The topics chosen make a very nice graduate course and an interesting book, adopting a rather different approach from most other texts. As this is a self-contained text it would have been a useful addition to have, at the end of each chapter, some historical comments about how the subject has evolved, and some guide to further reading.

J. R. L. WEBB

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Cuntz, J., Skandalis, G. and Tsygan, B. *Cyclic homology in non-commutative geometry* (Springer, 2004), 137 pp., 3 540 40469 4 (hardback), £65.50.

Cyclic homology was discovered in the early 1980s independently by Alain Connes and Boris Tsygan. It has since become one of the most important tools in Connes's noncommutative geometry, where it plays the role of (co)homology for noncommutative spaces. The main ingredients of cyclic homology are cyclic-type theories of an algebra A: the classical Hochschild homology $H_*(A, A)$, cyclic homology $HC_*(A)$, and the periodic cyclic homology $HP_*(A)$ together with their dual cohomology theories. The Hochschild and cyclic homology are $\mathbb{N}$-graded, whereas the periodic cyclic homology, similarly to topological K-theory of C^*-algebras, is $\mathbb{Z}/2\mathbb{Z}$-graded.

The basic idea of noncommutative geometry is to treat all algebras as algebras of functions on spaces; the space in question is often not known explicitly, but the properties of the algebra correspond to certain features of the space. For example, in the case of a manifold M, properties of M, and various analytic and geometric objects associated with it, can be described in terms of a suitable algebra of functions of M: measurable, continuous, smooth, holomorphic and so on. One can say that Hochschild and cyclic homology generalize in some sense the differential and integral calculus on M. An important theorem of Connes shows that periodic cyclic homology is an extension of de Rham cohomology. In the case of the algebra of smooth functions on a differentiable manifold M, where both theories can be defined, we have the following isomorphisms:

$$HP_0(C^\infty(M)) \simeq H^{\text{even}}(M) = \bigoplus_{i=0}^{k} H^{2i}(M),$$
$$HP_1(C^\infty(M)) \simeq H^{\text{odd}}(M) = \bigoplus_{i=0}^{m} H^{2i+1}(M),$$

where $2k$ (respectively, $2m + 1$) is the largest even (respectively, odd) integer not greater than the dimension of M. The advantage of cyclic theory is that it can be defined for algebras of functions that are not differentiable, whereas the de Rham theory exists in the smooth category.

This basic pattern was studied in a great variety of contexts, and has been extended to crossed product algebras, which are the non-commutative replacement for quotient spaces, groupoids, etc. The power of cyclic cohomology as a computational tool has been greatly augmented with the proof of the excision theorem for periodic cyclic homology and cohomology. This theorem, due to Cuntz and Quillen, states that any algebra extension $0 \to J \to A \to A/J \to 0$ leads to an exact sequence of length six in both periodic cyclic homology and cohomology, which is analogous to an excision sequence in K-theory.

The slim volume by Cuntz, Tsygan and Skandalis covers the foundations of the theory and its main results in the past two decades. Following the format of other books in the series it provides only sketches of proofs; however, this allows for a quick tour of the main points of the cyclic homology.

Cuntz's contribution, 'Cyclic theory, bivariant K-theory and the bivariant Chern character', provides a survey of the results achieved through a far-reaching development of the initial idea of Connes, whose construction of cyclic theory was motivated by the bivariant KK-theory of Kasparov and the problem of constructing a suitable target for a non-commutative generalization of the Chern character. The main object of study in this part of the book is the bivariant cyclic homology $HP(A, B)$, which unifies the periodic cyclic homology and cohomology in one functor that shares certain crucial features with Kasparov's KK-theory. The two theories are

(1) homotopy invariant (in the case of HP we need to insist on smooth homotopies);

(2) stable, i.e. the map $A \to A \otimes \mathfrak{K}$, where $\mathfrak{K}$ is the algebra of *smooth* compact operators (infinite matrices with rapidly decreasing coefficients), induces an isomorphism in both variables of the theory; and

(3) half-exact, i.e. any extension of algebras admitting a continuous linear splitting induces a sequence $HP(\cdot\,, J) \to HP(\cdot\,, A) \to HP(\cdot\,, A/J)$ which is exact in the middle (there is a similar exact sequence, with the arrows reversed, in the other variable).

These properties imply the existence of a Chern character from Cuntz's bivariant kk-theory to the bivariant periodic cyclic homology; this is guaranteed by the universal property satisfied by the kk-theory. The main part of Cuntz's article is devoted to detailed exposition of both theories and culminates in the construction of the Chern character. The main technical tool described by Cuntz is the X-complex, which he developed jointly with Quillen in their proof of the excision theorem in the bivariant periodic cyclic homology. An important part of Cuntz's paper is a modification of the theory to cover a large class of topological algebras called multiplicatively convex algebras. The last chapter of this part, written by Ralf Meyer, describes two extensions of the theory: the analytic bivariant theory, developed by Meyer, and the local cyclic homology, due to Michael Puschnigg. The analytic theory works for bornological algebras, where a bornology is a class of subsets of an algebra which specifies the bounded subsets. This theory covers a very wide range of situations, including Connes's entire cyclic cohomology as well as the usual periodic cyclic homology. Puschnigg's local theory is an attempt to modify the periodic theory to include C^*-algebras.

Boris Tsygan describes cyclic cohomology from the point of view of the cyclic category and the cyclic double complex. The main part of his article is devoted to algebraic index theorems of Bressler, Nest and Tsygan. These results, which extend work first started by Nest and Tsygan, provides a remarkable algebraic unification of index-type theorems, including the Atiyah–Singer theorem. Tsygan also describes his work with Tamarkin which provides a different proof and extensions of Kontsevitch's results on classification of deformation quantizations. The description of cyclic theory complements that of Cuntz and focuses on the features of cyclic theory from the point of view of homological algebra.

George Skandalis introduces a recent surprising result of Connes and Moscovici, who wanted to provide a local index formula in the context of the action of a countable group Γ by smooth diffeomorphisms on a smooth manifold M. In general, there is no Γ-invariant metric on M, and so there is no elliptic differential operator on M with a Γ-invariant symbol. Because of this, Connes and Moscovici were forced to construct a cyclic cocycle (which would provide the index formula) on the crossed product algebra $C_c^\infty(P) \rtimes \Gamma$, where P is the total space of the bundle of all metrics on M. The cyclic cocycle, which involves the Wodzicki residue, consists of thousands of terms even in the simplest case. To be able to cope with this large number of terms Connes and Moscovici introduce a Hopf algebra $\mathcal{H}$ which allows them to simplify the calculation. Moreover, they define cyclic cohomology for this algebra and prove that the required cyclic cocycle comes from a universal cocycle, which they construct, in $HC^*_{\mathrm{Hopf}}(\mathcal{H})$ via a classifying map

$$HC^*_{\mathrm{Hopf}}(\mathcal{H}) \to HC^*(C_c^\infty(\Gamma)) \rtimes \Gamma.$$

Connes and Moscovici also prove that $HC^*_{\mathrm{Hopf}}(\mathcal{H})$ is canonically isomorphic to the Gelfand–Fuchs cohomology.

Later, a remarkable connection was established between this Hopf algebra and the Hopf algebra constructed by Kreimer to catalogue computations of Feynman diagrams. The article of Skandalis is a translation (by Ponge and Wright) of his Bourbaki Seminar.

This book is a very useful reference to the basic properties of and the main results in modern cyclic theory and it goes far beyond Loday's monograph (J.-L. LODAY, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften, Volume 301 (Springer, 1992)). Although proofs are either not provided at all or are only sketched, there are ample references to the literature where details can be found.

J. BRODZKI

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HÉLEIN, F. *Harmonic maps, conservation laws and moving frames* (Cambridge University Press, 2002), 290 pp, 0 521 81160 0 (hardback), £55.

A harmonic map $u : M \to N$ between two Riemannian manifolds (M, g) and (N, h) is a solution (in local coordinates) of the system of elliptic equations

$$\Delta_g u^i + g^{\alpha\beta}(x)\Gamma^i_{jk}(u(x))\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta} = 0,$$

where Δ_g is the Laplacian corresponding to the metric g on the source manifold (M, g) and the Γ^i_{jk} are the Christoffel symbols of the target manifold (N, h). These are the Euler–Lagrange equations corresponding to the energy or Dirichlet integral

$$E[u] = \frac{1}{2}\int_M |\mathrm{d}u(x)|^2 \,\mathrm{d}v_g.$$

Harmonic maps arise in a variety of different situations in geometry and physics. A submanifold M of an affine Euclidean space has constant mean curvature if and only if its Gauss map is a harmonic map. A submanifold M of a manifold N is minimal if and only if the immersion of M into N is harmonic. Harmonic maps with values in a sphere have been used in condensed matter physics to model nematic liquid crystals. Harmonic maps between surfaces and Lie groups are studied in theoretical physics because of their close relation to anti self-dual Yang–Mills connections. For a review of the theory of harmonic maps up to 1988 the reader is referred to [**3**, **4**].

The book under review provides a self-contained, readable and accessible introduction to the analytical aspects of the theory of harmonic maps as well as an exposition of some recent results due to the author. Only a few basic facts from differential geometry and the calculus of variations are assumed and these can be found in, for example, [**6**].

Chapter 1 contains introductory material on the geometric and analytic setting of harmonic maps. First, the notion of the Laplacian associated with a Riemannian metric is defined, and then a discussion of smooth harmonic maps between two Riemannian manifolds follows. The variational framework is then developed, in which harmonic maps are thought of as critical points of the Dirichlet integral. Noether's theorem is proved followed by a section on Sobolev spaces and various notions of weakly harmonic maps.

Chapter 2 is devoted to harmonic maps with symmetries and in particular to harmonic maps between a surface and a sphere, a special case that arises frequently both in differential geometry and in physics. Here the conformal transformations of the domain manifold combined with the symmetries of the target create a rich setting for the study of harmonic maps. The chapter ends with a discussion of weak compactness and regularity results in two dimensions.

Chapter 3 discusses compensation phenomena in which certain quadratic expressions of first derivatives of functions have more regularity than expected. This is related to the compensated compactness method of Murat and Tartar [**11**, **14**] as well as later work by Müller [**9**] and Coifman, Lions, Meyer and Semmes [**2**] on compensation phenomena using Hardy spaces. A related phenomenon in the setting of hyperbolic partial differential equations has been discovered