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BIPARTITE SCORE SETS

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ABSTRACT. The question of what sets of integers may be the score sets of bipartite tournaments was posed recently by K. B. Reid. The main theorem of this paper establishes a sufficient condition for pairs of sets to be bipartite score sets. This simple condition yields an immediate affirmative answer for a large class of pairs of sets.

Preliminaries. A *bipartite tournament* is a complete asymmetric bipartite digraph. The outdegree of a vertex of a bipartite tournament is called a *score.* Two sequences $s_1 \leq s_2 \leq \cdots \leq s_p$ and $t_1 \leq t_2 \leq \cdots \leq t_q$ of integers are called *score sequences of a bipartite tournament,* if there exists a bipartite tournament with bipartition (X, Y) such that the vertices of X and Y may be labelled $x_1, x_2, \ldots, x_p$ and $y_1, y_2, \ldots, y_q$ respectively, and moreover, the outdegree of x_i is s_i and of y_j is t_j for all i and j. The sets $S = \{s_i : 1 \leq i \leq p\}$ and $T = \{t_j : 1 \leq j \leq q\}$ of elements of the score sequences are called *score sets.*

The question of what sets of integers may be the score sets of a bipartite tournament was posed by K. B. Reid at the fourth International Conference on Theory and Applications of Graphs in Kalamazoo. This paper provides an affirmative answer for a large class of sets and some simplification of the question for arbitrary sets.

Throughout the remainder of the paper it will be assumed that $S = \{s_1, s_2, \ldots, s_n = s\}$ and $T = \{t_1, t_2, \ldots, t_m = t\}$ are nonempty finite sets of non-negative integers such that $s_1 < s_2 < \cdots < s_n$ and $t_1 < t_2 < \cdots < t_m$ where s_1 and t_1 are not both zero. The latter condition is needed, since a bipartite tournament cannot contain vertices of score zero in both partitions. Further, to insure that $t \geq n$, it will be assumed that $n \leq m$ and that if $n = m$, then $t_1 \neq 0$.

MAIN THEOREM. *There exists a bipartite tournament with bipartition* (X, Y) *whose score sets are S and T, such that* $|X| > t$ *if and only if*

$$b = \sum_{i=1}^{n} s_i + (t - n + 1)s + \sum_{j=1}^{m} t_j + 1 - m(t + 1) \quad \text{is positive.}$$

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The computation of b can serve as a quick test for a pair of sets to be score sets. However, for an obvious collection of pairs of sets, the subsequent corollary eliminates even this task in providing an affirmative answer to the proposed question.

COROLLARY 1. *If $s \geq m-1$, then there exists a bipartite tournament with score sets S and T.*

Note that in the corollary above, m is the number of scores in the longer set T and s is the largest score in the shorter set S.

The proof of the theorem and corollary follows from a criterion for score sequences of n-partite tournaments established by J. W. Moon [2]. Since the full generality of the result is not needed, we state only the applicable portion.

THEOREM (Moon). *There exists a bipartite tournament with score sequences $p_1 \leq p_2 \leq \cdots \leq p_n$ and $q_1 \leq q_2 \leq \cdots \leq q_m$ if and only if*

$$\sum_{i=1}^{k} p_i + \sum_{j=1}^{r} q_j \geq kr$$

for all k and r such that $0 \leq k \leq n$ and $0 \leq r \leq m$ with equality when $k=n$ and $r=m$.

Proof of main theorem and corollaries

Proof of main theorem. ($\rightarrow$) Suppose there is a bipartite tournament with bipartition (X, Y) whose score sets are S and T such that $|X|>t$.

Let $p_1 \leq p_2 \leq \cdots \leq p_t \leq p_{t+1}$ be the initial $t+1$ terms of the score sequences corresponding to the partition X (Note that this requires $|X|>t$.) Likewise, let $q_1 \leq q_2 \leq \cdots \leq q_m$ be the first m scores of the sequence corresponding to the partition Y. It follows from Moon's theorem that

$$m(t+1) \leq \sum_{i=1}^{t+1} p_i + \sum_{j=1}^{m} q_j.$$

Since $s_1 < s_2 < \cdots < s_n$ are the scores of the vertices in X and each must appear in the score sequence, $p_i \leq s_i$ and $p_i \leq s$ for $1 \leq i \leq n$. Likewise, $q_j \leq t_j$ for $1 \leq j \leq m$. Hence,

$$m(t+1) \leq \sum_{i=1}^{n} s_i + (t+1-n)s + \sum_{j=1}^{m} t_j$$

and therefore

$$b = \sum_{i=1}^{n} s_i + (t+1-n)s + \sum_{j=1}^{m} t_j - m(t+1) + 1$$

is positive.

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($\leftarrow$) Suppose $b > 0$. Let

$$p_i = \begin{cases} s_i, & 1 \le i < n \\ s, & n \le i \le t+1 \end{cases} \quad \text{and} \quad q_j = \begin{cases} t_j, & 1 \le j < m \\ t, & m \le j \le u \end{cases}$$

where $u = m + b - 1$. Note that $m \le u$ and recall $n \le t$.

That there exists a bipartite tournament with bipartition (X, Y) whose score sets are S and T, such that $|X| = t+1$ and $|Y| = u$ will follow from showing that the sequences $p_1 \le p_2 \le \cdots \le p_{t+1}$ and $q_1 \le q_2 \le \cdots \le q_u$ satisfy Moon's criterion.

Let

$$S(k, r) = \sum_{i=1}^{k} p_i + \sum_{j=1}^{r} q_j.$$

We must show that $S(k, r) \ge kr$ for each k and r such that, $0 \le k \le t+1$ and $0 \le r \le u$ with equality when $k = t+1$ and $r = u$. Note that the requirements for $k = 0$ or $r = 0$ are equivalent to the convention that the elements of S and T are non-negative with at most one zero between them. The remaining possibilities for k and r lie in the four cases:

Case i: $1 \le k < n$ and $1 \le r < m$.

Case ii: $1 \le k < n$ and $m \le r \le u$.

Case iii: $n \le k \le t+1$ and $1 \le r < s$.

Case iv: $n \le k \le t+1$ and $s \le r \le u$.

Case i: Suppose $1 \le k \le n$ and $1 \le r < m$. If $s_1 \ne 0$, then, as $1 \le s_1 < s_2 < \cdots < s_n$, $s_i \ge i$ for each i, $1 \le i \le n$. Thus,

$$\sum_{i=1}^{k} s_i \ge \binom{k+1}{2}.$$

Also, as $0 \le t_1 < t_2 < \cdots < t_m$, $t_j \ge j - 1$ for each j, $1 \le j \le m$. Thus,

$$\sum_{j=1}^{r} t_j \ge \binom{r}{2}$$

and

$$S(k, r) \ge \binom{k+1}{2} + \binom{r}{2} \ge kr.$$

If $s_1 = 0$, then $t_1 \ne 0$ and reasoning as above yields

$$S(k, r) \ge \binom{k}{2} + \binom{r+1}{2} \ge kr.$$

Case ii: Suppose $1 \le k < n$ and $m \le r \le u$. Observe that $S(k, r) = S(k, m-1) + (r - m + 1)t$ and recall that $n \le t$, hence $k < t$. Now, applying case i, $S(k, r) \ge k(m-1) + (r - m + 1)k = kr$.

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Case iii: Suppose $n \le k \le t+1$ and $1 \le r < s$. Now, $S(k, r) = S(n-1, r) + (k-n+1)s$. Therefore, by cases i and ii and the hypothesis $r < s$,

$$S(k, r) \ge (n-1)r + (k-n+1)r = kr.$$

Case iv: Suppose $n \le k \le t+1$ and $s \le r \le u = m+b-1$. If $r \le m$, then

$$S(k, r) = \sum_{i=1}^{n} s_i + (k-n)s + \sum_{j=1}^{r} t_j.$$

Subtracting

$$b = \sum_{i=1}^{n} s_i + (t+1-n)s + \sum_{j=1}^{m} t_j + 1 - m(t+1)$$

yields

$$S(k, r) - b = m(t+1) - (t+1-k)s - \sum_{j=r+1}^{m} t_j - 1.$$

Since $t_j \le t$ for $1 \le j \le m$, $\sum_{j=r+1}^{m} t_j \le (m-r)t$ and it follows that

(1) $$S(k, r) - b \ge m + rt - (t+1-k)s - 1.$$

On the other hand, if $r > m$, then

$$S(k, r) = \sum_{i=1}^{n} s_i + (k-n)s + \sum_{j=1}^{m} t_j + (r-m)t.$$

Subtracting b yields (1) as above. Thus,

$$\begin{aligned} S(k, r) &\ge b + m - 1 + rt - (t-k)s - s \\ &\ge u - s + (r-s)(t-k) + rk \ge kr. \end{aligned}$$

Finally,

$$\begin{aligned} S(t+1, u) &= \sum_{i=1}^{n} s_i + (t+1-n)s + \sum_{j=1}^{m} t_j + (b-1)t \\ &= b - 1 + m(t+1) + (b-1)t \\ &= (t+1)(m+b-1). \end{aligned}$$

Proof of Corollary 1: Put

$$c = \sum_{i=1}^{n} s_i + \sum_{j=1}^{m-1} t_j - n(m-1).$$

We observe that $b \ge c$ with equality occurring only if $s = m-1$.

If $s_1 \ne 0$, then as in the previous proof $s_i \ge i$ for $1 \le i \le n$ and $t_j \ge j-1$ for $1 \le j \le m$. Hence,

$$c \ge \binom{n+1}{2} + \binom{m-1}{2} - n(m-1) \ge 0$$

with equality occurring only if $s_i = i$ for $1 \leq i \leq n$ and $t_j = j-1$ for $1 \leq j \leq m$. Thus, $b \geq c \geq 0$. If $b > 0$, the result follows by the Main theorem. If $b = 0$, then $b = c = 0$ and $m-1 = s = s_n = n$. This implies that $S = \{1, 2, \ldots, n\}$ and $T = \{0, 1, 2, \ldots, n\}$. It is a simple task to verify by Moon's criterion or by actual construction that the latter form score sequences of a bipartite tournament.

If $s_1 = 0$, it can be shown in a similar manner that $c \geq 0$ and that for $b = 0$; $S = \{0, 1, 2, \ldots, n-1\}$ and $T = \{1, 2, \ldots, n\}$. Again it is routine to show that S and T form score sequences.

Consider a set of non-negative integers $r_1 < r_2 < \cdots < r_n \neq 0$. Since $r_n \geq n-1$, Corollary 1 implies that the sets $S = \{r_n\}$ and $T = \{r_1, r_2, \ldots, r_n\}$ form the score sets of some bipartite tournament. Thus, we have the interesting corollary:

COROLLARY 2: *Any finite nonempty set of non-negative integers, except* $\{0\}$, *may be the union of the score sets of some bipartite tournament.*

A weak theorem. The maximum score t in T requires at least t vertices in the partition X. The Main theorem is restricted to those cases in which the partition X may be formed with more than t vertices. Unfortunately, it is possible that a bipartite tournament with score sets S and T exists which has exactly t vertices in partition X while none exists with more than t vertices. As examples, consider $S = \{0\}$ and $T = \{t\}$ where t is any positive integer or $S = \{1, 2\}$ and $T = \{1, 2, 3, 5\}$. The first example is obvious. The second may be verified by computation of b and application of Moon's theorem to the score sequences $1 \leq 2 \leq 2 \leq 2 \leq 2$ and $1 \leq 2 \leq 3 \leq 5$.

We observe that a bipartite tournament with score sets S and T which has exactly t vertices in partition X exists if and only if a bipartite tournament with score sets S and $T - \{t\}$ which has exactly t vertices in the first partition exists. In the latter case the size of the first partition has little relation to the scores of the vertices of the other partition.

To augment this discouraging observation, consider the sets $S = \{0, 2\}$ and $T = \{2, c, 8, 9, 11\}$ for $2 < c < 8$. Since $b = c - 7$ is not positive, the Main theorem permits only the possibility of exactly eleven vertices in partition X. As may be confirmed with the aid of the ensuing theorem, S and T are score sets of bipartite tournaments only for $c = 5$ and $c = 7$.

Thus, the existence of a simple necessary and sufficient condition for bipartite tournaments with score sets S and T when $b \leq 0$ is made unlikely. However, the subsequent theorem shows a necessary condition in terms of a 'special' linear combination of the possible scores. The theorem and its proof offer an elementary algorithm for the resolution of particular cases.

WEAK THEOREM. *Suppose* $b \leq 0$. *If there exists a bipartite tournament with score sets* S *and* T, *then* $s < m + b$ *and there are non-negative integers*

$a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$ such that

$$\sum_{i=1}^{n} a_i(s-s_i)+\sum_{j=1}^{m} b_j(t-t_j)=m+b-s-1 \quad \text{where} \quad \sum_{i=1}^{n} a_i=t-n.$$

Proof of weak theorem: Let $c_i(d_j)$ be the number of vertices of $X(Y)$ of score $s_i(t_j)$ for $1\le i\le n(1\le j\le m)$. Let $q=\sum_{j=1}^{m} d_j$ and note that $t=\sum_{i=1}^{n} c_i$. Note further that

$$\sum_{j=1}^{m} d_j t_j+\sum_{i=1}^{n} c_i s_i=qt.$$

From the definition of b it follows that

$$b+m-s-1=\sum_{i=1}^{n} s_i+(t-n)s+\sum_{j=1}^{m} t_j-mt.$$

Subtracting the previous line yields,

$$b+m-s-1=\sum_{i=1}^{n} (1-c_i)s_i+(t-n)s+\sum_{j=1}^{m} (1-d_j)t_j+(q-m)t.$$

Now,

$$\sum_{i=1}^{n} (1-c_i)=n-t \quad \text{and} \quad \sum_{j=1}^{m} (1-d_j)=m-q.$$

Thus,

$$b+m-s-1=\sum_{i=1}^{n} (c_i-1)(s-s_i)+\sum_{j=1}^{m} (d_j-1)(t-t_j).$$

Since $c_i\ge 1$ and $s\ge s_i$ for $1\le i\le n$ and $d_j\ge 1$ and $t\ge t_j$ for $1\le j\le m$, we have written $b+m-s-1$ as a sum of non-negative terms. Thus, $s<m+b$.

One special case of this type allows a quick affirmative answer.

WEAK COROLLARY: *If $b\le 0$ and $s+1=m+b$, then there exists a bipartite tournament with score sets S and T.*

Proof of weak corollary. Let $q_j=t_j$ for $1\le j\le m$ and

$$p_i=\begin{cases} s_i, & 1\le i<n \\ s, & n\le i<t \end{cases}.$$

To show that $p_1\le p_2\le\cdots\le p_t$ and $q_1\le q_2\le\cdots\le q_m$ are score sequences of a bipartite tournament we follow the proof of the main theorem through case iv, until we obtain the inequality $S(k, r)\ge b+m+rt-(t+1-k)s-1$. Since

$$b+m-s-1=0,\ S(k, r)\ge rt-(t-k)s\ge kr.$$

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There remains only to show that

$$\sum_{i=1}^{t} p_i + \sum_{j=1}^{m} q_j = mt.$$

Now,

$$\begin{aligned}\sum_{i=1}^{t} p_i + \sum_{j=1}^{m} q_j - mt &= \sum_{i=1}^{n} s_i + (t-n)s + \sum_{j=1}^{m} t_j - mt \\ &= b + m - s - 1 = 0.\end{aligned}$$

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