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Einstein–Maxwell Equations on Four-dimensional Lie Algebras

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Abstract. We classify up to automorphisms all left-invariant non-Einstein solutions to the Einstein–Maxwell equations on four-dimensional Lie algebras.

1 Introduction

A Riemannian 4-manifold (M, g) is called *Einstein* if the trace-free Ricci tensor is identically zero, that is, $\mathrm{Ric}_0 := \mathrm{Ric} - \frac{s}{4}g = 0$. From the viewpoint of general relativity, these are the Riemannian solutions of Einstein's field equations in vacuum. One can also consider the same equations in the presence of an *electro-magnetic field* F. In physics, F can be thought as a differential 2-form, which is closed and co-closed: $dF = 0$ and $d \star F = 0$, where $\star$ is the Hodge star operator (in particular, the manifold is assumed to be oriented in order to define $\star$). In this setting, the metric g and the 2-form F must satisfy the coupled system

$$\begin{aligned} \mathrm{Ric}_0 &= -[F \circ F]_0, \\ dF &= 0, \\ d \star F &= 0, \end{aligned}$$

known as the *Einstein–Maxwell equations*. Here $[F \circ F]_0 = F_{is}F_j^s - \frac{1}{4}F_{st}F^{st}g_{ij}$ is the trace-free part of the composition of F with itself, where F is thought as an endomorphism of the tangent bundle after raising an index. This term (up to a constant) is what physicists call the *stress-energy-tensor* of the electro-magnetic field.

Although the Einstein–Maxwell equations can be considered in any dimension $n \geq 4$, the four-dimensional case has a privileged status, because in this dimension, the equations imply that the solutions must have constant-scalar-curvature [12,18]. Also, in dimension four, if (g, F) is a solution of the Einstein–Maxwell equations and g is not Einstein, then F is determined uniquely up to a constant: $F := cF^+ + \frac{1}{c}F^-$, where $F^\pm = \frac{1}{2}(F \pm \star F)$ are the self-dual and the anti-self-dual parts of F [14]. Therefore, the Einstein–Maxwell equations can actually be thought of as having one unknown: the metric. We say that a metric is an *Einstein-Maxwell metric* if there is a 2-form F so that (g, F) is a solution of Einstein–Maxwell equations.

The Einstein–Maxwell equations also have some remarkable ties to Kähler geometry. First, any Kähler metric with constant-scalar-curvature (cscK for short) is an

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Einstein–Maxwell metric. Indeed, as LeBrun [12] observed, for a cscK metric, the 2-form F can be chosen as $F = \frac{1}{2}\omega + \rho_0$, where ω is the Kähler form and $\rho_0 := \mathrm{Ric}_0(J\cdot, \cdot)$ is the trace-free Ricci form of the metric. Second, more generally, a constant-scalar-curvature metric in a conformal class of a Kähler metric is Einstein–Maxwell if the conformal factor is a *holomorphy potential* [1, 13]. These observations lead to many examples of Einstein–Maxwell metrics. Any cscK metric on a complex surface is a solution. Recently, some conformally Kähler solutions have been discovered on Hirzebruch surfaces and more generally on so-called minimal ruled surfaces fibered over Riemann surfaces of any genus [2, 9, 12–14]. We also refer the reader to [5–7, 10, 11, 17] for more about obstructions to the existence of Einstein–Maxwell metrics.

In this paper, in pursuit of finding new examples, we look into four-dimensional Lie algebras. The four-dimensional Lie algebras were already classified by Mubarakzyanov [15] (a list can be found in [16]), and the (automorphism-reduced) form of left-invariant metrics on these algebras was computed by Karki in his thesis [8], where he also determined all left-invariant *Einstein metrics* on four-dimensional Lie algebras up to automorphisms of the Lie algebra. Here we find the left-invariant *non-Einstein* solutions to the Einstein–Maxwell equations (up to automorphisms).

Theorem 1.1 *The following are the four-dimensional Lie algebras admitting left-invariant non-Einstein solutions to the Einstein–Maxwell equations.*

(i) $2\mathcal{A}_2$: $[e_1, e_2] = e_2$ *and* $[e_3, e_4] = e_4$.
(ii) $\mathcal{A}_2 \oplus 2\mathcal{A}_1$: $[e_1, e_2] = e_2$.
(iii) $\mathcal{A}_{4,6}^{a,0}$: $[e_1, e_4] = ae_1$, $[e_2, e_4] = -e_3$, *and* $[e_3, e_4] = e_2$ *with* $a \neq 0$.
(iv) $\mathcal{A}_{4,9}^{-\frac{1}{2}}$: $[e_2, e_3] = e_1$, $[e_1, e_4] = \frac{1}{2}e_1$, $[e_2, e_4] = e_2$, *and* $[e_3, e_4] = -\frac{1}{2}e_3$.

Here we use the same notation for Lie algebras as in [16]. These solutions turn out to be Kähler with the fixed orientation $e^1 \wedge e^2 \wedge e^3 \wedge e^4$ except on $\mathcal{A}_{4,9}^{-\frac{1}{2}}$, which admits a solution metric that cannot be (left-invariant) Kähler with the fixed orientation (however, it is Kähler for the reverse orientation). That solution is actually a non-Kähler almost-Kähler metric (so the almost-complex structure J is non-integrable) with J-invariant Ricci tensor [4]. Indeed, a non-Kähler almost-Kähler metric with J-invariant Ricci tensor of constant scalar curvature is a solution to the Einstein–Maxwell equations, because the Ricci form is closed in that case [3]; hence the same argument applies for cscK metrics.

We also remark that $2\mathcal{A}_2$ is the only algebra which admits an left-invariant Einstein metric and also a non-Einstein solution to the Einstein–Maxwell equations. Furthermore, we remark that the corresponding Lie groups to all these Lie algebras admit no compact quotient.

2 Left-invariant Non-Einstein Solutions to the Einstein–Maxwell Equations

We present in this section the list of all four-dimensional Lie algebras admitting non-Einstein solutions to the Einstein–Maxwell equations. We give an explicit description of the solutions up to automorphisms of the Lie algebra. In order to do so, we went over the list of four-dimensional Lie algebras in [16] and their (automorphism-reduced) left-invariant Riemannian metrics (as in [8]) and then used a Maple program to determine solutions of the Einstein–Maxwell equations.

2.1 The Lie Algebra $2\mathcal{A}_2$

The structure equations of the Lie algebra $2\mathcal{A}_2$ are $[e_1, e_2] = e_2$ and $[e_3, e_4] = e_4$, where $\{e_i\}$ is a basis of $2\mathcal{A}_2$. This Lie algebra is not unimodular, so it does not admit a compact quotient. Up to automorphisms of the Lie algebra (and scaling), a left-invariant metric g is given by

$$g = \begin{bmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & a_3 & a_4 \\ a_1 & a_3 & a_5 & 0 \\ a_2 & a_4 & 0 & 1 \end{bmatrix},$$

where a_i are constants satisfying the conditions $a_5 - a_3^2 - a_1^2 > 0$ and

$$(a_4^2 - 1)a_1^2 - 2a_1a_2a_3a_4 + (a_3^2 - a_5)a_2^2 - a_4^2a_5 - a_3^2 + a_5 > 0.$$

Suppose that $F = \sum_{1\le i<j\le 4} a_{ij}e^{ij}$ is a 2-form where a_{ij} are constants and $\{e^i\}$ is the dual basis of $\{e_i\}$ and $e^{ij} = e^i \wedge e^j$. Noting that $de^i = -e^i[e_j, e_k]$, the condition $dF = 0$ implies that $a_{14} = a_{23} = a_{24} = 0$. Suppose that the orientation is $e^{1234} = e^1 \wedge e^2 \wedge e^3 \wedge e^4$. Then we have

$$\begin{aligned} \star F = {} & \frac{1}{\sqrt{\det g}}\left(a_4a_1a_{12} - a_2a_3a_{12} + a_2a_{13} + a_{34}\right)e^{12} \\ & - \frac{1}{\sqrt{\det g}}\left(a_2a_5a_{12} - -a_2a_3a_{13} - a_3a_{34}\right)e^{13} \\ & + \frac{1}{\sqrt{\det g}}\left(a_2a_4a_{13} + a_1a_{12} + a_4a_{34}\right)e^{14} \\ & - \frac{1}{\sqrt{\det g}}\left(a_4a_5a_{12} - a_4a_3a_{13} + a_1a_{34}\right)e^{23} \\ & + \frac{1}{\sqrt{\det g}}\left(a_4^2a_{13} + a_3a_{12} - a_2a_{34} - a_{13}\right)e^{24} \\ & + \frac{1}{\sqrt{\det g}}\left(a_4a_1a_{34} - a_3a_2a_{34} + a_5a_{12} - a_3a_{13}\right)e^{34}. \end{aligned}$$

Then $d \star F = 0$ implies the system of equations

$$\begin{aligned} a_2a_4a_{13} + a_1a_{12} + a_4a_{34} &= 0, \\ a_4a_5a_{12} - a_4a_3a_{13} + a_1a_{34} &= 0, \\ a_4^2a_{13} + a_3a_{12} - a_2a_{34} - a_{13} &= 0, \end{aligned}$$

which have the following *non-trivial* solutions, *i.e.*, $F \neq 0$.

Solution 1: $a_1 = a_4 = 0$, $a_3 = (a_2a_{34} + a_{13})/a_{12}$, with $a_{12} \neq 0$. Then $[F \circ F]_0(e_3, e_4) = 0$. On the other hand, the trace-free Ricci tensor $\mathrm{Ric}_0(e_3, e_4) = -a_2a_5a_{12}^2/2\det g$. Thus $a_2 = 0$. Then $[F \circ F]_0(e_1, e_3) = 0$, while $\mathrm{Ric}_0(e_1, e_3) = -a_{13}^2/2\det g$. Hence, $a_{13} = 0$.

We obtain then the solution to the Einstein–Maxwell equations given by the metric

(2.1) $$g = e^1 \otimes e^1 + e^2 \otimes e^2 + a_5\, e^3 \otimes e^3 + e^4 \otimes e^4,$$

and $F = a_{12}e^{12} + a_{34}e^{34}$ such that

(2.2) $$a_5 = \frac{1 + a_{34}^2}{1 + a_{12}^2} \neq 1.$$

Actually, when $a_5 = 1$ in (2.1), the metric g is then Einstein. In fact, the trace-free Ricci tensor of g is given by

$$\mathrm{Ric}_0 = \Big(\frac{1 - a_5}{2a_5}\Big)\, e^1 \otimes e^1 + \Big(\frac{1 - a_5}{2a_5}\Big)\, e^2 \otimes e^2 + \Big(\frac{a_5 - 1}{2}\Big)\, e^3 \otimes e^3 + \Big(\frac{a_5 - 1}{2a_5}\Big)\, e^4 \otimes e^4.$$

Moreover, we have

$$F^\pm = \frac{1}{2}\Big(\pm\frac{a_{34}}{\sqrt{a_5}} + a_{12}\Big)\, e^{12} + \frac{1}{2}\big(\pm\sqrt{a_5}a_{12} + a_{34}\big)e^{34}.$$

Furthermore, the metric g is Kähler with respect to the Kähler form

$$\omega = e^{12} + \sqrt{a_5}e^{34},$$

with the trace-free Ricci form given by

$$\rho_0 = \frac{1 - a_5}{a_5}e^{12} + \frac{a_5 - 1}{\sqrt{a_5}}e^{34}.$$

Using the relation (2.2), we then have

$$\frac{1}{2}\omega = \frac{1}{\big(\frac{a_{34}}{\sqrt{a_5}} + a_{12}\big)}F^+, \qquad \rho_0 = \Big(\frac{a_{34}}{\sqrt{a_5}} + a_{12}\Big)F^-.$$

Solution 2: $a_{12} = 0, a_1 = 0, a_4 = 0, a_{13} = -a_2a_{34}$. Then $[F \circ F]_0(e_1, e_2) = [F \circ F]_0(e_3, e_4) = 0$ implies that $a_2 = a_3 = 0$ and we again get the solution (2.1) (with $a_{12} = 0$).

Solution 3: $a_1 = -a_4({}_{13}a_2 + a_{34})/a_{12}, a_3 = (a_{13} + a_2a_{34} - a_{13}a_4^2)/a_{12}, a_5 = (a_{34}^2 + a_{13}^2 + 2a_{13}a_{34}a_2 - a_{13}^2a_4^2)/a_{12}^2$. Then $[F \circ F]_0 \equiv 0$; so any Einstein–Maxwell metric must be Einstein.

2.2 The Lie Algebra $\mathcal{A}_2 \oplus 2\mathcal{A}_1$

The structure equation is $[e_1, e_2] = e_2$. This Lie algebra is not unimodular, so it does not admit a compact quotient. Moreover, it does not admit any left-invariant Einstein metric [8]. Up to automorphisms of the Lie algebra (and scaling), a left-invariant metric g is given by

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a_1 & a_2 \\ 0 & a_1 & 1 & 0 \\ 0 & a_2 & 0 & 1 \end{bmatrix},$$

with the conditions $1 - a_1^2 > 0$ and $1 - a_1^2 - a_2^2 > 0$. The condition $dF = 0$ implies that $a_{23} = a_{24} = 0$. The condition $d \star F = 0$ implies the following non-trivial solutions.

Solution 1: $a_{13} = a_1 a_{12}, a_{14} = a_2 = 0$. To get a solution to the Einstein–Maxwell equations we need $a_1 = 0$, and so a solution is given by

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4$$

and $F = a_{12}e^{12} + a_{34}e^{34}$ such that

$$a_{34}^2 - a_{12}^2 = 1. \tag{2.3}$$

Moreover, we have

$$F^{\pm} = \frac{1}{2}(a_{12} \pm a_{34})e^{12} + \frac{1}{2}(\pm a_{12} + a_{34})e^{34}.$$

Furthermore, the metric g is Kähler with respect to the Kähler form

$$\omega = e^{12} + e^{34},$$

with the trace-free Ricci form given by $\rho_0 = -\frac{1}{2}e^{12} + \frac{1}{2}e^{34}$. Using the relation (2.3), we then have

$$\frac{1}{2}\omega = \frac{1}{(a_{12} + a_{34})}F^+, \quad \rho_0 = (a_{12} + a_{34})F^-.$$

Solution 2: $a_{12} = \frac{a_{14}}{a_2}, a_{13} = \frac{a_1 a_{14}}{a_2}$, with $a_2 \neq 0$. Maple shows that there is no solution to the Einstein–Maxwell equations.

2.3 The Lie Algebra $\mathcal{A}_{4,6}^{a,0}$

The structure equations are $[e_1, e_4] = ae_1$, $[e_2, e_4] = -e_3$, and $[e_3, e_4] = e_2$, with $a \neq 0$. This Lie algebra does not admit a compact quotient and does not admit any left-invariant Einstein metric [16]. Up to automorphisms of the Lie algebra, a left-invariant metric g is given by

$$g = \begin{bmatrix} 1 & a_1 & a_2 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the conditions $a_3 - a_3 a_1^2 - a_2^2 > 0$ and $1 - a_1^2 > 0$. The condition $dF = 0$ implies that $a_{12} = a_{13} = 0$. Then the condition $d \star F = 0$ implies

$$a_1^2 a_{34} - a_2 a_1 a_{24} + a_2 a_{14} - a_{34} = 0,$$
$$a_3 a_1 a_{14} - a_1 a_2 a_{34} + a_2^2 a_{24} - a_3 a_{24} = 0.$$

Then we distinguish two cases.

Case 1: $a_2 = 0, a_{34} = 0, a_{24} = a_1 a_{14}$. To get a solution to the Einstein–Maxwell equations, we need $a_1 = 0$, $a_3 = 1$, and $a_{23}^2 - a_{14}^2 = a^2$. Then the Einstein–Maxwell metric is

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4,$$

and $F = a_{14}e^{14} + a_{23}e^{23}$ such that

$$a_{23}^2 - a_{14}^2 = a^2, \tag{2.4}$$

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with $a \neq 0$. If $a = 0$, then g is Einstein. In addition, we have

$$F^+ = \frac{1}{2}(a_{14} + a_{23})e^{14} + \frac{1}{2}(a_{14} + a_{23})e^{23}$$

$$F^- = \frac{1}{2}(a_{14} - a_{23})e^{12} + \frac{1}{2}(-a_{14} + a_{23})e^{34}.$$

Furthermore, the metric g is Kähler with respect to the Kähler form

$$\omega = e^{14} + e^{23},$$

with the trace-free Ricci form given by $\rho_0 = -\frac{a^2}{2}e^{14} + \frac{a^2}{2}e^{34}$. Using the relation (2.4), we then have

$$\frac{1}{2}\omega = \frac{1}{(a_{14} + a_{23})}F^+, \quad \rho_0 = (a_{14} + a_{23})F^-.$$

Case 2: $a_2 \neq 0$, $a_{14} = \frac{a_{34}}{a_2}$ and $a_{24} = \frac{a_1 a_{34}}{a_2}$. Maple shows that there is no non-Einstein solution to the Einstein–Maxwell equations.

2.4 The Lie Algebra $\mathcal{A}_{4,9}^{-\frac{1}{2}}$

The structure equations of the Lie algebra $\mathcal{A}_{4,9}^{-\frac{1}{2}}$ are

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = \frac{1}{2}e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = -\frac{1}{2}e_3.$$

This Lie algebra does not admit a compact quotient and does not admit any left-invariant Einstein metric [16]. Up to automorphisms of the Lie algebra, a left-invariant metric g is given by

$$g = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 1 & a_2 & a_3 \\ 0 & a_2 & 1 & a_4 \\ 0 & a_3 & a_4 & 1 \end{bmatrix},$$

with the conditions $a_1 > 0, 1 - a_2^2 > 0, 1 - a_4^2 + 2a_2a_3a_4 - a_2^2 - a_3^2 > 0$. The condition $dF = 0$ implies that $a_{12} = 0$ and $a_{14} = \frac{1}{2}a_{23}$. Then we have two cases.

Case 1: If we suppose that $a_2a_3 \neq a_4$, then the condition $d \star F = 0$ implies that

$$a_{13} = \frac{4a_1a_2a_{24}a_3 - 4a_1a_{23}a_3^2 - 4a_1a_{24}a_4 + a_2^2a_{23} + 4a_1a_{23} - a_{23}}{a_2a_3 - a_4},$$

$$a_{34} = a_2a_{24} - a_{23}a_3.$$

Maple shows that there is no solution to the Einstein–Maxwell equations.

Case 2: If we suppose that $a_4 = a_2a_3$, then there are two solutions to $d \star F = 0$.
Solution 1: $a_{23} = 0$ and $a_{34} = a_2a_{24}$. To get a solution to the Einstein–Maxwell equations, we need $a_1 = 1$ and $a_2 = a_3 = 0$ Then the Einstein–Maxwell metric is $g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4$, with $F = a_{13}e^{13} + a_{24}e^{24}$ such that

$$a_{13}^2 - a_{24}^2 = \frac{3}{2}. \tag{2.5}$$

It turns out that the metric g is non-Kähler almost-Kähler with the orientation e^{1234}. Indeed, g is compatible with the closed 2-form $\omega = e^{13} - e^{24}$ inducing a non-integrable

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almost-complex structure J defined by $Je_1 = e_3$ and $Je_2 = -e_4$. Moreover, its Ricci tensor is J-invariant. Indeed, its trace-free Ricci form is given by $\rho_0 = \frac{3}{4}e^{13} + \frac{3}{4}e^{24}$. We then have

$$F^+ = \frac{1}{2}(a_{13} - a_{24})e^{13} + \frac{1}{2}(a_{24} - a_{13})e^{24},$$
$$F^- = \frac{1}{2}(a_{13} + a_{24})e^{13} + \frac{1}{2}(a_{24} + a_{13})e^{24}.$$

Furthermore, using the relation (2.5), we have

$$\frac{1}{2}\omega = \frac{2}{3}(a_{13} + a_{24})F^+, \quad \rho_0 = \frac{3}{2(a_{13} + a_{24})}F^-.$$

If we reverse the orientation to be $-e^{1234}$, then

$$F^+ = \frac{1}{2}(a_{13} + a_{24})e^{13} + \frac{1}{2}(a_{24} + a_{13})e^{24},$$
$$F^- = \frac{1}{2}(a_{13} - a_{24})e^{13} + \frac{1}{2}(a_{24} - a_{13})e^{24}.$$

Furthermore, the metric g is Kähler with respect to the Kähler form $\omega = e_{13} + e_{24}$, with the trace-free Ricci form given by $\rho_0 = \frac{3}{4}e^{13} - \frac{3}{4}e^{24}$. So using the relation (2.5), we then have

$$\frac{1}{2}\omega = \frac{1}{(a_{13} + a_{24})}F^+, \quad \rho_0 = (a_{13} + a_{24})F^-.$$

Solution 2: $a_{34} = a_2a_{24} - a_3a_{23}$, $a_1 = \frac{1-a_2^2}{4(1-a_3^2)}$ (with $a_3 \neq \pm 1$). Maple shows that there is no solution to the Einstein–Maxwell equations.

3 Non-existence of Einstein–Maxwell Metrics

In this section, we will explain briefly why all the other Lie algebras do not admit any non-Einstein Einstein–Maxwell metrics.

3.1 The Lie Algebra $\mathcal{A}_{4,1}$

The structure of the Lie algebra is

$$[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a_1 & 0 \\ 0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the condition that $a_2 - a_1^2 > 0$. A form F satisfies $dF = 0$ if

$$F = a_{14}e^{14} + a_{23}e^{23} + a_{24}e^{24} + a_{34}e^{34}.$$

Then the condition $d \star F = 0$ implies that $a_{34} = a_1a_{24}$ and $a_1a_{34} = a_2a_{24}$. Since $a_2 \neq 0$, we get $a_{34}\left(1 - \frac{a_1^2}{a_2}\right) = 0$; hence $a_{34} = a_{24} = 0$. We deduce that a solution to $dF = d\star F = 0$

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is given by $F = a_{14}e^{14} + a_{23}e^{23}$. Now the tensor $[F \circ F]_0$ satisfies $[F \circ F]_0(e_1, e_2) = 0$, while the trace free part of the Ricci tensor satisfies $\mathrm{Ric}_0(e_1, e_2) = -\frac{1}{2}\frac{a_1}{a_2 - a_1^2}$. Hence $a_1 = 0$ and so there is no solution to the Einstein–Maxwell equations.

3.2 The Lie Algebra $\mathcal{A}^p_{4,2}$

The structure of the Lie algebra is given by

$$[e_1, e_4] = pe_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3,$$

with $p \neq 0$. Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & a_1 & a_2 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the conditions $a_3 - a_3 a_1^2 - a_2^2 > 0$ and $a_3 > 0$. The equation $dF = 0$ implies $a_{23} = 0$, $a_{12}(p+1) = 0$, and $a_{12} + a_{13}(p+1) = 0$. We suppose first that $p \neq -1$. Then we get $a_{12} = a_{13} = 0$. The condition $d \star F = 0$ implies that

$$a_{34} - a_2 a_{14} + a_2 a_1 a_{24} - a_1^2 a_{34} = 0,$$
$$a_3 a_1 a_{14} - a_1 a_2 a_{34} + a_2^2 a_{24} - a_3 a_{24} = 0,$$
$$-a_3 a_1 a_{24} + a_3 a_{14} - a_2 a_{34} = 0.$$

Since $a_3 > 0$, from the third equation we get $a_{14} = a_1 a_{24} + \frac{a_2}{a_3} a_{34}$. Replacing it in the second equation, we get that $a_{24} = 0$, because $a_3 - a_3 a_1^2 - a_2^2 > 0$. Then it is easy to deduce that $a_{34} = a_{14} = 0$. We conclude that under the hypothesis $p \neq -1$, there is no non trivial F satisfying $dF = d \star F = 0$.

Now we suppose that $p = -1$. Then $dF = 0$ implies that $a_{23} = a_{12} = 0$. From $d \star F = 0$, it follows that

$$a_{34} - a_2 a_{14} + a_2 a_1 a_{24} - a_1^2 a_{34} = 0, \quad -a_3 a_1 a_{24} + a_3 a_{14} - a_2 a_{34} = 0.$$

We get $a_{14} = a_1 a_{24} + \frac{a_2}{a_3} a_{34}$ from the second equation. Replacing it in the first we obtain that $a_{34} = 0$. A solution F of $dF = d \star F = 0$ is of the form

$$F = a_{13}e^{13} + a_1 a_{24} e^{14} + a_{24} e^{24},$$

and then using Maple, it turns out that there are no solutions to the Einstein–Maxwell equations.

3.3 The Lie Algebra $\mathcal{A}_{4,3}$

The structure of the Lie algebra is

$$[e_1, e_4] = e_1, \quad [e_3, e_4] = e_2.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & a_1 & a_2 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the conditions that $a_3 - a_3 a_1^2 - a_2^2 > 0$ and $a_3 > 0$. The equation $dF = 0$ implies that $a_{12} = a_{13} = 0$. From $d \star F = 0$, it follows that

$$-a_1^2 a_{34} + a_2 a_1 a_{24} - a_2 a_{14} + a_{34} = 0, \quad a_3 a_1 a_{14} - a_1 a_2 a_{34} + a_2^2 a_{24} - a_3 a_{24} = 0.$$

We deduce then that $a_{34} = a_2 a_{14}$, $a_{24} = a_1 a_{14}$ and then Maple shows that there are no solutions to the Einstein–Maxwell equations.

3.4 The Lie Algebra $\mathcal{A}_{4,4}$

The structure of the Lie algebra is

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_1 + e_2, \quad [e_3, e_4] = e_2 + e_3.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & 0 \\ 0 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the conditions that $a_1 a_3 - a_2^2 > 0$ and $a_1 > 0$. Now $dF = 0$ implies that $a_{12} = a_{13} = a_{23} = 0$. From $d \star F = 0$, it follows that

$$a_1 a_{34} - a_2 a_{24} = 0, \quad a_2 a_{34} - a_3 a_{24} = 0, \quad a_{14}(a_1 a_3 - a_2^2) = 0.$$

Hence $a_{14} = a_{24} = a_{34} = 0$ and thus there is no non trivial solution F.

3.5 The Lie Algebra $\mathcal{A}_{4,5}^{a,b}$

The structure of the Lie algebra is

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = a e_2, \quad [e_3, e_4] = b e_3,$$

with $ab \neq 0$, $-1 \leq a \leq b \leq 1$. Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & a_1 & a_2 & 0 \\ a_1 & 1 & a_3 & 0 \\ a_2 & a_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the conditions that

$$1 - a_1^2 - a_2^2 - a_3^2 + 2a_1 a_2 a_3 > 0, \quad 1 - a_1^2 > 0, (1 - a_2^2)(1 - a_3^2) > 0.$$

Now $dF = 0$ implies the following solutions depending on a and b.

Case 1: $a \neq -1$, $b \neq -1$, and $a \neq -b$. In this case, we have $a_{12} = a_{13} = a_{23} = 0$. The condition $d \star F = 0$ implies that

$$a_1^2 a_{34} - a_3 a_1 a_{14} - a_1 a_2 a_{24} + a_2 a_{14} + a_3 a_{34} - a_{34} = 0,$$
$$-a_1 a_2 a_{34} - a_2 a_3 a_{14} + a_2^2 a_{24} + a_1 a_{14} + a_3 a_{34} - a_{24} = 0,$$
$$-a_3 a_1 a_{34} + a_3^2 a_{14} - a_3 a_2 a_{24} + a_1 a_{24} + a_2 a_{34} - a_{14} = 0.$$

It turns out that there is no non trivial solution F.

Case 2: $a \neq -1$ $b \neq -1$, and $a = -b$. In this case, we have $a_{12} = a_{13} = 0$. The condition $d \star F = 0$ implies that

$$a_1^2 a_{34} - a_3 a_1 a_{14} - a_1 a_2 a_{24} + a_2 a_{14} + a_3 a_{34} - a_{34} = 0,$$
$$-a_1 a_2 a_{34} - a_2 a_3 a_{14} + a_2^2 a_{24} + a_1 a_{14} + a_3 a_{34} - a_{24} = 0.$$

Then the non trivial solution is $a_{24} = a_1 a_{14}$ and $a_{34} = a_2 a_{14}$. Hence,

$$F = a_{14} e^{14} + a_{23} e^{23} + a_1 a_{14} e^{24} + a_2 a_{14} e^{34}.$$

Maple shows that there is no solution to the Einstein–Maxwell equations.

Case 3: $a \neq -1$, $b = -1$ and $a \neq -b$. In this case, $a_{12} = a_{23} = 0$. Then $d \star F = 0$ implies $a_{14} = a_1 a_{24}$, $a_{34} = a_3 a_{24}$ and it turns out that there is no solution of the Einstein–Maxwell equations.

Case 4: $a = -1$, $b \neq -1$, and $a \neq -b$. In this case, $a_{13} = a_{23} = 0$. Then $d \star F = 0$ implies $a_{14} = a_2 a_{34}$, $a_{24} = a_3 a_{34}$ and it turns out that there is no solution of the Einstein–Maxwell equations.

Case 5: $a = -1$, $b = -1$. So $dF = 0$ implies $a_{23} = 0$. The condition $d \star F = 0$ implies $a_{14} = (-a_3 a_1 a_{34} - a_3 a_2 a_{24} + a_1 a_{24} + a_2 a_{34})/(1 - a_3^2)$, and it turns out that there is no solution of the Einstein–Maxwell equations.

Case 6: $a = -1$, $b = 1$. The condition $dF = 0$ implies $a_{13} = 0$. The condition $d \star F = 0$ implies $a_{24} = (-a_1 a_2 a_{34} - a_2 a_3 a_{14} + a_1 a_{14} + a_3 a_{34})/(1 - a_2^2)$, and it turns out that there is no solution of the Einstein–Maxwell equations.

Case 7: $a = 1$, $b = -1$,. Then $dF = 0$ implies $a_{12} = 0$. The condition $d \star F = 0$ implies $a_{34} = (-a_3 a_1 a_{14} - a_2 a_1 a_{24} + a_2 a_{14} + a_3 a_{24})/(1 - a_1^2)$, and it turns out that there is no solution of the Einstein–Maxwell equations.

3.6 The Lie Algebra $\mathcal{A}_{4,6}^{a,b}$

The structure of the Lie algebra is

$$[e_1, e_4] = a e_1, \quad [e_2, e_4] = b e_2 - e_3, \quad [e_3, e_4] = e_2 + b e_3,$$

with $a \neq 0$, $b > 0$. Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & a_1 & a_2 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the conditions that $a_3 - a_3 a_1^2 - a_2^2 > 0$, $1 - a_1^2 > 0$. The condition $dF = 0$ implies that $a_{12}(a + b) = a_{13}$, $a_{13}(a + b) = -a_{12}$, and $b a_{23} = 0$. This implies that $a_{12} = a_{13} = 0 = a_{23} = 0$. The condition $d \star F = 0$ implies that

$$a_1^2 a_{34} - a_2 a_1 a_{24} + a_2 a_{14} - a_{34} = 0, \quad a_3 a_1 a_{24} - a_3 a_{14} + a_2 a_{34} = 0,$$
$$a_3 a_1 a_{14} - a_1 a_2 a_{34} + a_2^2 a_{24} - a_3 a_{24} = 0.$$

Then there is no non-trivial solution F.

3.7 The Lie Algebra $\mathcal{A}_{4,7}$

The structure of the Lie algebra is

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & a_4 & 0 & 0 \\ a_3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the conditions that $a_1 > 0$, $a_4 > 0$, $a_1 a_4 - a_2^2 - a_3^2 a_4 > 0$. The condition $dF = 0$ implies that $a_{12} = a_{13} = 0$, $a_{23} = \frac{1}{2} a_{14}$. Then $d \star F = 0$ implies that

$$\begin{aligned} a_4 a_1 a_{34} - a_3 a_4 a_{14} - a_2^2 a_{34} + a_3 a_2 a_{24} &= 0, \\ a_2 a_3 a_{34} - a_3^2 a_{24} + a_1 a_{24} - a_2 a_{14} &= 0, \\ -a_4 a_3 a_{34} + a_4 a_{14} - a_2 a_{24} - \tfrac{1}{4} a_1 a_{14} &= 0. \end{aligned}$$

Then there are two possible solutions.

Solution 1: $a_2 = a_{24} = 0$, $a_{34} = a_3 a_{14}/a_1$, and $a_4 = a_1^2/4(a_1 - a_3^2)$. Then the tensor $[F \circ F]_0 \equiv 0$, and so any Einstein–Maxwell metric is Einstein.

Solution 2: $a_1 = a_2 a_{14}/a_{24}$, $a_3 = a_2 a_{34}/a_{24}$, and $a_4 = a_2(a_{14}^2 + 4a_{24}^2)/4(-a_2 a_{34}^2 + a_{14} a_{24})$, with $a_{24} \neq 0$ and $-a_2 a_{34}^2 + a_{14} a_{24} \neq 0$; otherwise we are in the first case. We again get $[F \circ F]_0 \equiv 0$.

3.8 The Lie Algebra $\mathcal{A}_{4,8}$

The structure of the Lie algebra is

$$[e_2, e_3] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = -e_3.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 1 & a_2 & a_3 \\ 0 & a_2 & 1 & a_4 \\ 0 & a_3 & a_4 & 1 \end{bmatrix},$$

with the conditions that $a_1 > 0, 1 - a_2^2 > 0, 1 - a_4^2 + 2a_2 a_3 a_4 - a_2^2 - a_3^2 > 0$. The condition $dF = 0$ implies that $a_{12} = a_{13} = a_{14} = 0$. However, there is no non trivial solution to the equation $d \star F = 0$.

3.9 The Lie Algebra $\mathcal{A}_{4,9}^b$

The structure of the Lie algebra is

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = (b+1)e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = be_3,$$

with the conditions $-1 < b \le 1$ and $b \neq -\frac{1}{2}$. Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 1 & a_2 & a_3 \\ 0 & a_2 & 1 & a_4 \\ 0 & a_3 & a_4 & 1 \end{bmatrix},$$

with the conditions $a_1 > 0$, $1 - a_2^2 > 0$, $1 - a_4^2 + 2a_2a_3a_4 - a_2^2 - a_3^2 > 0$. From the condition $dF = 0$, we get $a_{12} = 0$, $a_{13} = 0$, $a_{14} = (1+b)a_{23}$. Then $d \star F = 0$ implies that

$$a_3 = \frac{a_2^2a_{34} + a_2a_{23}a_4 - a_{34}}{a_{23}}, \qquad a_{24} = a_2a_{34} + a_{23}a_4,$$
$$b = \frac{a_1a_2^2a_{34}^2 - a_1a_{23}^2a_4^2 + a_2^2a_{23}^2 + a_1a_{23}^2 - a_1a_{34}^2 - a_{23}^2}{a_{23}^2(1 - a_2^2)},$$

($a_{23} \neq 0$, otherwise F is trivial) Maple shows then that there is no solution to the Einstein–Maxwell equations.

3.10 The Lie Algebra $\mathcal{A}_{4,10}$

The structure of the Lie algebra is

$$[e_2, e_3] = e_1, \quad [e_2, e_4] = -e_3, \quad [e_3, e_4] = e_2,$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & 1 & a_4 \\ 0 & a_3 & a_4 & 1 \end{bmatrix},$$

with the conditions that $a_1 > 0$, $a_2 > 0$, $a_2 - a_2a_4^2 - a_3^2 > 0$. Now $dF = 0$ implies that $a_{12} = a_{13} = a_{14} = 0$. Then $d \star F = 0$ implies that $a_2 = \frac{a_{23}^2(1-a_4^2)}{a_{34}^2}$, $a_{24} = a_{23}a_4$, $a_3 = \frac{a_{23}(a_4^2-1)}{a_{34}}$, ($a_{34} \neq 0$, otherwise F is trivial). But then the determinant of g is 0.

3.11 The Lie Algebra $\mathcal{A}_{4,11}^a$

The structure of the Lie algebra is

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = 2ae_1, \quad [e_2, e_4] = ae_2 - e_3, \quad [e_3, e_4] = e_2 + ae_3,$$

with $a > 0$. Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & 1 & a_4 \\ 0 & a_3 & a_4 & 1 \end{bmatrix},$$

with the conditions that $a_1 > 0$, $a_2 > 0$, $a_2 - a_2a_4^2 - a_3^2 > 0$. The condition $dF = 0$ implies that $a_{12} = a_{13} = 0$ and $a_{14} = 2aa_{23}$. Moreover, the condition $d \star F = 0$ implies $a_1 = \frac{4a_2^2a^2}{a_2 - a_2a_4^2 - a_3^2}$, $a_{24} = a_{23}a_4$, $a_{34} = -\frac{a_{23}a_3}{a_2}$. Then $[F \circ F]_0 \equiv 0$ and hence any Einstein–Maxwell metric is Einstein.

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3.12 The Lie Algebra $\mathcal{A}_{4,12}$

The structure of the Lie algebra is

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = -e_2, \quad [e_2, e_4] = e_1.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & a_2 & 1 & a_4 \\ 0 & a_3 & a_4 & a_5 \end{bmatrix},$$

with the conditions that $a_1 > 0$, $a_1 - a_2^2 > 0$, $a_1 a_5 - a_1 a_4^2 - a_2^2 a_5 + 2a_2 a_3 a_4 - a_3^2 > 0$. Then $dF = 0$ implies that $a_{12} = 0$, $a_{13} = a_{24}$, $a_{14} = -a_{23}$. The condition $d \star F = 0$ implies the following different solutions.

Solution 1: $a_2 = a_{23} = a_{34} = a_4 = 0$ and $a_5 = \frac{a_3^2+1}{a_1}$. Then $[F \circ F]_0 \equiv 0$ and so any Einstein–Maxwell metric is Einstein.

Solution 2: $a_{24} = 0$, $a_{34} = -\frac{a_3 a_{23}}{a_1}$, $a_4 = \frac{a_2 a_3}{a_1}$, $a_5 = \frac{-a_1 a_2^2 + a_1^2 + a_3^2}{a_1}$. Then we have $[F \circ F]_0 \equiv 0$.

Solution 3: Suppose that $a_{24} \neq 0$, $a_{23} \neq -a_{24} a_4$, $a_{24} \neq a_{23} a_4$, $a_{23} \neq 0$. Then

$$a_1 = -\frac{a_{23}(a_{23} a_4 - a_{24})}{a_{24}(a_{24} a_4 + a_{23})}, \quad a_{34} = \frac{a_3 a_{24}(a_4 a_{24} + a_{23})}{a_{23} a_4 - a_{24}}, \quad a_2 = 0,$$

$$a_5 = -\frac{a_{24}^3 a_3^2 a_4 + a_{23}^3 a_4^2 + a_{23} a_{24}^2 a_3^2 - a_{23} a_{24}^2 a_4^2 - 2a_{23}^2 a_{24} a_4 + a_{24}^3 a_4 + a_{23} a_{24}^2}{a_{23} a_{24}(a_{23} a_4 - a_{24})}.$$

Then $[F \circ F]_0 \equiv 0$.

Solution 4: If we suppose that $a_{24} = 0$ then we are in the case of Solution 2.

Solution 5: If we suppose that $a_{23} = 0$, then we are in the case of Solution 1.

Solution 6: If we suppose that $a_{24} = a_{23} a_4$, then one of the following hold.

(1) $a_2 = a_4 = 0$, $a_{34} = -\frac{a_3 a_{23}}{a_1}$ and $a_5 = \frac{a_1^2 + a_3^2}{a_1}$. Then $[F \circ F]_0 \equiv 0$.

(2) $a_2 \neq 0$ and then

$$a_3 = \frac{(a_1^2 a_4^2 - a_1 a_2^2 + a_1^2 + a_2^2) a_4}{a_2(a_1 a_4^2 + 1)}, \quad a_{34} = -\frac{a_{23} a_4(-a2^2 a_4^2 + a_1 a_4^2 - a_2^2 + a_1)}{a_2(a_1 a_4^2 + 1)},$$

$$a_5 = \frac{\begin{array}{c} a_1^3 a_4^6 - 2a_1^2 a_2^2 a_4^4 - a_2^4 a_4^4 + 2a_1^3 a_4^4 + 3a_1 a_2^2 a_4^4 \\ - a_1^2 a_2^2 a_4^2 - 2a_2^4 a_4^2 + a_1^3 a_4^2 + 2a_1 a_2^2 a_4^2 - a_2^4 + a_2^2 a_4^2 + a_1 a_2^2 \end{array}}{a_2^2(a_1 a_4^2 + 1)^2}.$$

Then $[F \circ F]_0 \equiv 0$.

Solution 7: If we suppose that $a_{23} = -a_4 a_{24}$, then one of the following hold.

(1) $a_2 = 0$, $a_{34} = 0$, $a_5 = \frac{1+a_3^2}{a_1}$. Then $[F \circ F]_0(e_1, e_3) = 0$ implies that $a_3 = 0$ and there will be no solution to the Einstein–Maxwell equations.

(2) If $a_2 \neq 0$, then

$$a_3 = \frac{(a_1a_2^2 + a_1a_4^2 - a_2^2 + a_1)a_4}{a_2(a_4^2 + a_1)}, \qquad a_{34} = \frac{a_{24}(a_2^2a_4^2 + a_4^4 + a_2^2 + a_4^2)}{a_2(a_4^2 + a_1)},$$

$$a_5 = \frac{\begin{aligned}-a_2^4a_4^4 + 3a_1a_2^2a_4^4 + a_1a_4^6 + a_1^2a_2^2a_4^2 - 2a_2^4a_4^2 - 2a_2^2a_4^4 \\ + 2a_1a_2^2a_4^2 + 2a_1a_4^4 - a_2^4 - a_2^2a_4^2 + a_1a_2^2 + a_1a_4^2\end{aligned}}{a_2^2(a_4^2 + a_1)^2}.$$

Then $[F \circ F]_0 \equiv 0$.

Solution 8: $a_2 \neq 0$ and

$$a_3 = \frac{a_1^2a_{24}^2a_4 - a_1a_2^2a_{23}a_{24} + a_1^2a_{23}a_{24} + a_1a_{23}^2a_4 + a_2^2a_{23}a_{24} - a_1a_{23}a_{24}}{a_2(a_1a_{24}^2 + a_{23}^2)},$$

$$a_{34} = -\frac{a_1^2a_{24}^2a_{23}a_4 - a_2^2a_{23}^2a_{24} - a_2^2a_{24}^3 + a_1a_{23}^2a_{24} + a_{23}^3a_4 - a_{23}^2a_{24}}{a_2(a_1a_{24}^2 + a_{23}^2)},$$

$$\begin{aligned}a_2^2(a_1a_{24}^2 + a_{23}^2)a_5 = {} & a_1^3a_{24}^4a_4^2 - 2a_1^2a_2^2a_{23}a_{24}^3a_4 + 2a_1^3a_{23}a_{24}^3a_4 - a_1^2a_2^2a_{23}^2a_{24}^2 \\ & + 2a_1^2a_{23}^2a_{24}^2a_4^2 - 2a_1a_2^2a_{23}^3a_{24}a_4 + 2a_1a_2^2a_{23}a_{24}^3a_4 \\ & - a_2^4a_{23}^4 - 2a_2^4a_{23}^2a_{24}^2 - a_2^4a_{24}^4 \\ & + a_1^3a_{23}^2a_{24}^2 + 2a_1^2a_{23}^3a_{24}a_4 - 2a_1^2a_{23}a_{24}^3a_4 \\ & + a_1a_2^2a_{23}^4 + 4a_1a_2^2a_{23}^2a_{24}^2 + a_1a_2^2a_{24}^4 \\ & + a_1a_{23}^4a_4^2 + 2a_2^2a_{23}^3a_{24}a_4 - 2a_1^2a_{23}^2a_{24}^2 \\ & - 2a_1a_{23}^3a_{24}a_4 - a_2^2a_{23}^2a_{24}^2 + a_1a_{23}^2a_{24}^2.\end{aligned}$$

Then $[F \circ F]_0 \equiv 0$. Here $a_{23} \neq 0$ and $a_{24} \neq 0$, otherwise F vanishes.

3.13 The Lie Algebra $\mathcal{A}_{3,1} \oplus \mathcal{A}_1$

The structure of the Lie algebra is $[e_3, e_4] = e_1$. Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the condition $a_1 > 0$. The condition $dF = 0$ implies $a_{12} = 0$ and then $d \star F = 0$ implies $a_{34} = 0$. Maple then shows that there is no solution to the Einstein–Maxwell equation.

3.14 The Lie Algebra $\mathcal{A}_{3,2} \oplus \mathcal{A}_1$

The structure of the Lie algebra is

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} a_1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & a_3 \\ 0 & 0 & 1 & 0 \\ a_2 & a_3 & 0 & 1 \end{bmatrix},$$

with the conditions $a_1 > 0$, $a_1 - a_2^2 - a_1 a_3^2 > 0$. The condition $dF = 0$ implies $a_{12} = a_{14} = a_{24} = 0$. Then $d \star F = 0$ has no non trivial solution.

3.15 The Lie Algebra $\mathcal{A}_{3,3} \oplus \mathcal{A}_1$

The structure of the Lie algebra is

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & a_1 & 0 & a_2 \\ a_1 & 1 & 0 & a_3 \\ 0 & 0 & 1 & 0 \\ a_2 & a_3 & 0 & 1 \end{bmatrix},$$

with the conditions $1 - a_1^2 > 0, 1 - a_1^2 - a_2^2 - a_3^2 + 2a_1 a_2 a_3 > 0$. The condition $dF = 0$ implies $a_{12} = a_{14} = a_{24} = 0$. Then the only solution to $d \star F = 0$ is the trivial solution.

3.16 The Lie Algebra $\mathcal{A}_{3,4} \oplus \mathcal{A}_1$

The structure of the Lie algebra is

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & a_1 & 0 & a_2 \\ a_1 & 1 & 0 & a_3 \\ 0 & 0 & 1 & 0 \\ a_2 & a_3 & 0 & 1 \end{bmatrix},$$

with the conditions $1 - a_1^2 > 0, 1 - a_1^2 - a_2^2 - a_3^2 + 2a_1 a_2 a_3 > 0$. The condition $dF = 0$ implies $a_{14} = a_{24} = 0$. The condition $d \star F = 0$ implies that $a_{13} = -a_2 a_{34}$, $a_{23} = -a_3 a_{34}$. Then Maple shows that there is no solution to the Einstein–Maxwell equations.

3.17 The Lie Algebra $\mathcal{A}_{3,5} \oplus \mathcal{A}_1$

The structure of the Lie algebra is

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = a e_2,$$

with $0 < |a| < 1$. Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & a_1 & 0 & a_2 \\ a_1 & 1 & 0 & a_3 \\ 0 & 0 & 1 & 0 \\ a_2 & a_3 & 0 & 1 \end{bmatrix},$$

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with the conditions $1 - a_1^2 > 0, 1 - a_1^2 - a_2^2 - a_3^2 + 2a_1a_2a_3 > 0$. The condition $dF = 0$ implies $a_{12} = a_{14} = a_{24} = 0$ Then the only solution to $d \star F = 0$ is the trivial solution.

3.18 The Lie Algebra $\mathcal{A}_{3,6} \oplus \mathcal{A}_1$

The structure of the Lie algebra is

$$[e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & 1 & 0 \\ a_1 & a_3 & 0 & 1 \end{bmatrix},$$

with the conditions $a_2 > 0$, $a_2 - a_3^2 - a_2a_1^2 > 0$. The condition $dF = 0$ implies $a_{14} = a_{24} = 0$. The condition $d \star F = 0$ implies the following solutions

Solution 1: $a_3 = 0, a_{13} = -a_1a_{34}, a_{23} = 0$. Then $a_1 = 0, a_2 = 1, a_{12} = \pm a_{34}$ and hence $[F \circ F] \equiv 0$ and so any Einstein–Maxwell metric is Einstein.

Solution 2: $a_3 \neq 0, a_{13} = \frac{a_1a_{23}}{a_3}, a_{34} = -\frac{a_{23}}{a_3}$. Then Maple shows that there is no solution to the Einstein–Maxwell equations.

3.19 The Lie Algebra $\mathcal{A}_{3,7} \oplus \mathcal{A}_1$

The structure of the Lie algebra is

$$[e_1, e_3] = ae_1 - e_2, \quad [e_2, e_3] = e_1 + ae_2,$$

with $a > 0$. Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & 1 & 0 \\ a_1 & a_3 & 0 & 1 \end{bmatrix},$$

with the conditions $a_2 > 0$, $a_2 - a_3^2 - a_2a_1^2 > 0$. The condition $dF = 0$ implies that $a_{12} = a_{14} = a_{24} = 0$. Then the only solution to $d \star F = 0$ is the trivial solution.

3.20 The Lie Algebra $\mathcal{A}_{3,8} \oplus \mathcal{A}_1$

The structure of the Lie algebra is

$$[e_1, e_3] = -2e_2, \quad [e_1, e_2] = e_1, \quad [e_2, e_3] = e_3.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & 0 & a_5 \\ 0 & 0 & a_3 & a_6 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix},$$

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with the conditions $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1(a_2(a_3 - a_6^2) - a_3 a_5^2) - a_4^2 a_3 a_2 > 0$. The condition $dF = 0$ implies that $a_{14} = a_{24} = a_{34} = 0$. Then the condition $d \star F = 0$ implies that F is trivial.

3.21 The Lie Algebra $\mathcal{A}_{3,9} \oplus \mathcal{A}_1$

The structure of the Lie algebra is

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

Up to automorphisms of the Lie algebra, a metric g is given by

$$g = \begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & 0 & a_5 \\ 0 & 0 & a_3 & a_6 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix},$$

with the conditions $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1(a_2(a_3 - a_6^2) - a_3 a_5^2) - a_4^2 a_3 a_2 > 0$. The condition $dF = 0$ implies that $a_{14} = a_{24} = a_{34} = 0$. Then the only solution to $d \star F = 0$ is the trivial solution.

3.22 The Abelian Algebra

Any metric can be reduced to the flat euclidean metric.

4 Appendix

We reproduce below the essential part of Maple code used to solve the Einstein–Maxwell equations. We take here for example the Lie algebra $\mathcal{A}_{3,1} \oplus \mathcal{A}_1$.

```
# To define the structure equations of the Lie algebra:
brac :=(x, y) → vector(n, [(x[3]*y[4]-x[4]*y[3]),0,0,0]);
# To define the metric:
G := Matrix([[a1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]])
# To define the coefficients of the Levi-Civita connection with respect to a metric G:
LeviCivita := (x, y, z) → (1/2)*evalm(innerprod(brac(x, y), G,
z)+innerprod(brac(z, x), G, y)-innerprod(brac(y, z), G, x));
# To define the Levi-Civita connection of two vectors with respect to a G-orthonormal basis {v[i]}:
LC := (x, y) → evalm(LeviCivita(x, y, evalm(v[1]))*v[1]+LeviCivita(x, y,
evalm(v[2]))*v[2]+LeviCivita(x, y, evalm(v[3]))*v[3]+LeviCivita(x, y,
evalm(v[4]))*v[4]);
# To define the Riemannian tensor of the metric G:
Rc := (x, y, z) → -evalm(simplify(LC(x, LC(y, z))-LC(y, LC(x, z))-LC(brac(x,
y), z)));
RiemC := (x, y, z, w) → simplify(innerprod(Rc(x, y, z), G, w));
# To define the Ricci tensor, the Riemannian scalar R and the trace free part of the Ricci tensor Ric0 of the
metric G:
Ricci := (x, y)→ evalm(simplify(RiemC(x, evalm(v[1]), y, evalm(v[1]))+RiemC(x,
evalm(v[2]), y, evalm(v[2]))+RiemC(x, evalm(v[3]), y, evalm(v[3]))+RiemC(x,
evalm(v[4]), y, evalm(v[4]))));
R := evalm(simplify(Ricci(evalm(v[1]), evalm(v[1]))+Ricci(evalm(v[2]),
evalm(v[2]))+Ricci(evalm(v[3]), evalm(v[3]))+Ricci(evalm(v[4]), evalm(v[4]))));
Rico := (x, y)→evalm(simplify(Ricci(x, y)-(1/4)*R*innerprod(x, G, y)));
# To define the Hodge star of a 2-form:
DGsetup([x1, x2, x3, x4]):
g := evalDG(a1*(dx1 &t dx1)+(dx2 &t dx2)+(dx3 &t dx3)+(dx4 &t dx4));
```

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```
HodgeStar(g, a13*(dx1 &w dx3)+a14*(dx1 &w dx4)+a24(dx2 &w dx4)+a23*(dx2 &w
dx3)+a34*(dx3 &w dx4));
# To define the trace free part [F ∘ F]_0 of a 2-form F:
K := Transpose(Multiply(F, MatrixInverse(G)));
F ∘ F := simplify(Multiply(Transpose(K), F));
Tr(F ∘ F) := innerprod(evalm(v[1]), F ∘ F, evalm(v[1]))+innerprod(evalm(v[2]),
F ∘ F, evalm(v[2]))+innerprod(evalm(v[3]), F ∘ F,
evalm(v[3]))+innerprod(evalm(v[4]), F ∘ F, evalm(v[4]));
[F ∘ F]_0 := simplify(F ∘ F-(Tr(F∘F)/4)*G);
# To solve the Einstein–Maxwell equations:
solve(Rico(evalm(e[1]),evalm(e[1]))+[F ∘ F]_0[1,1]=0,
Rico(evalm(e[1]),evalm(e[2]))+[F ∘ F]_0[1,2]=0,
Rico(evalm(e[1]),evalm(e[3]))+[F ∘ F]_0[1,3]=0,
Rico(evalm(e[1]),evalm(e[4]))+[F ∘ F]_0[1,4]=0,
Rico(evalm(e[2]),evalm(e[2]))+[F ∘ F]_0[2,2]=0,
Rico(evalm(e[2]),evalm(e[3]))+[F ∘ F]_0[2,3]=0,
Rico(evalm(e[2]),evalm(e[4]))+[F ∘ F]_0[2,4]=0,
Rico(evalm(e[3]),evalm(e[3]))+[F ∘ F]_0[3,3]=0,
Rico(evalm(e[3]),evalm(e[4]))+[F ∘ F]_0[3,4]=0,
Rico(evalm(e[4]),evalm(e[4]))+[F ∘ F]_0[4,4]=0,a1, a13, a14, a23, a24);
```

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