# PROXIMITY AND SIMILARITY OF OPERATORS II

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**1. Introduction.** In this paper we continue the examination of the question of similarity of operators $A$ and $B$ begun in reference [3]. In that article, a similarity result was obtained based on a measure of closeness, or *proximity*, of the uniformly continuous semigroups $e^{tA}$ and $e^{tB}$, $t \geq 0$. The operators considered were elements of $\mathscr{B}(\mathscr{H})$, the algebra of bounded operators on a Hilbert space $\mathscr{H}$. We now wish to relax this requirement and replace $\mathscr{B}(\mathscr{H})$ by a complex Banach algebra $\mathscr{B}$, with unit $I$. In Section 2 we give a necessary condition for the similarity of $A, B \in \mathscr{B}$. We then give a condition sufficient to guarantee $A$ and $B$ are approximately similar (as defined in reference [5]). In Section 3 we restrict our attention to the case where $\mathscr{B} = \mathscr{B}(\mathscr{H})$. There we give a condition which guarantees $A, B \in \mathscr{B}(\mathscr{H})$ are intertwined by a Fredholm operator. This leads naturally into a discussion of proximity-similarity in the Calkin algebra $\mathscr{A}$. This is the subject of Section 4. Following reference [7] we define a metric $\rho$ on $\mathscr{N}(\mathscr{A})$, the normal elements of $\mathscr{A}$. We show $(\mathscr{N}(\mathscr{A}), \rho)$ is a complete metric space and that the unitary orbit of $a \in \mathscr{A}$ is the $\rho$-connected component of $a$ in $\mathscr{N}(\mathscr{A})$.

**2. Proximity-similarity in a Banach algebra.** In this section we investigate the proximity-similarity question in a complex Banach algebra $\mathscr{B}$ with unit. In Lemma 1, we give a necessary condition for operators $A, B \in \mathscr{B}$ to be similar. In Proposition 3 we give a condition which guarantees $A$ and $B$ are approximately similar (defined below). The proof of this proposition relies on properties of the Laplace transform, established in Lemma 2. Our notation is as follows. The set of invertible elements in $\mathscr{B}$ will be denoted by $\mathscr{G}(\mathscr{B})$. If $A \in \mathscr{B}$ then $\sigma(A)$ is the spectrum of $A$ and $r(A)$ is the spectral radius of $A$. Finally, the uniformly continuous group $e^{tA}$, $t \in \mathbb{R}$, will be denoted by $\{A_t\}$.

The following results are straightforward (we omit the proofs), but often used in what follows. We collect them here for reference.

Let $A \in \mathscr{B}$ and let $X \in \mathscr{G}(\mathscr{B})$. Then

$$\lim_{t \to \infty} \|e^{tA}\|^{1/t} = r(e^A), \tag{1}$$

$$\|XA\| \geq \frac{\|A\|}{\|X^{-1}\|}, \qquad \|AX\| \geq \frac{\|A\|}{\|X^{-1}\|}. \tag{2}$$

We now begin our discussion with two definitions. For $A$, $B$, and $T$ in $\mathscr{B}$ we define

$$\alpha_T(A, B) = \limsup_{t \to \infty} \frac{1}{t} \log \|A_{-t} T B_t\|$$

and

$$\alpha(A, B) = \inf_{T \in \mathscr{G}(\mathscr{B})} \alpha_T(A, B).$$

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Part (d) of the next lemma gives a necessary condition, in terms of $\alpha(A, B)$, for $A$ and $B$ to be similar in the algebra $\mathscr{B}$.

LEMMA 1. *Supppose $A$, $B$ and $T$ are in $\mathscr{B}$ with $T \neq 0$.*
(a) $\alpha_T(A, B) \leq \log r(e^{-A})r(e^B)$.
(b) *If $S \in \mathscr{G}(\mathscr{B})$, then* $\alpha_{TS}(A, B) = \alpha_T(A, SBS^{-1})$.
(c) *If $r(T) > 0$, then* $-\alpha_I(B, A) \leq \alpha_T(A, B)$.
(d) *If $A$ is similar to $B$, then* $\alpha(A, B) = 0$.

*Proof.* Part (a) follows from the inequality

$$\|A_{-t}TB_t\|^{1/t} \leq \|A_{-t}\|^{1/t}\,\|T\|^{1/t}\,\|B_t\|^{1/t}, \qquad \forall t > 0$$

and equation (1).

If $S \in \mathscr{G}(\mathscr{B})$, equation (2), yields

$$\frac{1}{t}\log\frac{\|A_{-t}TSB_t\|}{\|S\|} \leq \frac{1}{t}\log\|A_{-t}TSB_tS^{-1}\| \leq \frac{1}{t}\log\|A_{-t}TSB_t\|\,\|S^{-1}\|.$$

Noting that $A_{-t}TSB_tS^{-1} = A_{-t}(SBS^{-1})_t$, we obtain, after taking the limit supremum,

$$\alpha_{TS}(A, B) \leq \alpha_T(A, SBS^{-1}) \leq \alpha_{TS}(A, B).$$

Assume $r(T) > 0$. We have, for all $t > 0$,

$$B_{-t}TB_t = (B_{-t}A_t)(A_{-t}TB_t).$$

Hence, using $\sigma(B_{-t}TB_t) = \sigma(T)$, we have

$$0 < r(T) \leq \|B_{-t}TB_t\| \leq \|B_{-t}A_t\|\,\|A_{-t}TB_t\|.$$

This implies that

$$\frac{1}{t}\log r(T) \leq \frac{1}{t}\log\|B_{-t}A_t\| + \frac{1}{t}\log\|A_{-t}TB_t\|$$

and therefore

$$0 \leq \alpha_I(B, A) + \alpha_T(A, B).$$

Finally, we prove (d). If $A$ is similar to $B$ then there is an $S$ in $\mathscr{G}(\mathscr{B})$ such that $B = SAS^{-1}$. Part (b) of this lemma shows that

$$\alpha_T(A, B) = \alpha_T(A, SAS^{-1}) = \alpha_{TS}(A, A).$$

From this and part (c) we have

$$\alpha_T(A, B) = \alpha_{TS}(A, A) \geq -\alpha_I(A, A) = 0.$$

But, $\alpha_{S^{-1}}(A, B) = \alpha_I(A, A) = 0$, and hence $\alpha(A, B) = 0$.

REMARKS. 1. The estimate $\alpha_T(A, B) \leq \log r(e^{-A})r(e^B)$ is easily expressible in terms of the real parts of the spectra of $A$ and $B$.

2. Note that the condition

(C1) $\alpha(A, B) = 0$

is far from being a sufficient condition for similarity. For example, if $A$ and $B$ are self-adjoint operators on a Hilbert space, then, for all $T \neq 0$, $\alpha_T(iA, iB) = 0$.

3. We showed in part (d) that if $A$ is similar to $B$, then the infimum defining $\alpha$ is actually attained. The answer to the following question may prove interesting. Is there always a $T \in \mathcal{G}(\mathcal{B})$ such that $\alpha(A, B) = \alpha_T(A, B)$?

Now for $A$, $B$ and $T$ in $\mathcal{B}$ we define the normalized Laplace transform of the function $t \mapsto A_{-t}TB_t$ by

$$F(x) = x\int_0^\infty e^{-xt}A_{-t}TB_t\,dt.$$

The integral converges in the norm topology for all $x \in \mathbb{C}$ with $\mathfrak{R}(x) > \alpha_T(A, B)$. In this article we shall only consider the case $x \in \mathbb{R}$.

Lemma 2. *If $x > \alpha_T(A, B)$, then*

$$AF(x) - F(x)B = x(T - F(x)). \tag{3}$$

*Proof.* Let $s \in \mathbb{R}$. Then

$$A_{-s}F(x)B_s = x\int_0^\infty e^{-xt}A_{-(s+t)}TB_{(s+t)}\,dt$$

$$= xe^{sx}\int_s^\infty e^{-xt}A_{-t}TB_t\,dt.$$

It follows that

$$\frac{d}{ds}(A_{-s}F(x)B_s) = x^2e^{sx}\int_s^\infty e^{-xt}A_{-t}TB_t\,dt - xA_{-s}TB_s.$$

However,

$$\frac{d}{ds}(A_{-s}F(x)B_s) = A_{-s}F(x)B_sB - AA_{-s}F(x)B_s.$$

By acknowledging the equality of these two expressions at $s = 0$, we conclude that $AF(x) - F(x)B = x(T - F(x))$.

It is apparent that, if $F(x)$ is well behaved near $x = 0$, we should be able to establish at least an approximate similarity between $A$ and $B$ by utilizing equation (3). We do just that in Proposition 3, but first we make the notion of approximate similarity precise. Following reference **[6]**, we say that $A$ is *asymptotically similar* to $B$, if there are sequences $\{S_n\}$ and $\{T_n\}$ in $\mathcal{G}(\mathcal{B})$ such that

$$\lim_{n\to\infty} \|S_nAS_n^{-1} - B\| = \lim_{n\to\infty} \|A - T_nBT_n^{-1}\| = 0.$$

As introduced in reference **[5]**, $A$ and $B$ are said to be *approximately similar* if the above equations are valid for sequences $\{S_n\}$ and $\{T_n\}$ with $\|S_n\| \cdot \|S_n^{-1}\|$ and $\|T_n\| \cdot \|T_n^{-1}\|$ uniformly bounded in $n$.

The next proposition gives a somewhat stronger version of a result in reference **[3]**.

Proposition 3. *Suppose $A$, $B$ and $T$ are in $\mathcal{B}$. If*

$$\limsup_{t\to\infty} \|I - A_{-t}TB_t\| = \delta_0 < 1, \tag{4}$$

*then $A$ and $B$ are approximately similar.*

*Proof.* If (4) holds then there is a $\delta$ in $[\delta_0, 1)$ and a $t_0 > 0$ such that $\|I - A_{-t}TB_t\| < \delta$, for all $t \geq t_0$. It then follows that $\{A_{-t}TB_t\}$ is bounded for $t \geq 0$. Hence $\alpha_T(A, B) \leq 0$. Thus the Laplace transform is defined for all $x > 0$. Now

$$I - F(x) = x\int_0^\infty e^{-xt}(I - A_{-t}TB_t)\,dt,$$

whence

$$\begin{aligned}\|I - F(x)\| &\leq x\int_0^\infty e^{-xt}\,\|I - A_{-t}TB_t\|\,dt\\ &\leq x\int_0^{t_0} e^{-xt}\,\|I - A_{-t}TB_t\|\,dt + \delta x\int_{t_0}^\infty e^{-xt}\,dt\\ &\leq Mx + \delta e^{-xt_0}\\ &\leq Mx + \delta,\end{aligned}$$

where the constant $M$ is independent of $x$. This shows that there are positive numbers $a$ and $b$ with $b < 1$ such that for every $x$ in $(0, a)$ we have $\|I - F(x)\| < b$. In particular, for $x$ in $(0, a)$, $F(x)$ is invertible, and

$$\|F(x)\| < 1 + b, \qquad \|F(x)^{-1}\| \leq \frac{1}{1-b}.$$

From Lemma 2 we have $AF(x) - F(x)B = x(T - F(x))$ so that

$$\lim_{x\downarrow 0}\|A - F(x)BF(x)^{-1}\| = \lim_{x\downarrow 0}\|F(x)^{-1}AF(x) - B\| = 0.$$

REMARK. Note that if, for example, $\mathscr{B}$ is a dual space, then compactness arguments can be used to show $A$ is similar to $B$.

**3. Proximity-similarity for operators on a Hilbert space.** The proof of Proposition 3 can easily be modified to prove that if $X$ and $Y$ are in $\mathscr{G}(\mathscr{B})$ and

$$\limsup_{t\to\infty}\|I - A_{-t}YB_tX\| < 1, \tag{5}$$

then $A$ is approximately similar to $B$. If $\mathscr{B}$ is the algebra $\mathscr{B}(\mathscr{X})$ of operators on a Banach space $\mathscr{X}$, then (5) is equivalent to the following statement.

There is a $\delta$ in $(0, 1)$ and a $t_0 > 0$ such that for every $f \in \mathscr{X}$,

$$\sup_{t\geq t_0}\|(I - A_{-t}YB_tX)f\| < \delta\,\|f\|. \tag{6}$$

In case $\mathscr{X}$ is a Hilbert space, it follows from reference [4] that (6) is itself equivalent to:

for each $t > t_0$ there is an operator $D_t$ in $\mathscr{B}(\mathscr{X})$ with $\|D_t\| \leq \delta_0$ such that

$$X^{-1} - A_{-t}YB_t = D_tX^{-1}. \tag{7}$$

We now investigate the consequences of (7) when $X^{-1}$ is replaced by a Fredholm operator. We first prove a lemma.

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LEMMA 4. *Let $\mathscr{X}$ be a Hilbert space. Let $S$ be a bounded operator on $\mathscr{X}$ with closed range. Suppose $\{L_t\}$ is a family of bounded operators on $\mathscr{X}$ such that for all $t$, $L_t((S\mathscr{H})^\perp) = \{0\}$. If the mapping $t \mapsto L_tS$ is norm continuous then so is the map $t \mapsto L_t$.*

*Proof.* The operator $S$ acts invertibly from the orthogonal complement of the kernel of $S$ to the range of $S$. Hence there is a number $\alpha > 0$ such that for all $u \in (\ker S)^\perp$, $\|u\| \leq \alpha \|Su\|$. Let $f \in \mathscr{H}$ with $\|f\| = 1$. Write $f = Sg + h$ where $g \perp \ker S$ and $h \perp S\mathscr{H}$. We now have

$$\begin{aligned}\|(L_t - L_s)f\| &= \|(L_t - L_s)Sg\| \\ &\leq \|(L_t - L_s)S\| \, \|g\| \\ &\leq \alpha \, \|(L_t - L_s)S\| \, \|Sg\| \\ &\leq \alpha \, \|(L_t - L_s)S\| \, \|f\|.\end{aligned}$$

Therefore $\lim_{t \to s} \|L_t - L_s\| = 0$.

We are now in a position to prove the main result of this section.

PROPOSITION 5. *Suppose $\mathscr{H}$ is a separable Hilbert space and $A$, $B$, $G$ and $Y$ are bounded operators on $\mathscr{H}$ with $G$ a Fredholm operator. If there is a $\delta$ in $(0, 1)$ and a $t_0 > 0$ such that for all $t \geq t_0$ and all $f$ in $\mathscr{H}$, $\|(G - A_{-t}YB_t)f\| \leq \delta \, \|Gf\|$ then there is a Fredholm operator $F$ with index $(F) =$ index $(G)$, such that $AF = FB$.*

*Proof.* For each $t \geq t_0$, there is an operator $D_t$ with $\|D_t\| \leq \delta$ and $D_t(G\mathscr{H})^\perp = \{0\}$ such that $G - A_{-t}YB_t = D_tG$. (See the comment preceding equation (7).) Now the map $t \mapsto D_tG$ is norm continuous on $[t_0, \infty)$ and so Lemma 4 is applicable. Consequently $t \mapsto D_t$ is norm continuous on $[t_0, \infty)$. This allows us to define

$$H(x) = x \int_{t_0}^{\infty} e^{-xt} D_t \, dt,$$

the integral, as before, being with respect to the norm topology. If $F(x)$ is the normalized Laplace transform of $A_{-t}YB_t$, it follows that, for all $x > 0$,

$$G - F(x) = H(x)G + x \int_0^{t_0} e^{-xt}(G - A_{-t}YB_t) \, dt = H(x)G + O(x).$$

Now $x > 0$ implies $\|H(x)\| \leq \delta$. Therefore there is a sequence $\{x_n\}$ with $x_n \to 0$ such that both $\{F(x_n)\}$ and $\{H(x_n)\}$ are weakly convergent. Denoting their weak limits by $F$ and $G$, respectively, we have $G - F = HG$ and $\|H\| \leq \delta < 1$. This shows that $I - H$ is invertible; hence $F = (I - H)G$ is a Fredholm operator of the same index as $G$. Moreover, the equation $AF(x_n) - F(x_n)B = x_n(Y - F(x_n))$, which follows from Lemma 2, allows us to prove $AF = FB$.

**4. Proximity-similarity in the Calkin algebra.** Proposition 5 related directly to similarity in the Calkin algebra, $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, where $\mathscr{K}(\mathscr{H})$ is the ideal of compact operators in $\mathscr{B}(\mathscr{H})$. In this section, we will investigate other proximity-similarity results with regard to the Calkin algebra which are in keeping with the operator norm techniques

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of Section 3. We will employ the following notational conventions:

$\pi$ is the canonical homomorphism of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$.

$\|\cdot\|'$ represent the norm in $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$.

For $A \in \mathscr{B}(\mathscr{H})$, $\|A\|_e = \|\pi(A)\|'$ is the essential norm of $A$.

Since $G$ is a Fredholm operator if and only if $\pi(G)$ is invertible in $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, we may define

$$c(A, B) = \inf\{\|(\pi(G))^{-1}\|' \limsup_{t\to\infty} \|G - A_{-t}YB_t\| : Y \in \mathscr{B}(\mathscr{H}) \text{ and } G \text{ is a Fredholm operator}\}$$

Proposition 6. *Let $A$ and $B$ be bounded operators on $\mathscr{H}$. If $c(A, B) < 1$ then $\pi(A)$ and $\pi(B)$ are similar in the Calkin algebra.*

*Proof.* By assumption, there are numbers $\delta$ in $(0, 1)$ and $t_0 \geq 0$ and operators $Y$ and $G$ in $\mathscr{B}(\mathscr{H})$, with $G$ a Fredholm operator, such that, for all $t \geq t_0$, we have

$$\|(\pi(G))^{-1}\|' \|G - A_{-t}YB_t\| \leq \delta,$$

and so, for $x > 0$,

$$\|(\pi(G))^{-1}\|' \|G - F(x)\| \leq \delta. \tag{8}$$

As before, there is a sequence $\{x_n\}$ with $x_n \to 0$ such that $F(x_n)$ converges weakly to some operator $F$ and, using equation (3), we obtain

$$AF - FB = 0. \tag{9}$$

Moreover, (8) yields $\|(\pi(G))^{-1}\|' \|G - F\| \leq \delta$ which, in turn, implies that $\|(\pi(G))^{-1}\|' \|\pi(G) - \pi(F)\|' \leq \delta$. Hence $\|I - (\pi(G))^{-1}\pi(F)\|' \leq \delta$. Thus $(\pi(G))^{-1}\pi(F)$, and therefore $\pi(F)$, is invertible in the Calkin algebra. This proves that $F$ is a Fredholm operator. Applying the canonical homomorphism $\pi$ to both sides of equation (9) completes the proof.

Remark. The more general question as to whether the condition $\limsup_{t\to\infty} \|I - A_{-t}YB_t\|_e < 1$ implies similarity for $\pi(A)$ and $\pi(B)$ remains open.

We now wish to discuss another Calkin algebra result. However, in order to relate it to previous work, we temporarily return to a more general setting. Let $\mathscr{A}$ be a unital C*-algebra and let $\mathscr{N}(\mathscr{A})$ be the set of normal elements of $\mathscr{A}$. If $A \in \mathscr{A}$, the *unitary orbit* of $A$ is the set

$$\mathscr{U}(A) = \{UAU^{-1} : U \in \mathscr{A}, U \text{ unitary}\}.$$

For $A \in \mathscr{A}$ and $(s, t) \in \mathbb{R}^2$ define

$$A(s, t) = \exp(i[s\mathscr{R}(A) + t\mathscr{I}(A)]). \tag{10}$$

This defines a norm continuous, unitary valued function on $\mathbb{R}^2$. If $A \in \mathscr{N}(\mathscr{A})$ then $\{A(s, t)\}$ is a unitary group. For $A$ and $B$ in $\mathscr{N}(\mathscr{A})$ define

$$\rho(A, B) = \sup\{\|A(s, t) - B(s, t)\| : s, t \geq 0\}.$$

It was shown in reference [7] that in the case $\mathscr{A} = \mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, $(\mathscr{N}(\mathscr{A}), \rho)$ is a complete metric space and that the connected component of each element is its unitary orbit. We will show that an analogous result holds true in the Calkin algebra

$\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$. Although the pattern of the proof follows that in [7], some care is needed in the Calkin algebra setting. We therefore include many of the details.

PROPOSITION 7. *Let $\mathscr{A}$ be the Calkin algebra. Suppose A and B are in $\mathscr{N}_e = \pi^{-1}(\mathscr{N}(\mathscr{A}))$ and set $a = \pi(A)$, $b = \pi(B)$.*

(a) *If $\rho(a, b) < 1$, then A is unitarily equivalent to a compact perturbation of B.*

(b) *$(\mathscr{N}(\mathscr{A}), \rho)$ is a complete metric space.*

(c) *$\mathscr{U}(a)$ is the $\rho$-connected component of a in $\mathscr{N}(\mathscr{A})$.*

(d) *A is unitarily equivalent to a compact perturbation of B if and only if there is a finite set $\{B_0, B_1, \ldots, B_n\}$ in $\mathscr{N}_e$ such that $B_0 = A$, $B_n = B$, and $\rho(\pi(B_{i-1}), \pi(B_i)) < 1$ for $1 \le i \le n$.*

*Proof.* (a) Suppose $\rho(a, b) = \rho < 1$. For $x$ and $y > 0$ let

$$L(x, y) = xy \int_0^\infty \int_0^\infty e^{-(sx+ty)} A(-s, -t) \,.\, B(s, t)\, ds\, dt.$$

As in the one parameter setting, we have $\|1 - L(x, y)\|_e \le \rho$. Set $\lambda(x, y) = \pi(L(x, y))$ and $f(u, v) = a(-u, -v)\lambda(x, y)b(u, v)$. The integral formula above carries over via $\pi$ to $\lambda(x, y)$. On the one hand,

$$\frac{\partial f}{\partial u}(0, 0) = i(\lambda(x, y)\mathscr{R}(b) - \mathscr{R}(a)\lambda(x, y))$$

and

$$\frac{\partial f}{\partial v}(0, 0) = i(\lambda(x, y)\mathscr{I}(b) - \mathscr{I}(a)\lambda(x, y)),$$

while differentiation of the integral expression yields

$$\frac{\partial f}{\partial u}(0, 0) = x\lambda(x, y) - xy \int_0^\infty e^{-yt} a(0, -t) b(0, t)\, dt;$$

$$\frac{\partial f}{\partial v}(0, 0) = y\lambda(x, y) - xy \int_0^\infty e^{-xs} a(-s, 0) b(s, 0)\, dt.$$

Now,

$$\left\| xy \int_0^\infty e^{-yt} a(0, -t) b(0, t)\, dt \right\| \le x \quad \text{and} \quad \left\| xy \int_0^\infty e^{-xs} a(-s, 0) b(s, 0)\, dt \right\| \le y.$$

Since

$$\left(\frac{\partial f}{\partial u} + i \frac{\partial f}{\partial v}\right)\Bigg|_{(0,0)} = i(\lambda(x, y)b - a\lambda(x, y)),$$

we see that

$$\|\lambda(x, y)b - a\lambda(x, y)\|' \to 0 \quad \text{as} \quad (x, y) \to (0, 0).$$

Since $\|I - \lambda(x, y)\|' \le \rho < 1$, $a$ is approximately similar to $b$. It then follows from [1] that $A$ and $B$ have the same essential spectrum and index function on the essential resolvent. Thus the Brown–Douglas–Filmore Theorem [2] shows that $A$ is unitarily equivalent to a compact perturbation of $B$.

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(b) Note that $\rho$ is simply the restriction of the distance function on the Banach space of bounded continuous functions $C(\mathbb{R}_+ \times \mathbb{R}_+, \mathscr{A})$. If $\{a_n\}$ is a $\rho$-Cauchy sequence, let $f$ be its limit in $C(\mathbb{R}_+ \times \mathbb{R}_+, \mathscr{A})$. Since the unitary elements in $\mathscr{A}$ form a closed set, $\{f(s, t)\}$ is a continuous unitary semigroup. But then $\{f(s, 0)\}$ and $\{f(0, s)\}$ are continuous unitary group representations of $\mathbb{R}_+$ in $\mathscr{A}$. Let $iu$ and $iv$ be their respective generators, where $u$ and $v$ are self adjoint. Since $f(s, 0)f(0, t) = f(0, t)f(s, 0) = f(s, t)$, we have $uv = vu$. Let $a = u + iv$. Then $a(s, t)$, as defined in (10), is equal to $f(s, t)$ and hence $(\mathcal{N}(\mathscr{A}), \rho)$ is a complete metric space.

(c) We saw in part (a) of this proposition that if $\rho(a, b) < 1$, then $\mathscr{U}(a) = \mathscr{U}(b)$. In particular, each $\mathscr{U}(a)$ is open in the $\rho$-metric. If $\{a_n\}$ is a sequence in $\mathscr{U}(a)$ with $\rho(a_n, b) \to 0$, then for sufficiently large $n$, $a_n$ and $b$ are unitarily equivalent. Thus $b \in \mathscr{U}(a)$, so that $\mathscr{U}(a)$ is both open and closed in the $\rho$-metric. The proof of this part of the proposition will be completed by showing that $\mathscr{U}(a)$ is arcwise connected. Let $b \in \mathscr{U}(a)$ where $b = \pi(B)$, $a = \pi(A)$. There is a unitary operator $U$ and a compact operator $K$ on $\mathscr{H}$ such that $A = UBU^* + K$. Let $r \mapsto U_r$, $0 \le r \le 1$, be a $\|\,.\,\|$-continuous path such that $U_0 = I$ and $U_1 = U$, and let $A_r = U_rBU_r^* + K$, so that $A_0 = B + K$ and $A_1 = A$. Letting $a_r = \pi(A_r)$ and $u_r = \pi(U_r)$ we see that

$$\begin{aligned} a_r(s, t) &= \pi\{\exp(i[s\mathscr{R}(U_rBU_r^* + K) + t\mathscr{I}(U_rBU_r^* + K)])\} \\ &= \exp(i[u_r(s\mathscr{R}(b) + t\mathscr{I}(b))u_r^*]) = u_r b(s, t)u_r^*. \end{aligned}$$

Now, for any $r$ and $q$ in $[0, 1]$,

$$\begin{aligned} \rho(a_r, a_q) &= \sup_{s,t>0} \|u_r b(s, t)u_r^* - u_q b(s, t)u_q^*\|' \\ &= \sup_{s,t>0} \|u_r b(s, t)(u_r^* - u_q^*) + (u_r - u_q)b(s, t)u_q^*\|' \\ &\le 2\,\|u_r - u_q\|' \le 2\,\|U_r - U_q\|. \end{aligned}$$

We thus have a $\rho$-continuous arc in $\mathscr{U}(a)$ connecting $a$ to $b$.

(d) In view of part (a) of this proposition, we need only show that if $A$ and $B$ are essentially normal operators with $\pi(A)$ and $\pi(B)$ unitarily equivalent, then there is a finite set $\{B_0, B_1, \ldots, B_n\}$ in $\mathcal{N}_e$ such that $B_0 = A$, $B_n = B$, and $\rho(\pi(B_{i-1}), \pi(B_i)) < 1$ for $1 \le i \le n$. As in part (c), there is a $\rho$-continuous map $\gamma: [0, 1] \to \mathcal{N}(\mathscr{A}(\mathscr{H}))$ with $\gamma(0) = \pi(A)$ and $\gamma(1) = \pi(B)$. Since the range of $\gamma$ is $\rho$-compact, it is covered by a finite number of open spheres with radii less than 1/2 and centers in the range of $\gamma$.

REMARK. (1) In Proposition 7 we exhibited a metric on the set of normal elements in the Calkin algebra, in which closeness implies unitary equivalence. In that metric the scalar multiples of the identity are the only isolated points. Indeed, if $\pi(A)$ is a non-scalar, normal element of $\mathscr{A}(\mathscr{H})$ and $U$ is any unitary operator on $\mathscr{H}$ sufficiently close to $I$, then $\rho(\pi(A), \pi(U^*AU)) < 1$. The same type of metric was shown to exist in **[7]** on the set $\mathcal{N}(\mathscr{B}(\mathscr{H}))$. It would be interesting to know if such a "continuous" metric leading to similarity exists on $\mathscr{B}(\mathscr{H})$. One of several ill-fated attempts at finding such a metric has led to the following curious result.

*Let $A$ and $B$ be elements of a Banach algebra $\mathscr{B}$. Suppose that*

$$\limsup_{t\to\infty} \|A_t - B_t\|^{1/t} < \frac{1}{r(e^{-A})}.$$

*Then $A = B$.*

Indeed, it follows that there is a number $t_0 > 0$ and a number $\delta$ in $(0, 1)$ such that for all $t > t_0$,

$$\|1 - A_{-t}B_t\|^{1/t} \leq \|A_{-t}\|^{1/t}\,\|A_t - B_t\|^{1/t} < \delta.$$

Repeating the procedure used in Proposition 3, we find a constant $M$ such that

$$\|I - F(x)\| \leq Mx + x\int_{t_0}^{\infty} e^{-xt}\,\delta^t\,dt$$

$$= Mx + (e^{-xt_0}\,\delta^{t_0})\left(\frac{x}{x - \log\delta}\right).$$

Thus $\lim_{x\to 0} F(x) = I$. But $AF(x) - F(x)B = x(I - F(x))$, so that $A = B$.

(2) Note that the operators $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are similar in the algebra of all $2 \times 2$ matrices, but not in the algebra of diagonal matrices. In fact, if $X$ and $Y$ are invertible diagonal matrices, then

$$\lim_{t\to\infty} \|I - XA_{-t}YB_t\| = \infty.$$

Although the validity of these observations is made transparent by the commutativity of the set of diagonal matrices, one can illustrate the same type of behavior in the algebra of lower triangular matrices. In the general setting of two operators $A$ and $B$ in $\mathscr{B}(\mathscr{X})$, it might prove interesting to find the smallest norm closed algebra $\mathscr{C}$ containing $A$, $B$, and $I$ for which

$$\inf_{X,Y\in\mathscr{G}(\mathscr{C})}\ \limsup_{t\to\infty} \|I - XA_{-t}YB_t\| = \inf_{S,T\in\mathscr{G}(\mathscr{B}(\mathscr{X}))}\ \limsup_{t\to\infty} \|I - SA_{-t}TB_t\|.$$

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