}}\right)^{\alpha} dx\right]^{1/\alpha} \\
&\le \frac{2\alpha-1}{\beta-1-\varepsilon}\left(\int_{a'}^{b'} \frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^\beta}\,dx\right)^{1-1/\alpha} \\
&\quad\times\left(\int_{a'}^{b'} \frac{|h''(x)|^\alpha}{[h_\eta(x)]^{\beta-\alpha}}\,dx\right)^{1/\alpha}.
\end{aligned}$$

From the last inequality we have

$$\begin{aligned}
\int_{a'}^{b'} \frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^\beta}\,dx &\le \left(\frac{2\alpha-1}{\beta-1-\varepsilon}\right)^\alpha \int_{a'}^{b'} \frac{|h''(x)|^\alpha}{[h_\eta(x)]^{\beta-\alpha}}\,dx \\
&\le \left(\frac{2\alpha-1}{\beta-1-\varepsilon}\right)^\alpha \int_{a}^{b} \frac{|h''(x)|^\alpha}{[h(x)]^{\beta-\alpha}}\,dx \\
&< \infty.
\end{aligned}$$

On taking $\varepsilon \to 0$ together with $a' \searrow a$ and $b' \nearrow b$ we have

$$\int_{a}^{b} \frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^\beta}\,dx = \frac{2\alpha-1}{\beta-1}\int_{a}^{b} \frac{[h'(x)^2]^{\alpha-1}h''(x)}{[h_\eta(x)]^{\beta-1}}\,dx, \tag{2.4}$$

$$\int_{a}^{b} \frac{[h'(x)^2]^\alpha}{[h_\eta(x)]^\beta}\,dx \le \left(\frac{2\alpha-1}{\beta-1}\right)^\alpha \int_{a}^{b} \frac{|h''(x)|^\alpha}{[h_\eta(x)]^{\beta-\alpha}}\,dx. \tag{2.5}$$

Relations (2.1) and (2.2) for $\gamma = 1$ follow from (2.4) and (2.5) on letting $\eta \searrow 0$.

If $1/(\beta+1) < \gamma < 1$, (2.2) follows from Hölder's inequality and (2.2) for $\gamma = 1$, that is,

$$\begin{aligned}
\int_{a}^{b} \frac{[h'(x)^2]^{\gamma\alpha}}{[h(x)]^{\gamma(\beta+1)-1}}\,dx &= \int_{a}^{b} \frac{[h'(x)^2]^{\gamma\alpha}}{[h(x)]^{\gamma\beta}}[h(x)]^{1-\gamma}\,dx \\
&\le \left(\int_a^b h(x)\,dx\right)^{1-\gamma}\left(\int_a^b \frac{[h'(x)^2]^\alpha}{[h(x)]^\beta}\,dx\right)^\gamma \\
&\le \left(\frac{2\alpha-1}{\beta-1}\right)^{\gamma\alpha}\left(\int_a^b h(x)\,dx\right)^{1-\gamma}\left(\int_a^b \frac{|h''(x)|^\alpha}{[h(x)]^{\beta-\alpha}}\,dx\right)^\gamma.
\end{aligned}$$

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PROOF OF THEOREM 1. As in [5], the continuity of f' implies that $R \setminus \{x \mid f'(x) = 0\}$ is the union of at most a countable number of mutually disjoint open intervals (a_n, b_n) such that f satisfies the hypotheses of Lemma 1 on each closed interval $[a_n, b_n]$. By applying (2.2) and Hölder's inequality we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{[f'(x)^2]^{\gamma\alpha}}{[f(x)]^{\gamma(\beta+1)-1}}\,dx &= \sum_n \int_{a_n}^{b_n} \frac{[f'(x)^2]^{\gamma\alpha}}{[f(x)]^{\gamma(\beta+1)-1}}\,dx \\
&\le \left(\frac{2\alpha-1}{\beta-1}\right)^{\gamma\alpha} \sum_n \left(\int_{a_n}^{b_n} f(x)dx\right)^{1-\gamma} \\
&\quad \times \left(\int_{a_n}^{b_n} \frac{|f''(x)|^{\alpha}}{[f(x)]^{\beta-\alpha}} dx\right)^{\gamma} \\
&\le \left(\frac{2\alpha-1}{\beta-1}\right)^{\gamma\alpha} \left(\sum_n \int_{a_n}^{b_n} f(x)dx\right)^{1-\gamma} \\
&\quad \times \left(\sum_n \int_{a_n}^{b_n} \frac{|f''(x)|^{\alpha}}{[f(x)]^{\beta-\alpha}} dx\right)^{\gamma} \\
&\le \left(\frac{2\alpha-1}{\beta-1}\right)^{\gamma\alpha} \left(\int_{-\infty}^{+\infty} \frac{|f''(x)|^{\alpha}}{[f(x)]^{\beta-\alpha}} dx\right)^{\gamma}.
\end{aligned}$$

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