PARTICLES WITH 2 STATES AND LORENTZ TRANSFORMATIONS

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In this note, we draw attention to a natural connection between a group closely related to the homogeneous Lorentz group, and the most general set of measurements possible on particles with only two discrete states. We may think of these two states as "spin up" and "spin down", represented by the vectors $\alpha = (1, 0)$ and $\beta = (0, 1)$, respectively.

Without invoking the full formalism of quantum mechanics, it is nevertheless reasonable to require that a generalized "set of measuring instruments" satisfy the following conditions:

1) A contains at least one member (called σ_3) which can distinguish the two spin states of the particle, i.e., gives two different results when applied to states α and β; we take $\sigma_3 \alpha = +\alpha$, $\sigma_3 \beta = -\beta$.

2) A contains a member which can "flip a spin up". Let σ_+ be this member, then $\sigma_+ \beta = \alpha$ and $\sigma_+ \alpha = (\sigma_+)^2 \beta = 0$, because (by assumption) the particle has only two spin states.

3) A contains a "down flip" operator $\sigma_- : \sigma_- \alpha = \beta$, $\sigma_- \beta = (\sigma_-)^2 \alpha = 0$.

4) We now follow the example of quantum mechanics by requiring closure of the set A under a number of operations, which we think of as "composite measurements". If $a \in A$ and $b \in A$ are measurements, then we require $a+b$ and $a \cdot b$ to belong to A also, where addition and multiplication are defined in the conventional way for linear operators on the two-dimensional vector space spanned by the basis vectors α and β. We also require, for all complex numbers λ, that $a \in A$ implies $\lambda a \in A$. A thus becomes the associative algebra of all complex 2-by-2 matrices, with the general element (a_0, a_+, a_-, a_3 = complex nos, σ_0 = unit 2-by-2 matrix)

$$a = a_0 \sigma_0 + a_+ \sigma_+ + a_- \sigma_- + a_3 \sigma_3. \tag{1}$$

It is essential that this is an associative algebra rather than merely a Lie algebra. In the Lie algebra, the only multiplicative connecting operation is commutation $[a, b] = ab - ba$, and this does not permit relations such as $(\sigma_+)^2 = (\sigma_-)^2 = 0$; these relations restrict the irreducible representations of the algebra to be two-dimensional, i.e., restrict the results of measurements to 2 basic states. By contrast, the Lie algebra spanned by σ_3, σ_+ and σ_- has irreducible representations of every dimension $n = 2j+1$.

We now consider two observers, A and B, each equipped with a complete set of measuring instruments and carrying out observations on one and the same particle. The outcomes of the individual measurements will differ, in general. But *both* observers must come to the conclusion that the particle being measured is capable of assuming two states only. This means that the measuring instruments a of observer A span the algebra $\mathfrak{A}$, and so do the measuring instruments b of observer B. Furthermore, we should expect that there is a correspondence between a and b which preserves addition, multiplication and multiplication by scalars, i.e., the transformation from observer A to observer B should be described by an *automorphism* of the algebra $\mathfrak{A}$ of their measuring instruments.

According to a basic theorem in associative algebras [1], every automorphism of a complete matrix algebra is an *inner* automorphism, of form

$$a \to b = \varphi(a) = uau^{-1} \tag{2}$$

where u is a non-singular element of $\mathfrak{A}$, i.e. a 2-by-2 matrix with non-zero determinant. Furthermore, for every complex number μ, $v = \mu u$ induces the same mapping $a \to b$ as u. We may therefore restrict ourselves, without loss of generality, to 2-by-2 matrices u with det $(u) = 1$.

The group of automorphisms of the algebra $\mathfrak{A}$ is therefore the Lie group $SL(2, C)$. This is a six-parameter group (the condition det $(u) = 1$ imposes 2 real conditions on a matrix with complex coefficients). *This 6-parameter Lie group is the covering group for the homogeneous Lorentz group* [1].

The agreement of the number of parameters (6) may be dismissed as just coincidental. But the fact that the automorphism group of the algebra is a covering group for the Lorentz group $L(3, 1)$, rather than the four-dimensional rotation group $L(4, 0)$ or the Lorentz-type group $L(2, 2)$, seems significant to us. It seems striking that consideration of measurements on particles with exactly 2 states leads so naturally to transformations between observers in a $3+1$ dimensional continuum. One may be tempted to surmise that a world made up of elementary particles of spin $\frac{1}{2}$ must of necessity exhibit 3 dimensions of space and one dimension of time, for macroscopic measurements.

Reference

[1] H. Weyl, The Classical Groups (Princeton Mathematical Series, Princeton, N.J.).

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