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On the Dichotomy of the Evolution Families: A Discrete-Argument Approach

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Abstract. We establish a discrete-time criteria guaranteeing the existence of an exponential dichotomy in the continuous-time behavior of an abstract evolution family. We prove that an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ acting on a Banach space X is uniformly exponentially dichotomic (with respect to its continuous-time behavior) if and only if the corresponding difference equation with the inhomogeneous term from a vector-valued Orlicz sequence space $l^{\Phi}(\mathbb{N}, X)$ admits a solution in the same $l^{\Phi}(\mathbb{N}, X)$. The technique of proof effectively eliminates the continuity hypothesis on the evolution family (*i.e.,* we do not assume that $U(\,\cdot\,,s)x$ or $U(t,\,\cdot\,)x$ is continuous on $[s,\infty)$, and respectively $[0,t]$). Thus, some known results given by Coffman and Schaffer, Perron, and Ta Li are extended.

1 Introduction

The approach proposed by O. Perron in 1930 to characterize the asymptotic behavior of the solutions of differential systems has come into widespread usage. More precisely, we refer the reader to the classical work of Perron entitled "Die Stabilitatsfrage bei Differentialgeighungen" [13], where he characterizes the exponential stability of the solutions of the linear systems

$$\frac{dx}{dt} = A(t)x, \quad t \in [0,\infty), x \in \mathbb{R}^n$$

($A(\,\cdot\,)$ is here a continuous and bounded matrix-valued function) in terms of the existence of bounded solutions of the equation $\frac{dx}{dt} = A(t)x + f(t)$, where f is a continuous and bounded function on $\mathbb{R}_+$. Relevant results concerning the extension of Perron's method in the more general context of infinite-dimensional Banach space were obtained by J. L. Daleckij and M. G. Krein in [4], and J. L. Massera and J. J. Schaffer in [9]. The subject was extensively analyzed for the general case of an abstract (strongly continuous, exponentially bounded) evolution family, by Y. Latushkin [2, 7, 8], N. van Minh [11, 12], S. Montgomery-Smith [7], P. Preda [10, 16], P. Randolph [8], and R. Schnaubelt [11, 19].

For the case of discrete-time systems analogous results were obtained first by Ta Li (a former student of Perron) in 1934 (see [20]). Following Perron's work, Ta Li establishes a connection between the condition that the inhomogeneous equation has some bounded solution for every bounded "second member", on the one hand, and a certain form of stability for the solutions of the homogeneous equation, on the other. This concept was called "admissibility" and it was extended to the the case of discrete-time systems in infinite dimensional Banach spaces, by C. V. Coffman and J. J. Schaffer in 1967 (for details, we refer the reader to [3]). In the same line of research,

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D. Henry analyzes in his excellent monograph from 1981 (see [5]) the existence of the exponential dichotomy of the linear differential equations $\dot{x}+A(t)x=0$, where A_0 is sectorial in a Banach space X, and the mapping $t \mapsto A(t)-A_0$ is bounded and locally Holder continuous. In that context, the discrete dichotomy of a sequence $\{T_n\}_{n\in\mathbb{Z}}$ of bounded linear operators has been studied in terms of the existence and uniqueness of bounded solutions for $x_{n+1}=T_nx_n+f_n$, for every bounded sequence $\{f_n\}_{n\in\mathbb{Z}}$ in X. Thus, he emphasized the relation between the discrete dichotomy and the exponential dichotomy of an evolution family. More recently, other interesting results on the discrete-time systems were pointed out by A. Ben-Artzi and I. Gohberg [1], J. P. La Salle [6], M. Pinto [14], C. Preda [15], and A. L. Sasu and B. Sasu [18]. Also, applications of this "discrete-time theory" in the study of the continuous-time behavior for the solutions of the linear infinite-dimensional differential equations have been presented by C. Preda in [15] and Przyluski and Rolewicz in [17].

Following the above line of results, we prove in this paper that an evolution family $\mathcal{U}=\{U(t,s)\}_{t\geq s\geq 0}$ acting on a Banach space X is uniformly exponentially dichotomic (with respect to its continuous-time behavior) if and only if the corresponding difference equation with the inhomogeneous term from a vector-valued Orlicz sequence space $l^{\Phi}(\mathbb{N},X)$ admits a solution in the same $l^{\Phi}(\mathbb{N},X)$. It is worth noting that the class of vector-valued Orlicz sequence spaces is very large, and thus the present approach generalizes the above works and also allows the reader to choose the "test functions" in various ways, accordingly to what is needed. Also, the technique of proof effectively eliminates the continuity hypothesis on the evolution family (*i.e.,* we do not assume that $U(\cdot,s)x$ or $U(t,\cdot)x$ is continuous on $[s,\infty)$, and respectively $[0,t]$).

2 Preliminaries

Let $B(X)$ be the Banach algebra of all linear and bounded operators acting on the Banach space X. We will refer in the section next to the classic vector-valued sequence spaces:

$$l^p(\mathbb{N},X)=\Big\{f\colon \mathbb{N}\to X:\sum_{n=0}^{\infty}\|f(n)\|^p<\infty\Big\},\quad p\in[1,\infty),$$
$$l^{\infty}(\mathbb{N},X)=\Big\{f\colon \mathbb{N}\to X:\sup_{n\in\mathbb{N}}\|f(n)\|<\infty\Big\}.$$

It is well known that $l^p(\mathbb{N},X)$, $l^{\infty}(\mathbb{N},X)$ are Banach spaces endowed with the respectively norms:

$$\|f\|_p=\Big(\sum_{n=0}^{\infty}\|f(n)\|^p\Big)^{1/p};\quad \|f\|_{\infty}=\sup_{n\in\mathbb{N}}\|f(n)\|.$$

For convenience, we have incorporated a very sketchy introduction about the scalar-valued Orlicz sequence space into this section. Let $\varphi\colon\mathbb{R}_+\to\mathbb{R}_+$ be a function that is non-decreasing, with $\varphi(t) > 0$, for each $t > 0$. Define

$$\Phi(t) = \int_0^t \varphi(s)ds,$$

the associated Young function. For $f : \mathbb{N} \to \mathbb{R}$ a scalar-valued sequence we define

$$m^\Phi(f) = \sum_{k=0}^{\infty} \Phi(|f(k)|).$$

The set $l^\Phi(\mathbb{N}, \mathbb{R})$ of all f for which there exists a scalar $j > 0$ such that $m^\Phi(jf) < \infty$ is clearly a vector space. Equipped with the Luxemburg norm

$$\|f\|_\Phi = \inf\left\{ j > 0 : m^\Phi\left(\tfrac{1}{j}f\right) \le 1 \right\}$$

the space $(l^\Phi(\mathbb{N}, \mathbb{R}), \|\cdot\|_\Phi)$ becomes a Banach space.

Remark 2.1 It is easy to check that $\chi_{\{0,\dots,n\}} \in l^\Phi(\mathbb{N}, \mathbb{R})$ and also

$$\|\chi_{\{0,\dots,n\}}\|_\Phi = \frac{1}{\Phi^{-1}(\frac{1}{n+1})} \quad \text{for all } n \in \mathbb{N}.$$

(As usual χ_A denotes the characteristic function (indicator) of some set A.)

Example 2.2 Setting the above Young function $\Phi(t) = t^p$, we obtain that $l^p(\mathbb{N}, \mathbb{R})$ is a scalar-valued Orlicz sequence space for all $p \in [1, \infty)$.

Remark 2.3 If $l^\Phi(\mathbb{N}, \mathbb{R}) = l^p(\mathbb{N}, \mathbb{R})$, then $\lim_{t\to 0} \frac{\Phi(t)}{t^p} = 1$.

Proof If $l^\Phi(\mathbb{N}, \mathbb{R}) = l^p(\mathbb{N}, \mathbb{R})$, then $\|\chi_{\{0,\dots,n\}}\|_\Phi = \|\chi_{\{0,\dots,n\}}\|_p$, for all $n \in \mathbb{N}$, and by Remark 2.1 we have that

$$\Phi^{-1}\left(\frac{1}{n+1}\right) = \left(\frac{1}{n+1}\right)^{\frac{1}{p}} \quad \text{for all } n \in \mathbb{N}.$$

Let $x \in (0, 1]$ and $m = \left[\frac{1}{x}\right] \in \mathbb{N}^*$, where $[a]$ denotes the greatest integer that is less than or equal to a. Using the fact that Φ^{-1} is nondecreasing we have that

$$\left(\frac{1}{m+1}\right)^{\frac{1}{p}} = \Phi^{-1}\left(\frac{1}{m+1}\right) \le \Phi^{-1}(x) \le \Phi^{-1}\left(\frac{1}{m}\right) = \left(\frac{1}{m}\right)^{\frac{1}{p}},$$

which implies that

$$\left[\frac{1}{\left(\left[\frac{1}{x}\right] + 1\right)x}\right]^{\frac{1}{p}} \le \frac{\Phi^{-1}(x)}{x^{\frac{1}{p}}} \le \left[\frac{1}{x\left[\frac{1}{x}\right]}\right]^{\frac{1}{p}} \quad \text{for all } x \in (0, 1].$$

Hence

$$\lim_{x\to 0} \frac{\Phi^{-1}(x)}{x^{\frac{1}{p}}} = 1 \quad \text{and} \quad \lim_{u\to 0} \frac{\Phi(u)}{u^p} = \lim_{u\to 0} \frac{1}{\left[\frac{\Phi^{-1}(\Phi(u))}{(\Phi(u))^{\frac{1}{p}}}\right]^p} = 1. \qquad \blacksquare$$

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Example 2.4 Consider $\varphi, \Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\varphi(t) = \sum_{m=1}^{\infty} \frac{t^{\frac{1}{m}}}{m^2}, \quad \Phi(t) = \int_0^t \varphi(s)ds = \sum_{m=1}^{\infty} \frac{t^{1+\frac{1}{m}}}{m(m+1)}.$$

We claim that $l^\Phi(\mathbb{N},\mathbb{R}) \neq l^p(\mathbb{N},\mathbb{R})$ no matter how we choose $p \in [1,\infty)$. Indeed,

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0, \text{ and } \lim_{t\to 0} \frac{\Phi(t)}{t^p} = \infty, \text{ for each } p \in [1,\infty),$$

and using the above remark, our claim follows easily.

Let now $l^\Phi(\mathbb{N},\mathbb{R})$ be a scalar-valued Orlicz sequence space. We denote by

$$l^\Phi(\mathbb{N}, X) = \{f\colon N \mapsto X : (\|f(n)\|)_{n\in\mathbb{N}} \text{ belongs to } l^\Phi(\mathbb{N},\mathbb{R})\}.$$

We will call $l^\Phi(\mathbb{N}, X)$ as a *vector-valued Orlicz sequence space.*

Remark 2.5 $l^\Phi(\mathbb{N}, X)$ is a Banach space endowed with the norm

$$\|f\|_{l^\Phi(\mathbb{N},X)} = \| \, \|f(\cdot)\| \, \|_\Phi.$$

Remark 2.6 For any scalar-valued Orlicz sequence space $l^\Phi(\mathbb{N},\mathbb{R})$, we have that:

(i) $l^\Phi(\mathbb{N},\mathbb{R}) \subset l^\infty(\mathbb{N},\mathbb{R})$;
(ii) $\|f\|_\infty \leq \frac{1}{\|\chi_{\{0\}}\|_\Phi}\|f\|_\Phi$.

Lemma 2.7 *If Φ is the Young function of the scalar-valued Orlicz sequence space $l^\Phi(\mathbb{N},\mathbb{R})$, then the followings statements hold:*

(i) *The map $a_\Phi : \mathbb{N} \to R_+^*$ given by $a_\Phi(n) = (n+1)\Phi^{-1}(\frac{1}{n+1})$ is nondecreasing;*
(ii) *$\sum_{k=0}^{n} |f(k)| \leq a_\Phi(n)\|f\|_\Phi$, for all $t > 0$, $f \in l^\Phi(\mathbb{N},\mathbb{R})$.*

Proof (i) First let us prove that the map $b\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ given by $b(u) = \frac{\Phi(u)}{u}$ is nondecreasing. If $0 < u_1 \leq u_2$, then

$$\begin{aligned}\frac{\Phi(u_1)}{u_1} &= \frac{1}{u_1}\int_0^{u_1} \varphi(s)ds = \frac{1}{u_1}\int_0^{u_2} \varphi(\frac{u_1}{u_2}v)\frac{u_1}{u_2}dv \\ &= \frac{1}{u_2}\int_0^{u_2} \varphi(\frac{u_1}{u_2}v)dv \leq \frac{1}{u_2}\int_0^{u_2} \varphi(v)dv = \frac{\Phi(u_2)}{u_2}.\end{aligned}$$

In order to prove that a_Φ is nondecreasing, we will choose randomly $n \in \mathbb{N}$, and we observe that $0 < w_2 := \Phi^{-1}(\frac{1}{n+2}) \leq \Phi^{-1}(\frac{1}{n+1}) := w_1$, and thus $b(w_2) \leq b(w_1)$. Having in mind that

$$b(w_1) = \frac{1}{a_\Phi(n+1)} \quad \text{and} \quad b(w_2) = \frac{1}{a_\Phi(n)},$$

it results that a_Φ is a nondecreasing function.

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(ii) Consider $f \in l^{\Phi}(\mathbb{N}, \mathbb{R}), n \in \mathbb{N}^*, c > 0$ such that $m^{\Phi}(\frac{1}{c} f) \le 1$. Then we have that

$$\Phi\left(\frac{1}{c(n+1)} \sum_{k=0}^{n} |f(k)|\right) \le \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(\frac{1}{c}|f(k)|\right) \le \frac{1}{n+1},$$

and so

$$\sum_{k=0}^{n} |f(k)| \le (n+1)\Phi^{-1}\left(\frac{1}{n+1}\right) c,$$

which implies that

$$\sum_{k=0}^{n} |f(k)| \le (n+1)\Phi^{-1}\left(\frac{1}{n+1}\right) \|f\|_{\Phi} = a_{\Phi}(n)\|f\|_{\Phi}$$

for all $n \in \mathbb{N}$, $f \in l^{\Phi}(\mathbb{N}, \mathbb{R})$. ∎

Remark 2.8 Using a simple translation argument we may state that

$$\sum_{k=n_0}^{n_0+n} |f(k)| \le a_{\Phi}(n)\|f\|_{\Phi}$$

for all $n_0, n \in \mathbb{N}$, $f \in L^{\Phi}(\mathbb{N}, \mathbb{R})$.

Definition 2.9 A family of bounded linear operators acting on X and denoted by $\mathcal{U} = \{U(t,s)\}_{t \ge s \ge 0}$ is called an evolution family if the following statements hold:

- $U(t,t) = I$ (where I is the identity operator on X) for all $t \ge 0$;
- $U(t,s) = U(t,r)U(r,s)$ for all $t \ge r \ge s \ge 0$;
- there exist $M > 0, \omega > 0$ such that

$$\|U(t,s)\| \le M e^{\omega(t-s)} \quad \text{for all} \quad t \ge s \ge 0.$$

Definition 2.10 A family of bounded linear operators $\{P(t)\}_{t \ge 0}$ acting on X is called a family of projectors if

- $P^2(t) = P(t)$ for all $t \ge 0$;
- $P(\cdot)x$ is bounded for all $x \in X$.

We also denote $Q(t) = I - P(t), t \ge 0$.

Definition 2.11 The evolution family $\mathcal{U} = \{U(t,s)\}_{t \ge s \ge 0}$ is said to be uniformly exponentially dichotomic (u.e.d) if there exist a family of projectors $\{P(t)\}_{t \ge 0}$ and two constants $N > 0, \nu > 0$ such that the following conditions hold:

- $U(t,s)P(s) = P(t)U(t,s)$ for all $t \ge s \ge 0$;
- $U(t,s)\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$ is an isomorphism for all $t \ge s \ge 0$;
- $\|U(t,s)x\| \le N e^{-\nu(t-s)}\|x\|$ for all $x \in ImP(s),\ t \ge s \ge 0$;
- $\|U(t,s)x\| \ge \frac{1}{N} e^{\nu(t-s)}\|x\|$ for all $x \in \operatorname{Ker} P(s),\ t \ge s \ge 0$.

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If the first two conditions from the above definition hold, we will denote

$$U_1(t,s) = U(t,s)|_{ImP(s)}, \quad U_2(t,s) = U(t,s)|_{\operatorname{Ker} P(s)}.$$

Remark 2.12 The evolution family is $\mathcal{U}$ is u.e.d. if and only if there exist the constants $N_1 > 0, N_2 > 0, \nu_1 > 0, \nu_2 > 0$ such that

$$\|U_1(t,s)\| \leq N_1 e^{-\nu_1(t-s)} \quad \text{and} \quad \|U_2(t,s)\| \geq N_2 e^{\nu_2(t-s)},$$

for all $t \geq s \geq 0$.

Definition 2.13 Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family and assume that there exist a family of projectors $\{P(t)\}_{t\geq 0}$ such that

- $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s \geq 0$;
- $U(t,s)\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$ is an isomorphism for all $t \geq s \geq 0$;

Then we say that a vector-valued Orlicz sequence space $l^{\Phi}(\mathbb{N}, X)$ is admissible to $\mathcal{U}$ if the following statements hold:

- $\sum_{k=n}^{\infty} \|U_2^{-1}(k,n)Q(k)f(k)\| < \infty$ for all for all $f \in l^{\Phi}(\mathbb{N}, X)$;
- $x_f\colon \mathbb{N} \to X$, defined by $x_f(n) = \sum_{k=0}^{n} U_1(n,k)f(k) - \sum_{k=n}^{\infty} U_2^{-1}(k,n)Q(k)f(k)$, lies in $l^{\Phi}(\mathbb{N}, X)$.

3 The Main Result

Lemma 3.1 *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family and assume that there exist a family of projectors $\{P(t)\}_{t\geq 0}$ such that*

- *$U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s \geq 0$;*
- *$U(t,s)\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$ is an isomorphism for all $t \geq s \geq 0$;*

If there exists a vector-valued Orlicz sequence space $l^{\Phi}(\mathbb{N}, X)$ that is admissible to $\mathcal{U}$, then there exists $K > 0$ such that $\|x_f\|_{l^{\Phi}(\mathbb{N},X)} \leq K\|f\|_{l^{\Phi}(\mathbb{N},X)}$.

Proof We define the operator $V_m\colon l^{\Phi}(\mathbb{N}, X) \to l^1(\mathbb{N}, X)$, given by

$$(V_m f)(k) = \begin{cases} U_2^{-1}(k,m)Q(k)f(k), & k \geq m, \\ 0, & k < m. \end{cases}$$

It is easy to see that V_m is a linear operator for each $m \in \mathbb{N}$. Now take $m \in \mathbb{N}$, $\{f_n\}_{n\in\mathbb{N}} \subset l^{\Phi}(\mathbb{N}, X),\ f \in l^{\Phi}(\mathbb{N}, X),\ g \in l^1(\mathbb{N}, X)$ such that

$$f_n \xrightarrow{l^{\Phi}(\mathbb{N},X)} f,\ V_m f_n \xrightarrow{l^1(\mathbb{N},X)} g.$$

By Remark 2.6 we have that

$$f_n(k) \longrightarrow f(k),\ (V_m f_n)(k) \longrightarrow g(k) \text{for all } k \in \mathbb{N},$$

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and hence $V_m f = g$. Thus V_m is bounded, for all $m \in \mathbb{N}$. We now define the linear operator $W: l^{\Phi}(\mathbb{N}, X) \to l^{\Phi}(\mathbb{N}, X)$ given by

$$(Wf)(m) = \sum_{k=0}^{m} U_1(m,k)P(k)f(k) - \sum_{k=m}^{\infty} U_2^{-1}(k,m)Q(k)f(k).$$

We take $\{g_n\}_{n\in\mathbb{N}} \subset l^{\Phi}(\mathbb{N}, X)$, $g \in l^{\Phi}(\mathbb{N}, X)$, $h \in l^{\Phi}(\mathbb{N}, X)$ such that

$$g_n \overset{l^{\Phi}(\mathbb{N},X)}{\longrightarrow} g, \; Wg_n \overset{l^{\Phi}(\mathbb{N},X)}{\longrightarrow} h.$$

Then we have

$$\begin{aligned}
&\|(Wg_n)(m) - (Wg)(m)\| \\
&\quad \leq \sum_{k=0}^{m} \|U_1(m,k)P(k)(g_n(k) - g(k))\| + \sum_{k=m}^{\infty} \|U_2^{-1}(k,m)Q(k)(g_n(k) - g(k))\| \\
&\quad \leq \Big(\sum_{k=0}^{m} \|U_1(m,k)P(k)\|\Big) \frac{1}{\|\chi_{\{0\}}\|_{\Phi}} \|g_n - g\|_{l^{\Phi}(\mathbb{N},X)} + \|V_m(g_n - g)\|_1,
\end{aligned}$$

for all $m, n \in \mathbb{N}$. It follows, again using Remark 2.6, that $Wg = h$. Thus we obtain that

$$\|x_f\|_{l^{\Phi}(\mathbb{N},X)} = \|Wf\|_{l^{\Phi}(\mathbb{N},X)} \leq \|W\| \|f\|_{l^{\Phi}(\mathbb{N},X)} \text{ for all } f \in l^{\Phi}(\mathbb{N}, X). \qquad \blacksquare$$

Lemma 3.2 *Let $g\colon \{(t,t_0) \in \mathbb{R}^2 : t \geq t_0 \geq 0\} \to \mathbb{R}_+$ be a function such that the following properties hold:*

(i) $g(t,t_0) \leq g(t,s)g(s,t_0)$ *for all* $t \geq s \geq t_0 \geq 0$;
(ii) $\sup_{0\leq t_0 \leq t \leq t_0+1} g(t,t_0) < \infty$;
(iii) *there exists a sequence* $h\colon \mathbb{N} \to \mathbb{R}$, $\lim_{n\to\infty} h(n) = 0$ *and* $g(m+n,n) \leq h(m)$ *for all* $m, n \in \mathbb{N}$.

Then there exist two constants $N, \nu > 0$ such that

$$g(t,t_0) \leq Ne^{-\nu(t-t_0)} \textit{ for all } t \geq t_0 \geq 0.$$

Proof Let

$$a = \sup_{0\leq t_0\leq t\leq t_0+1} g(t,t_0), \; m_0 = \min\{m \in \mathbb{N}^* : h(m) \leq \frac{1}{e}\}.$$

Conditions (i) and (ii) imply that $\sup_{0\leq t_0\leq t\leq t_0+2m_0} g(t,t_0) \leq a^{2m_0}$.

Fix $t_0 \geq 0$, $t \geq t_0 + 2m_0$, $m = [\frac{t}{m_0}]$, $n = [\frac{t_0}{m_0}]$, where $[s]$ denotes the largest integer less than or equal to $s \in \mathbb{R}$. One can see that

$$m_0 m \leq t < m_0(m+1), \; m_0 n \leq t_0 < m_0(n+1), \; m \geq n+2,$$

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and thus

$$\begin{aligned} g(t,t_0) &\le g(t,m_0m)g(m_0m,m_0(n+1))g(m_0(n+1),t_0) \\ &\le a^{4m_0}\prod_{k=n+2}^{m} g(m_0k,m_0(k-1)) \le a^{4m_0}\prod_{k=n+2}^{m} h(m_0) \\ &\le a^{4m_0}e^{-(m-n-1)} \le a^{4m_0}e^{-\frac{t-t_0}{m_0}+2}. \end{aligned}$$

Taking into account that

$$g(t,t_0) \le a^{2m_0} \le a^{2m_0}e^2e^{-\frac{t-t_0}{m_0}} \quad \text{for all} \ \ t_0 \ge 0,\ t \in [t_0, t_0+2m_0],$$

we easily obtain that

$$g(t,t_0) \le Ne^{-\nu(t-t_0)} \text{ for all } t \ge t_0 \ge 0,$$

where

$$N = \max\{a^{4m_0}e^2, a^{2m_0}e^2\},\ \nu = 1/m_0. \qquad \blacksquare$$

Theorem 3.3 *Let $\mathcal{U} = \{U(t,s)\}_{t\ge s\ge 0}$ be an evolution family and assume that there exists a family of projectors $\{P(t)\}_{t\ge 0}$ such that*

- *$U(t,s)P(s) = P(t)U(t,s)$, for all $t \ge s \ge 0$;*
- *$U(t,s)\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$ is an isomorphism for all $t \ge s \ge 0$.*

Then $\mathcal{U}$ is uniformly exponentially dichotomic if and only if there exists $l^{\Phi}(\mathbb{N},X)$ a vector-valued Orlicz sequence space that is admissible to $\mathcal{U}$.

Proof *Necessity.* It follows easily from Definition 2.13 that $l^1(\mathbb{N},X)$ is admissible to $\mathcal{U}$.

Sufficiency. Let $m \in \mathbb{N},\ x \in X$ and $f\colon \mathbb{N} \to X,\ f = \chi_{\{m\}}x$. It is easy to verify that $f \in l^{\Phi}(\mathbb{N},X)$ and $\|f\|_{l^{\Phi}(\mathbb{N},X)} = \|\chi_{\{0\}}\|_{\Phi}\|x\|$ and

$$\begin{aligned} (x_f)(k) &= \sum_{j=0}^{k} U_1(k,j)P(j)f(j) - \sum_{j=k}^{\infty} U_2^{-1}(j,k)Q(j)f(j) \\ &= \begin{cases} U_1(k,m)P(m)x, & k > m, \\ -U_2^{-1}(m,k)Q(m)x, & k < m, \end{cases} \end{aligned}$$

and thus we have

$$\begin{aligned} \|U_1(k,m)P(m)x\| &\le \|x_f\|_{\infty} \le \frac{1}{\|\chi_{\{0\}}\|_{\Phi}}\|x_f\|_{l^{\Phi}(\mathbb{N},X)} \\ &\le K\frac{1}{\|\chi_{\{0\}}\|_{\Phi}}\|f\|_{l^{\Phi}(\mathbb{N},X)} = K\|x\|, \quad \text{while} \quad k > m. \end{aligned}$$

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Also,

$$\|U_2^{-1}(m,k)Q(m)x\| \le \|x_f\|_\infty \le \frac{1}{\|\chi_{\{0\}}\|_\Phi}\|x_f\|_{l^\Phi(\mathbb{N},X)}$$
$$\le K\frac{1}{\|\chi_{\{0\}}\|_\Phi}\|f\|_{l^\Phi(\mathbb{N},X)} = K\|x\|, \quad \text{while} \quad k<m.$$

It is now clear that

$$\|U_1(m,n)\| \le K,\ \|U_2^{-1}(m,n)\| \le K \quad \text{for all} \quad m,n\in\mathbb{N}, \quad \text{with} \quad m\ge n.$$

Let $m, n_0 \in \mathbb{N},\ x \in ImP(n_0),\ f\colon \mathbb{N}\to X$ given by

$$f(n)=\begin{cases} U_1(n,n_0)x, & n\in\{n_0,\dots,n_0+m\},\\ 0, & n\notin\{n_0,\dots,n_0+m\}.\end{cases}$$

Then $f\in l^\Phi(\mathbb{N},X),\ \|f\|_{l^\Phi(\mathbb{N},X)} \le K\frac{1}{\Phi^{-1}(\frac{1}{m+1})}\|x\|$, and $f(n)\in ImP(n)$ for all $n\in\mathbb{N}$. It follows that

$$(x_f)(n)=\sum_{k=0}^{n}U_1(n,k)f(k)=\begin{cases}0, & n<n_0\\ (n-n_0+1)U_1(n,n_0)x, & n\in\{n_0,\dots,n_0+m\}\\ (m+1)U_1(m,n_0)x, & n\ge n_0+m+1\end{cases}$$

Thus we have

$$\begin{aligned}&\frac{(m+1)(m+2)}{2}\|U_1(m+n_0,n_0)x\|\\ &=\sum_{n=n_0}^{n_0+m}(n-n_0+1)\|U_1(m+n_0,n_0)x\|\\ &\le K\sum_{n=n_0}^{n_0+m}(n-n_0+1)\|U_1(n,n_0)x\| = K\sum_{n=n_0}^{n_0+m}\|x_f(n)\| \le Ka_\Phi(m)\|x_f\|_{l^\Phi(\mathbb{N},X)}\\ &\le K^2a_\Phi(m)\|f\|_{l^\Phi(\mathbb{N},X)} \le K^3\|x\|(m+1).\end{aligned}$$

We obtain that

$$\|U_1(m+n_0,m)\| \le \frac{2K^3}{m+2} \ \text{ for all }\ m,n_0\in\mathbb{N}.$$

By Lemma 3.2 it results that there exist two constants $N_1, \nu_1 > 0$ such that

$$\|U_1(t,t_0)\| \le N_1e^{-\nu_1(t-t_0)} \ \text{ for all }\ t\ge t_0\ge 0.$$

Consider again $m, n_0\in\mathbb{N}$, $x\in \operatorname{Ker} P(m+n_0)$, $g\colon\mathbb{N}\to X$ given by

$$g(n)=\begin{cases}U_2^{-1}(m+n_0,n)x, & n\in\{n_0,\dots,n_0+m\},\\ 0, & n\notin\{n_0,\dots,n_0+m\}.\end{cases}$$

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Then $g \in l^{\Phi}(\mathbb{N}, X)$, $\|g\|_{l^{\Phi}(\mathbb{N},X)} \leq K \frac{1}{\Phi^{-1}(\frac{1}{m+1})}\|x\|$ and $g(n) \in \operatorname{Ker} P(n)$, for all $n \in \mathbb{N}$. A simple computation shows that

$$\begin{aligned}(x_g)(n) &= -\sum_{k=n}^{n_0+m} U_2^{-1}(k,n)U_2^{-1}(m+n_0,k)x = -\sum_{k=n}^{n_0+m} U_2^{-1}(m+n_0,n)x \\ &= -(n_0+m-n+1)U_2^{-1}(m+n_0,n)x, \text{ for all } n \in \{n_0,\dots,n_0+m\},\end{aligned}$$

and hence

$$\begin{aligned}&\frac{(m+1)(m+2)}{2}\|U_2^{-1}(m+n_0,n_0)x\| \\ &\quad= \sum_{n=n_0}^{n_0+m}(n_0+m-n+1)\|U_2^{-1}(m+n_0,n_0)x\| \\ &\quad\leq K\sum_{n=n_0}^{n_0+m}(n_0+m-n+1)\|U_2^{-1}(m+n_0,n)x\| \\ &\quad= K\sum_{n=n_0}^{n_0+m}\|x_g(n)\| \leq Ka_{\Phi}(m)\|x_g\|_{l^{\Phi}(\mathbb{N},X)} \\ &\quad\leq K^2 a_{\Phi}(m)\|g\|_{l^{\Phi}(\mathbb{N},X)} \leq K^3 a_{\Phi}(m)\frac{1}{\Phi^{-1}(\frac{1}{m+1})}\|x\| \leq \frac{(2m+1)K^3}{a_{\Phi}(m)\frac{1}{\Phi^{-1}(\frac{1}{m+1})}}\|x\|.\end{aligned}$$

We can state that

$$\|U_2^{-1}(n_0+m,n_0)\| \leq \frac{2K^3}{a_{\Phi}(m)\frac{1}{\Phi^{-1}(\frac{1}{m+1})}}\|x\| \text{ for all } m, n_0 \in \mathbb{N}.$$

In order to apply Lemma 3.2 again, we observe that

$$U_2^{-1}(t,t_0) = U_2(t_0,[t_0])U_2^{-1}([t_0]+2,[t_0])U_2([t_0]+2,t)$$

for all $0 \leq t_0 \leq t \leq t_0+1$. This implies that

$$\sup_{0 \leq t_0 \leq t \leq t_0+1}\|U_2^{-1}(t,t_0)\| \leq M^2 e^{3\omega}K.$$

Hence we can find two constants $N_2, \nu_2 > 0$ such that

$$\|U_2^{-1}(t,t_0)\| \leq N_2 e^{-\nu_2(t-t_0)} \text{ for all } t \geq t_0 \geq 0.$$

By Remark 2.12, it follows that $\mathcal{U}$ is u.e.d. ■

Remark 3.4 By Example 2.2 we obtain, in the conditions of the above theorem, that $\mathcal{U}$ is uniformly exponentially dichotomic if and only if there exists $p \in [1,\infty]$ such that $l^p(\mathbb{N}, X)$ is admissible to $\mathcal{U}$. Also, Example 2.4 shows that the present approach can bring other interesting situations beside the classical $l^p(\mathbb{N}, X)$-admissibility.

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