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ISOMETRIES OF BERGMAN SPACES OVER BOUNDED RUNGE DOMAINS

CLINTON J. KOLASKI

1. Introduction.

1.1. The isometries of the Hardy spaces $H^p (0 < p < \infty, p \neq 2)$ of the unit disc were determined by Forelli in [**2**]. Generalizations to several variables: For the polydisc the isometries of H^p onto itself were characterized by Schneider [**9**]. For the unit ball the case $p > 2$ was then done by Forelli [**3**]; Rudin [**8**] removed the restriction $p > 2$ by proving a theorem on equimeasurability. Finally, Koranyi and Vagi [**6**] noted that the methods developed by Forelli, Rudin and Schneider applied to bounded symmetric domains.

In this note it will be shown that their methods also apply to the Bergman spaces over bounded Runge domains. The isometries which are onto are completely characterized; the special cases of the ball and polydisc are particularly nice and are given separately.

1.2. Definitions. Let $\mathbf{C}^n$ denote the vector space of all ordered n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers, with inner product

(1) $\quad \langle z, w\rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n$

and norm

(2) $\quad |z| = \langle z, z\rangle^{1/2}.$

Given a bounded domain $\Omega \subset \mathbf{C}^n$, let $H(\Omega)$ denote the class of holomorphic functions with domain Ω; let $\mathrm{Aut}(\Omega)$ denote the class of all holomorphic one-to-one maps of Ω onto Ω (the "automorphisms" of Ω); let m_Ω denote Lebesgue measure on Ω normalized so that $m_\Omega(\Omega) = 1$; and, for $0 < p \leqq \infty$, let

(3) $\quad B^p(\Omega) = L^p(\Omega) \cap H(\Omega)$

denote the Bergman space over Ω. It is well-known that $B^p(\Omega)$ is a closed subspace of $L^p(\Omega) = L^p(m_\Omega)$.

2. Isometries on Bergman spaces.

2.1. THEOREM. *Let Ω be a bounded Runge domain and let $0 < p < \infty$, $p \neq 2$.*

(i) *If T: $B^p(\Omega) \to B^p(\Omega)$ is a linear isometry and if $T1$ is denoted by*

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g, then there exists a holomorphic map Φ taking Ω onto a dense subset of Ω such that,

$$(4) \qquad (Tf)(z) = g(z)\cdot(f\circ\Phi)(z)$$

for all $z \in \Omega$ and all $f \in B^p(\Omega)$, moreover

$$(5) \qquad \int_\Omega (h\circ\Phi)|g|^p dm_\Omega = \int_\Omega h dm_\Omega$$

for every bounded Borel function h on Ω.

(ii) *Conversely, if Φ is a holomorphic map of Ω into Ω, and if $g \in B^p(\Omega)$ satisfies* (5) *for every continuous function h on Ω, then* (4) *defines an isometry of $B^p(\Omega)$.*

(iii) *If the linear isometry T is onto $B^p(\Omega)$, then $\Phi \in \mathrm{Aut}(\Omega)$ and g is related to Φ by*

$$(6) \qquad |g|^p = |J_\Phi|$$

where J_Φ is the Jacobian of Φ.

Conversely, if $\Phi \in \mathrm{Aut}(\Omega)$ and if $g \in B^p(\Omega)$ is related to Φ by (6), *then* (5) *holds and the isometry given by* (4) *is onto $B^p(\Omega)$.*

2.2. *Remarks.* (a) An isometry of $B^p(B_n)$ determines a holomorphic map $\Phi: B_n \to B_n$. If we could show that Φ is proper, then the theorem of Alexander [**1**] tells us that Φ is an automorphism of $B_n (n \geqq 2)$. Consequently, our isometry would be completely described by Theorem (3.2).

Conversely, let Φ be a holomorphic map of B_n into B_n. If we can find a g such that the map $f \to g(f\circ\Phi)$ is an isometry onto $B^p(B_n)$ then Φ must be an automorphism of B_n. This may be of use in resolving certain conjectures about inner maps of B_n.

(b) The following example shows that $\Phi(\Omega)$ may be a proper subset of Ω. Let $n = 1$ and let our domain be the unit disc $U = \{|z| < 1\}$. Define Φ on U by

$$(7) \qquad \Phi(z) = -1 + 2(1 + iz)/\psi(z)$$

where

$$(8) \qquad \psi(z) = 1 + iz + \sqrt{2(1 - z^2)}.$$

Here we choose the principal branch for our square root.

It is easily shown that Φ maps U in a one-to-one manner onto $U\backslash[0, 1)$, and that

$$(9) \qquad d\Phi/dz = 2\sqrt{2}(z + i)/[\psi^2(z)\sqrt{1 - z^2}].$$

By (7) and (8),

$$\lim_{z\to i} \Phi(z) = -1 \quad\text{and}\quad \frac{1}{\psi(z)} = \frac{\Phi(z) + 1}{2(1 + iz)};$$

hence $1/|\psi|$ is bounded on U. If we set $g(z) = [d\Phi/dz]^2$, then (by (9)), $g \in B^1(U)$. Since $|g| = |J_\Phi|$, the pair (Φ, g) satisfies (5) and determines (by (4)) an isometry of $B^1(U)$.

(c) Using (34) and (35), the argument given in the proof of (iii) shows $|g|^p = |J_\Phi|$ whenever Φ is one-to-one. For a given isometry T, is Φ necessarily one-to-one? A second question is suggested by (ii). Does every holomorphic map $\Phi: \Omega \to \Omega$ generate an isometry of $B^p(\Omega)$? We shall now answer both questions negatively by showing the map $\Phi(z) = z^2$, $z \in U$, generates an isometry of $B^p(U)$ ($p \geqq 1, p \neq 2$) if and only if $p = 1$.

Let μ be the Borel measure induced by Φ:

$$(10) \quad \mu(X) = m_U[\Phi^{-1}(X)]$$

for Borel subsets X of U. Then

$$(11) \quad \int_U h d\mu = \int_U h(\Phi) dm_U$$

where h is any Borel measurable function on U. Using (11) it is easy to show

$$(12) \quad \int_U z^i \bar{z}^j d\mu = \int_U z^i \bar{z}^j \left(\frac{1}{2|z|}\right) dm_U$$

for all $i, j = 0, 1, 2, \ldots$. Thus by the Stone-Weierstrass theorem

$$(13) \quad d\mu = \left(\frac{1}{2|z|}\right) dm_U.$$

For the moment let us suppose that there is a $g \in B^p(U)$ which is related to Φ by (5). If $X = \Phi^{-1}(Y)$ where Y is a Borel subset of U, then by (5), (13) and (11),

$$(14) \quad \int_X |g|^p dm_U = \int_Y dm_U = \int_Y 2|z| d\mu = \int_X 2|\Phi| dm_U.$$

For $0 < r < 1$, let $X_r = \Phi^{-1}(Y_r)$ where Y_r is the disc $|z| < r^2$. Then by (14) and the fact that g is continuous

$$(15) \quad |g(0)|^p = \operatorname*{Lim}_{r \downarrow 0} \frac{1}{m_U(X_r)} \int_{X_r} |g|^p dm_U = \operatorname*{Lim}_{r \downarrow 0} \frac{m_U(Y_r)}{m_U(X_r)} = 0.$$

Since g is holomorphic, (15) implies

$$(16) \quad g(z) = z^k h(z)$$

where $h(0) \neq 0$ and k is a positive integer.

It follows easily from (14) and (16) that $p = 1$ ($p \geqq 1, p \neq 2$) so Φ cannot generate an isometry on $B^p(U)$ when $p \neq 1, 2$.

On the other hand, if we set $g = 2\Phi$, and $p = 1$, then the pair (Φ, g) satisfies (5), and so Φ generates on isometry on $B^1(U)$.

(d) With regard to (iii) we note that when the group $\mathrm{Aut}(\Omega)$ is known, it may be possible to find a g for each automorphism Φ.

For example, if $\Phi \in \mathrm{Aut}(U^n)$, then [**7**, p. 167]

$$(17)\quad \Phi(z_1, \dots, z_n) = (\phi_1(z_{i_1}), \dots, \phi_n(z_{i_n}))$$

where $\phi_1, \dots, \phi_n$ are conformal maps of U onto U and $(i_1, \dots, i_n)$ is a permutation of $(1, \dots, n)$. It follows from (iii) that

$$(18)\quad g(z_1, \dots, z_n) = \theta[\phi_1'(z_{i_1}) \dots \phi_n'(z_{i_n})]^{2/p}$$

where θ is a unimodular complex number and ϕ_i' is the derivative of ϕ_i. This is similar to Schneider's theorem [**9**] for the isometries of $H^p(U^n)$.

Thus every $\Phi \in \mathrm{Aut}(U^n)$ generates an isometry onto $B^p(U^n)$. It is shown in Theorem (3.2) that a similar result holds for the ball B_n. One might conjecture that this property holds whenever the group $\mathrm{Aut}(\Omega)$ acts transitively on Ω, but the answer is not known at present.

3. B^p spaces over balls and polydiscs.

3.1. By (2) the open unit ball B_n in $\mathbf{C}^n$ is the set of all $z \in \mathbf{C}^n$ with $|z| < 1$. We recall that the Poisson-Bergman kernel for B_n is the map $\chi: B_n \times B_n \to (0, \infty)$ given by

$$(19)\quad \chi(z, w) = [(1 - |w|^2)/|1 - \langle z, w\rangle|^2]^{N+1}.$$

It is known [**5**, p. 20] that

$$(20)\quad |J_\Phi(z)| = \chi(z, \Phi^{-1}(0))$$

for all $z \in B_n$, and for every $\Phi \in \mathrm{Aut}(B_n)$.

3.2. THEOREM. *Let $0 < p < \infty$, $p \neq 2$. If T is a linear isometry of $B^p(B_n)$ onto $B^p(B_n)$, then there is a $\Phi \in \mathrm{Aut}(B_n)$ such that*

$$(21)\quad Tf = g(f \circ \Phi)$$

for all $f \in B^p(B_n)$. Moreover, g is related to Φ by

$$(22)\quad g(z) = \theta\left\{\frac{1 - |a|^2}{(1 - \langle z, a\rangle)^2}\right\}^{(n+1)/p} \quad (z \in B_n)$$

where θ is a unimodular complex number and $\Phi(a) = 0$.

Conversely, if $\Phi \in \mathrm{Aut}(B_n)$, and if g is related to Φ by (22), then (21) defines a linear isometry T of $B^p(B_n)$ onto $B^p(B_n)$.

Proof. For the converse, let $\Phi \in \mathrm{Aut}(B_n)$ and let g be defined by (22). Equating (19), (20) and (22) shows that the pair (Φ, g) satisfies (6). Thus, (22) defines an isometry by the converse in (iii) of Theorem (2.1).

For the first part, let T be an isometry onto $B^p(B_n)$. Then, by (i) and (iii) of Theorem (2.1), (21) holds and g is related to Φ by (6).

Having already established our converse, it suffices to show that g is unique (modulo a unimodular constant).

By (19), $\chi > 0$. Thus, equating (6) and (20) shows

$$(23) \quad |g(z)|^p = \chi(z, a) > 0 \qquad (z \in B_n)$$

where $\Phi(a) = 0$.

We also know from part (i) of Theorem (2.1) that g is holomorphic in B_n.

Now suppose h is a holomorphic function on B_n for which (23) holds. Then $h(z) \neq 0$, so g/h is a holomorphic function on B_n which satisfies

$$(24) \quad \left|\frac{g(z)}{h(z)}\right| \equiv 1 \quad (z \in B_n).$$

Hence, by the maximum principle, $g = \theta h$ for some unimodular complex number θ. This completes the proof of Theorem (3.2).

We shall now consider the case where our domain is the polydisc U^n. If $\Phi \in \mathrm{Aut}(U_n)$, then (recall (17))

$$\Phi(z_1, \ldots, z_n) = (\phi_1(z_{i_1}), \ldots, \phi_n(z_{i_n})).$$

Define $g(=g_m)$ on U^n by

$$(25) \quad g(z_1, \ldots, z_n) = \theta[\psi_1(z_{i_1}) \ldots \psi_n(z_{i_n})]^{1/p}$$

where

$$(26) \quad \psi_k(\zeta) = \left\{\frac{1 - a_k\bar{a}_k}{(1 - \bar{a}_k\zeta)^2}\right\}^2 \quad (\zeta \in U, 1 \leqq k \leqq n)$$

and $\theta \in \mathbf{C}$, $|\theta| = 1$, $\Phi(a_1, \ldots, a_n) = 0$.

A slight modification of the proof of Theorem (3.2) yields:

3.3. THEOREM. *If one replaces B_n by U^n in Theorem* (3.2), *then the resulting statement is true when g is related to Φ by* (25).

4. The proof of theorem (2.1).

4.1. As remarked earlier our proof utilizes the concepts developed in [**2**], [**3**], [**6**], [**8**] and [**9**]. The following lemma is due to Schneider [**9**], but the argument given in [**6**, Lemma 2] is clearer.

4.2. LEMMA. *If $0 \neq f \in H(\Omega)$ (recall Definition* (1.2)), *and if h: $\Omega \rightarrow \mathbf{C}$ satisfies*

$$(27) \quad fh^k \in H(\Omega)$$

for $k = 1, 2, \ldots$, then $h \in H(\Omega)$.

4.3. *The proof of Theorem* 2.1. Let Ω be a bounded Runge domain, let T be a linear isometry of $B^p(\Omega)$ $(0 < p < \infty, p \neq 2)$ and set $g = T1$. De-

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fine the measure ν on Ω by $d\nu = |g|^p \, dm_\Omega$. For $f \in B^\infty(\Omega)$ define

(28) $Af = Tf/g$.

Since $0 \neq g \in B^p(\Omega)$, $g(z) \neq 0$ a.e.$[m_\Omega]$. Thus (28) is well defined. It follows (as in [**8**, p. 226]) that A is a multiplicative linear map of $B^\infty(\Omega)$ into $L^\infty(\nu) = L^\infty(m_\Omega)$ which preserves sup norms. Moreover, if $f \in B^\infty(\Omega)$, then

(29) $g(Af)^k = gA(f^k) = T(f^k) \in B^p(\Omega)$

for all $k = 1, 2, \ldots$. Hence, by Lemma (4.2), $Af \in B^\infty(\Omega)$.

Thus A is a multiplicative linear sup norm isometry of $B^\infty(\Omega)$ into $B^\infty(\Omega)$ with $A1 = 1$.

Let $u_i(z) = z_i$ (the i-th coordinate function). Define $\Phi: \Omega \to \mathbf{C}^n$ by

(30) $\Phi = (Au_1, \ldots, Au_n)$.

It follows (as in [**8**]) that

(31) $\nu(\Phi^{-1}E) = m_\Omega(E)$

for every Borel set $E \subset \mathbf{C}^n$. This gives (5).

We shall first show that $\Phi(\Omega)$ is a dense subset of $\bar{\Omega}$. If we set $X = \Phi^{-1}(\Omega)$, then (31) gives

(32) $1 = m_\Omega[\Omega] = \nu(X)$.

Our definition of ν implies $\nu(\Omega) = 1$ and $m_\Omega \ll \nu$. Hence, by (32)

(33) $\nu(\Omega \backslash X) = 0 = m_\Omega(\Omega \backslash X)$.

Since Φ is continuous, (33) implies that X is a dense open subset of Ω. It now follows from the definition of X that

(34) $\Phi(\Omega) \subset \bar{\Omega}$.

Finally, setting $E = \Phi(\Omega)$ in (31) shows

(35) $m_\Omega[\Phi(\Omega)] = \nu(\Omega) = 1$.

It follows from (34) and (35) that $\Phi(\Omega)$ is dense in Ω.

Let $f \in B^p(\Omega)$ be given and fixed. Since Ω is a bounded Runge domain we may choose a sequence of polynomials, P_n, such that $P_n \to f$ (and so $TP_n \to Tf$) in the $L^p(\Omega)$ norm. Since point evaluations (i.e., the maps $f \to f(z)$, $z \in \Omega$) are bounded linear functionals on $B^p(\Omega)$, we find

(36) $(TP_n)(z) \to (Tf)(z) \qquad (z \in \Omega)$.

Now the multiplicativity of A combined with the definition of Φ, shows that (4) holds for all polynomials and all z in X. Hence, by (36),

(37) $(Tf)(z) = g(z) \cdot (f \circ \Phi)(z) \qquad (z \in X)$.

It remains to be shown that $X = \Omega$. If $g(z) \neq 0$, choose $\{z_n\} \subset X$

such that $z_n \to z$. Then

$$(38) \quad (f \circ \Phi)(z_n) = (Tf)(z_n)/g(z_n) \to (Tf)(z)/g(z)$$

and it follows that (4) holds. Moreover, since Ω is a domain of holomorphy, (38) shows that $\Phi(z) \in \Omega$.

Finally, suppose there is a $z_0 \in \{g = 0\}$ such that $\Phi(z_0) \in \partial\Omega$. Choose a small disc $\bar{\Delta} \subset \Omega$ with $\bar{\Delta} \cap \{g = 0\} = \{z_0\}$. Then $\Phi(\partial\Delta)$ is a compact subset of Ω, $\Phi(z_0) \in \partial\Omega$, and since Ω is Runge, there exists a polynomial P satisfying

$$(39) \quad |P(\Phi(z_0))| > \mathrm{Sup}_{w \in \Phi(\partial\Delta)} \{|P(w)|\}.$$

Consider $T(P^n) = g \cdot (P^n \circ \Phi)$. If n is sufficiently large, the function $g \cdot (P^n \circ \Phi)$ will not obey the maximum principle on $\bar{\Delta}$ near z_0. The contradiction completes the proof of part (i).

The proof of part (ii) is trivial.

To prove (iii) suppose T is onto. By (i) its inverse is given by $T^{-1}f = h(f \circ \Psi)$, with $h = T^{-1}1$. If $f \in B^p(\Omega)$, then

$$(40) \quad f = TT^{-1}f = g(h \circ \Phi)(f \circ \Psi \circ \Phi) = f \circ \Psi \circ \Phi.$$

Similarly, $f = T^{-1}Tf = f \circ \Phi \circ \Psi$ and so $\Phi \in \mathrm{Aut}(\Omega)$ with $\Psi = \Phi^{-1}$.

To prove (6), let f be continuous on Ω. Setting $h = f \circ \Phi^{-1}$ in (5) shows

$$(41) \quad \int_\Omega f|g|^p dm_\Omega = \int_\Omega f \circ \Phi^{-1} dm_\Omega = \int_\Omega f|J_\Phi| dm_\Omega$$

where J_Φ is the Jacobian of Φ. Since (41) holds for all continuous f, we find $|g|^p = |J_\Phi|$ a.e. $[m_\Phi]$, and (6) follows from the continuity of g and J_Φ.

The proof of the converse in (iii) is straightforward. This completes the proof of Theorem (2.1).

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The University of Texas at San Antonio,
San Antonio, Texas